INRIA Rocquencourt, Lundi 2 Février 1998

Excursions of Birth and Death Processes, Orthogonal Polynomials, and Continued Fractions

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Modelling the superposition of data connections on an ATM link

Rate Envelope Multiplexing (REM)

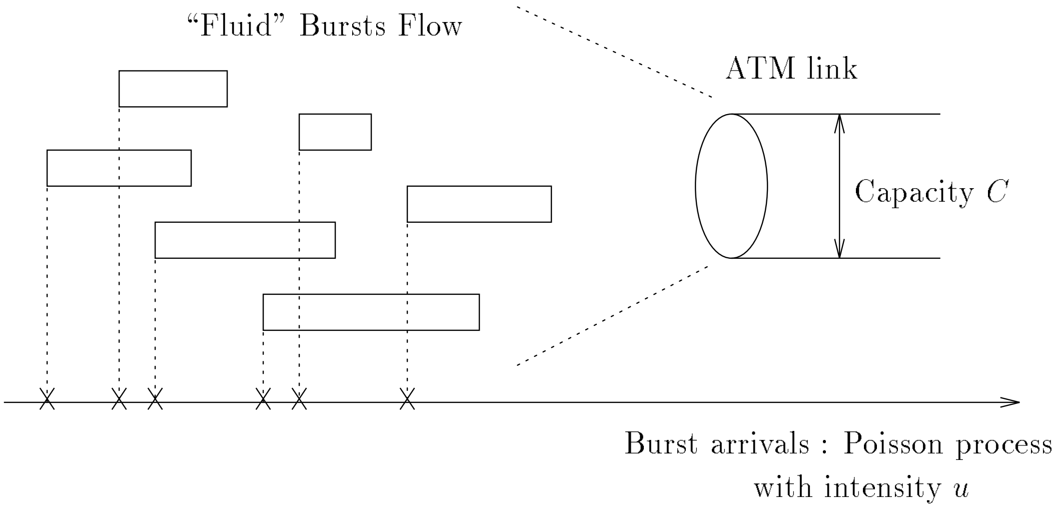

Figure 1: Superposition of fluid bursts on an ATM link

$\Rightarrow$ Excursions in $G/G/\infty$ queues.

Under exponential assumptions

$\Rightarrow$ **Excursions of birth and death processes.**

Quality of service parameters

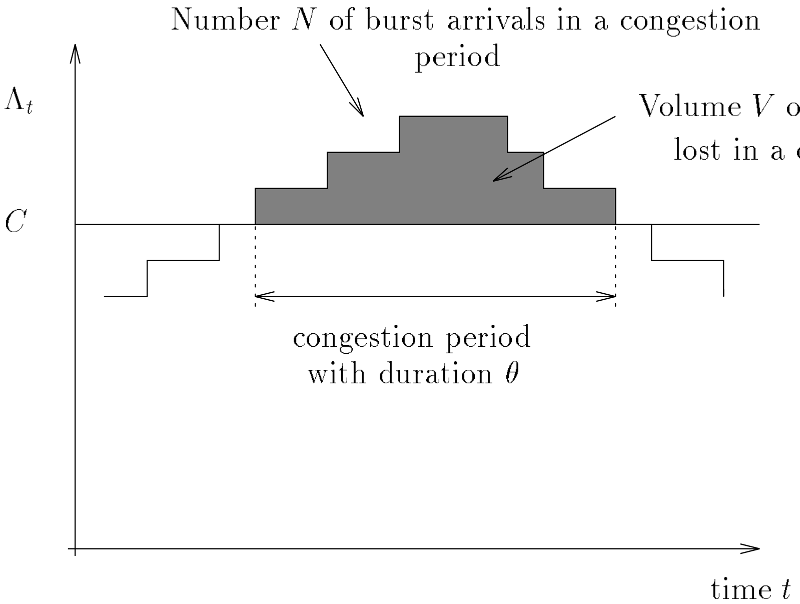

Figure 2: Superposition of fluid bursts on an ATM link

- the duration θ of an excursion above level C,
- the volume V of lost information,
- the number of bursts arriving in a busy period.

Result of Karlin and McGregor

A birth and death process $\{\Lambda_t\}$ with rates

$$q_{m,m+1} = \lambda_m > 0,\ q_{m,m-1} = \mu_m,\ q_{m,m} = -\left(\lambda_m + \mu_m\right) \text{ for } m \geq 0; \qquad (1)$$

$q_{m,n} = 0$ otherwise. (The rate μ_0 is equal to 0 and $\mu_m > 0$ for $m > 0$.)

Karlin and McGregor result : there exist

- an OPS $\{Q_n(x)\}$,
- a positive spectral measure $\mu(dx)$ of total mass one and not supported by a finite set of points,

so that

$$\text{for } m,n \geq 0,\ p_{m,n}(t) \stackrel{\text{def}}{=} \mathbb{P}\{\Lambda_t = n/\Lambda_0 = m\} = \pi_n \int_0^\infty e^{-tx} Q_n(x) Q_m(x) \mu(dx) \qquad (2)$$

and

$$\int_0^\infty Q_m^2(x)\mu(dx) = \frac{1}{\pi_m} \text{ with } \pi_0 = 1 \text{ and } \pi_m = \frac{\lambda_0 \dots \lambda_{m-1}}{\mu_1 \dots \mu_m} \text{ for } m \geq 1. \qquad (3)$$

The polynomials $\{Q_n(x)\}$ form an OPS w.r.t. $\mu(dx)$ and satisfy

$$\begin{cases} Q_0(x) \equiv 1, \quad Q_{-1}(x) \equiv 0, \\ \lambda_n Q_{n+1}(x) + (x - \lambda_n - \mu_n) Q_n(x) + \mu_n Q_{n-1}(x) = 0 \text{ for } n \geq 0. \end{cases} \qquad (4)$$

The OPS $\{Q_n(x)\}$= fundamental OPS.

Continued fractions

Let $\tilde{p}_0(z) = \int_0^\infty e^{-zt} p_{0,0}(t)dt$.

$$\text{For } \Re(z) > 0,\ \tilde{p}_0(z) = \cfrac{1}{z + \lambda_0 - \cfrac{\lambda_0\mu_1}{z + \lambda_1 + \mu_1 - \cfrac{\lambda_1\mu_2}{z + \lambda_2 + \mu_2 - \ddots}}}, \qquad (5)$$

Karlin and Mc Gregor result and the continued fraction $\tilde{p}_0(z)$ are used to study the transience and the recurrence of $\{\Lambda_t\}$ in a purely analytic way.

Properties of the continued fractions

The successive denominators $\{P_n^-(z)\}$ of the continued fraction $\tilde{p}_0(z)$ satisfy

$$\begin{cases} P_0^-(z) \equiv 1, \quad P_{-1}^-(z) \equiv 0, \\ P_{n+1}^-(z) + (-z - \lambda_n - \mu_n)P_n^-(z) + \lambda_{n-1}\mu_n P_{n-1}^-(z) = 0 \text{ for } n \geq 0. \end{cases} \tag{6}$$

Connection with the fundamental OPS

$$\begin{cases} P_0^-(-z) \equiv 1, \quad P_{-1}^-(-z) \equiv 0, \\ P_n^-(-z) = \lambda_0 \dots \lambda_{n-1} Q_n(z). \end{cases} \tag{7}$$

$\{P_n^-(-z)\}$ **form an OPS with respect to the measure** $\mu(dx)$

The J (Jacobi) fraction $\tilde{p}_0(z) =$ even part of a RITZ^{-1} fraction of the form

$$\tilde{P}(z) = \frac{\alpha_1|}{|z} + \frac{\alpha_2|}{|1} + \frac{\alpha_3|}{|z} + \frac{\alpha_4|}{|1} + \cdots,$$

whose even part $\tilde{P}^e(z)$ is given by

$$\tilde{P}^e(z) = \frac{\alpha_1|}{|z+\alpha_2} - \frac{\alpha_2\alpha_3|}{|z+\alpha_3+\alpha_4} - \frac{\alpha_4\alpha_5|}{|z+\alpha_5+\alpha_6} - \frac{\alpha_6\alpha_7|}{|z+\alpha_7+\alpha_8} - \cdots$$

Here,

$$\alpha_1 = 1, \ \alpha_{2k} = \lambda_{k-1}, \text{ and } \alpha_{2k+1} = \mu_k \text{ for } k \geq 1. \tag{8}$$

$\Rightarrow \tilde{P}(z)$ **is an** S **fraction.**

Stieltjes moment problem

The measure $\mu(dx)$ is one solution of the Stieltjes problem corresponding to the polynomials $\{P_m^-(-z)\}$.

$\tilde{p}_0(z)$ can be expressed as the Stieltjes transform of $\mu(dx)$

$$z \in \mathbb{C}\backslash(-\infty, 0),\ \tilde{p}_0(z) = \int_0^\infty \frac{1}{z+x}\mu(dx). \tag{9}$$

The measure $\mu(dx)$ is a priori not the unique solution to the Stieltjes moment problem.

Assumption (C_1)

$$\sum_{m=0}^{\infty} \pi_m < \infty.$$

The Markov chain $\{\Lambda_t\}$ is ergodic.

The stationary distribution $\{p_m\}$

$$p_0 = \frac{1}{\sum_{m=0}^\infty \pi_m} \text{ and for } \ m \geq 1,\ p_m = \frac{\pi_m}{\sum_{m=0}^\infty \pi_m}$$

Numerators of the continued fraction

The successive numerators $P_m^+(z)$ of the continued fraction $\tilde{p}_0(z)$ satisfy

$$\begin{cases} P_0^+(z) \equiv 0, \quad P_1^+(z) \equiv 1, \\ \text{and for } n \geq 0, \ \lambda_{n+1} \dfrac{P_{n+2}^+(z)}{\lambda_1 \ldots \lambda_{n+1}} + (-z - \lambda_{n+1} - \mu_{n+1}) \dfrac{P_{n+1}^+(z)}{\lambda_1 \ldots \lambda_n} \\ \qquad\qquad + \mu_{n+1} \dfrac{P_n^+(z)}{\lambda_1 \ldots \lambda_{n-1}} = 0. \end{cases} \tag{10}$$

Polynomials associated with the basic OPS $\{Q_n(x)\}$

$\{Q_n(x;\gamma)\}$, $\gamma = 0, 1, 2, \ldots$ are defined by the recursion

$$\begin{cases} Q_{-1}(z;\gamma) \equiv 0, \quad Q_0(z;\gamma) \equiv 1, \\ \text{for } n \geq 0, \ \lambda_{n+\gamma} Q_{n+1}(z;\gamma) + (z - \lambda_{n+\gamma} - \mu_{n+\gamma}) Q_n(z;\gamma) \\ \qquad\qquad + \mu_{n+\gamma} Q_{n-1}(z;\gamma) = 0. \end{cases} \tag{11}$$

These polynomials are also referred to as polynomials of the second kind.

The associated polynomials $\{Q_n(x;\gamma)\}$ form an OPS.

$$P_n^+(z) = \lambda_1 \ldots \lambda_{n-1} Q_{n-1}(-z;1) \tag{12}$$

Convergence of the continued fraction

$$\alpha_1 = \frac{1}{a_1} \text{ and } \alpha_n = \frac{1}{a_{n-1}a_n} \text{ for } n > 1. \tag{13}$$

In the present case, we have

$$\begin{aligned} a_2 &= \frac{1}{\lambda_0},\ a_{2m} = \frac{\mu_1 \ldots \mu_{m-1}}{\lambda_0 \ldots \lambda_{m-1}} = \frac{1}{\lambda_{m-1}\pi_{m-1}} \text{ for } m \geq 2, & (14) \\ a_1 &= 1,\ a_{2m+1} = \frac{\lambda_0 \ldots \lambda_{m-1}}{\mu_1 \ldots \mu_m} = \pi_m \text{ for } m \geq 1. & (15) \end{aligned}$$

Stieltjes' work :

$$\begin{aligned} S &= \sum_{n=1}^{\infty} a_n, & (16) \\ B &= \sum_{n=1}^{\infty} (a_1 + \ldots + a_{2n-1})a_{2n}, & (17) \\ C &= \sum_{n=1}^{\infty} (a_2 + \ldots + a_{2n})a_{2n+1}, & (18) \end{aligned}$$

The convergence of the continued fraction $\tilde{p}_0(z)$ depends on the series S, B, and C.

Convergence of the continued fraction (contn'd)

Convergence results and properties for the poles of the continued fraction:

Result 1. The zeros of the successive even denominators $\tilde{P}^-_{2n}(z)$ $(= P^-_n(z))$ and odd denominators $\tilde{P}^-_{2n+1}(z)$ of the continued fraction $\tilde{P}(z)$ are real, negative and simple. (This entails that the zeros of the polynomials $Q_n(x)$ are real, positive and simple).

Result 2. If $S < \infty$, then the odd and even parts of the continued fraction $\tilde{P}(z)$ converge, possibly to different functions, which are meromorphic with poles on the negative real axis; the poles s_k, $k = 1, 2, \ldots$ of $\tilde{p}_0$ satisfy the property

$$\sum_{k=2}^{\infty} \frac{1}{s_k} < \infty. \tag{19}$$

Result 3. If $S = \infty$, then the odd and even parts of the continued fraction $\tilde{P}(z)$ converge to the same function, which is analytic in the whole complex plane deprived of the negative real axis; the successive numerators and denominators do not have in general finite limits; however there two (and only two) cases where the even and the odd polynomials have finite limits

Result 3.1. if $B < \infty$, $P^-_n(z)$ converges to a finite limit $\tilde{P}^-(z)$ and the poles of $\tilde{P}(z)$ satisfy inequality (19);

Result 3.2. if $C < \infty$, the odd polynomials $\tilde{P}^-_{2n+1}(z)$ of the continued fraction $\tilde{P}(z)$ have a finite limit and the poles of $\tilde{P}(z)$ satisfy inequality (19).

Probabilistic Interpretation of the Series S, B, and C

- θ_m representing the duration of an excursion by the process $\{\Lambda_t\}$ above the level $m-1$;

$$\theta_m = \inf\{t > 0 : \Lambda_t = m - 1 \;/\; \Lambda_0 = m\}. \tag{20}$$

- τ_m be the passage time from state $m-1$ to state m;

$$\tau_m = \inf\{t > 0 : \Lambda_t \geq m \mid \Lambda_0 = m - 1\}. \tag{21}$$

Let $\mathcal{U}_m$ and $\mathcal{D}_m$ denote the upcrossing time from state 0 to state m and the downcrossing time from state m to state 0, respectively.

$$\mathcal{U}_m = \sum_{n=1}^{m} \tau_n \text{ and } \mathcal{D}_m = \sum_{n=1}^{m} \theta_n. \tag{22}$$

The random variable $\mathcal{C}_m =$ the time between two visits by the process $\{\Lambda_t\}$ at state 0, knowing that the process $\{\Lambda_t\}$ hits the state m.

Lemma 1 *Under condition* (C_1), *the respective mean values* $\overline{\theta}_m$ *and* $\overline{\tau}_m$ *of the random variables* θ_m *and* τ_m *are given by*

$$\overline{\theta}_m = \frac{1}{\lambda_{m-1}\pi_{m-1}} \sum_{n=m}^{\infty} \pi_n, \tag{23}$$

$$\overline{\tau}_m = \frac{1}{\lambda_{m-1}\pi_{m-1}} \sum_{n=0}^{m-1} \pi_n. \tag{24}$$

Probabilistic interpretation

Theorem 1 *Under condition* (C_1), *the series* S, B, *and* C *are related to the respective mean values* $\bar{\mathcal{U}}_m$, $\bar{\mathcal{D}}_m$ *and* $\bar{\mathcal{C}}_m$ *of the random variables* $\mathcal{U}_m$, $\mathcal{D}_m$ *and* $\mathcal{C}_m$ *for* $m \geq 1$ *as follows:*

$$B = \lim_{m\to\infty} \bar{\mathcal{U}}_m, \tag{25}$$

$$C = \lim_{m\to\infty} \bar{\mathcal{D}}_m, \tag{26}$$

$$S = \frac{1}{p_0} + \lim_{m\to\infty} \bar{\mathcal{C}}_m, \tag{27}$$

Impact of the ergodicity assumption (C_1) **on the properties of the continued fraction** $\tilde{p}_0$**.**

Since from the ergodic assumption

$$\lim_{t\to\infty} p_{0,0}(t) = \pi_0 > 0 \Rightarrow \tilde{p}_0(0) = \infty$$

From the theory of continued fractions

$$\tilde{p}_0(z) = \sum_{m=1}^{\infty} \frac{\lambda_0 \dots \lambda_{m-2}\mu_1 \dots \mu_{m-1}}{P^-_{m-1}(z)P^-_m(z)}. \tag{28}$$

$$P^-_m(0) = \lambda_0 \dots \lambda_{m-1} \Rightarrow \sum_{m=1}^{\infty} \frac{\mu_1 \dots \mu_{m-1}}{\lambda_0 \dots \lambda_{m-1}} = \infty$$

$\Rightarrow S = \infty$, which entails, under condition (C_1), $B = \infty$.

Consequences of condition (C_1)

The solution to the Stieltjes moment problem is unique and the measure $\mu(dx)$ is the unique solution to the Stieltjes moment problem.

The odd and even parts of the S-fraction $\tilde{P}(z)$ converge to the same limit, which is an analytic function in the complex plane deprived of the non positive real axis.

$\mu(dx)$ has a mass at the origin (equal to $1/\sum_{m=0}^{\infty} \pi_m = p_0 > 0$).

Condition (C_1) is not sufficient to ensure $C < \infty$ or $C = \infty$.

$$(\lambda_m = u,\ \mu_m = m^2,\ C < \infty)$$

$$(\lambda_m = u,\ \mu_m = m,\ C = \infty)$$

Laplace transforms of transient characteristics

Let $\tau_{m,n}$ denote the first passage time from state m to state n, $n > m$. ($\tau_m \stackrel{\text{def}}{=} \tau_{m-1,m}$.)

Proposition 1 *The Laplace transform $\tilde{\tau}_{m,n}(z)$ of the passage time $\tau_{m,n}$ from state m to state n, $n > m$ is given by*

$$\text{for } \Re(z) \geq 0, \ \tilde{\tau}_{m,n}(z) = \frac{Q_m(-z)}{Q_n(-z)}. \tag{29}$$

Corollary 1 *The Laplace transform of the random variable $\mathcal{U}_m$ is given by*

$$\text{for } \Re(z) \geq 0, \ \tilde{\mathcal{U}}_m(z) = \frac{1}{Q_m(-z)}. \tag{30}$$

Proposition 2 *The Laplace transform $\tilde{\theta}_m$ of the random variable θ_m for $m \geq 1$ can formally be represented by a continued fraction as*

$$\tilde{\theta}_m(z) \ = \ -\frac{1}{\lambda_{m-1}} \ \mathop{\Phi}_{n=1}^{\infty} \ \frac{-\lambda_{m+n-2}\mu_{m+n-1}}{z + \lambda_{m+n-1} + \mu_{m+n-1}} \text{ for } \Re(z) \geq 0. \tag{31}$$

In the following, we introduce the scale function of the process $\{\Lambda_t\}$,

$$\mathbb{P}_{i_0}\{T_i < T_j\} = \frac{\sigma(i_0) - \sigma(j)}{\sigma(i) - \sigma(j)}. \tag{32}$$

where $T_i = \inf\{t > 0 : \Lambda_t = i\}$.

A possible choice for the scale function is

$$\sigma(0) = 0, \ \sigma(1) = \frac{\mu_1}{\lambda_0(\mu_1 + \lambda_1)}, \ \sigma(m) = \sum_{\ell=1}^{m-1} \frac{\mu_1 \dots \mu_l}{\lambda_0 \dots \lambda_\ell} \text{ for } m > 1. \tag{33}$$

Convergence and properties of the continued fraction $\tilde{\theta}_m(z)$

The J-fraction $\tilde{\theta}_m(z)$ is the even part of the $RITZ^{-1}$ fraction $\tilde{\Theta}_m(z)$ of the form

$$\tilde{\Theta}_m(z) = \frac{\alpha_1^m|}{|z} + \frac{\alpha_2^m|}{|1} + \frac{\alpha_3^m|}{|z} + \frac{\alpha_4^m|}{|1} + \cdots,$$

whose even part $\tilde{\Theta}_m^e(z)$ is given by

$$\tilde{\Theta}_m^e(z) = \frac{\alpha_1^m|}{|z+\alpha_2^m} - \frac{\alpha_2^m\alpha_3^m|}{|z+\alpha_3^m+\alpha_4^m} - \frac{\alpha_4^m\alpha_5^m|}{|z+\alpha_5^m+\alpha_6^m} - \frac{\alpha_6^m\alpha_7^m|}{|z+\alpha_7^m+\alpha_8^m} - \cdots$$

In the present case, we have

$$\begin{cases} \alpha_1^m = \mu_m \text{ and } \alpha_2^m = \lambda_m + \mu_m; \\ \alpha_{2k}^m\alpha_{2k+1}^m = \lambda_{m+k-1}\mu_{m+k} \text{ and } \alpha_{2k+1}^m + \alpha_{2(k+1)}^m = \lambda_{m+k} + \mu_{m+k} \text{ for } k \geq 1. \end{cases} \tag{34}$$

Lemma 2 *The coefficients α_k^m are given by $\alpha_1^m = \mu_m$ and for $k \geq 1$*

$$\alpha_{2k}^m = \lambda_{m+k-1}\frac{\sum_{\ell=0}^{k}\frac{\mu_m\ldots\mu_{m+\ell-1}}{\lambda_m\ldots\lambda_{m+\ell-1}}}{\sum_{\ell=0}^{k-1}\frac{\mu_m\ldots\mu_{m+\ell-1}}{\lambda_m\ldots\lambda_{m+\ell-1}}} = \lambda_{m+k-1}\frac{\sigma(m+k-1)-\sigma(m-1)}{\sigma(m+k-2)-\sigma(m-1)} \tag{35}$$

$$\alpha_{2k+1}^m = \mu_{m+k}\frac{\sum_{\ell=0}^{k-1}\frac{\mu_m\ldots\mu_{m+\ell-1}}{\lambda_m\ldots\lambda_{m+\ell-1}}}{\sum_{\ell=0}^{k}\frac{\mu_m\ldots\mu_{m+\ell-1}}{\lambda_m\ldots\lambda_{m+\ell-1}}} = \mu_{m+k}\frac{\sigma(m+k-2)-\sigma(m-1)}{\sigma(m+k-1)-\sigma(m-1)}, \tag{36}$$

where σ is the scale function.

The above result shows that the coefficients $\alpha_k^m > 0$ and then that $\tilde{\Theta}_m(z)$ is an S fraction.

Convergence of the continued fraction

Define the sequence $\{a_k^m\}$ by

$$a_1^m = \frac{1}{\alpha_1^m} \text{ and } \alpha_k^m = \frac{1}{a_{k-1}^m a_k^m} \text{ for } k \geq 2. \tag{37}$$

For $k \geq 0$

$$a_{2k}^m = \frac{\mu_m \dots \mu_{m+k-1}}{\lambda_m \dots \lambda_{m+k-1}} \frac{1}{A_{k-1}^m A_k^m} \text{ and } a_{2k+1}^m = \frac{\lambda_m \dots \lambda_{m+k-1}}{\mu_m \dots \mu_{m+k}} \left[A_k^m\right]^2. \tag{38}$$

where

$$A_{-1}^m = 0, \; A_0^m = 1 \text{ and } A_k^m = \sum_{\ell=0}^{k} \frac{\mu_m \dots \mu_{m+\ell-1}}{\lambda_m \dots \lambda_{m+\ell-1}} \text{ for } k \geq 1.$$

We have (?)

$$S_m \stackrel{def}{=} \sum_{k=1}^{\infty} a_k^m \geq \sum_{k=1}^{\infty} a_{2k+1}^m = \sum_{k=1}^{\infty} \frac{\lambda_m \dots \lambda_{m+k-1}}{\mu_m \dots \mu_{m+k}} \left[A_k^m\right]^2,$$

Since $\frac{\mu_1 \dots \mu_{m-1}}{\lambda_0 \dots \lambda_{m-1}} A_k^m$ behaves as $\sigma(m+k)$ as $k \to \infty$ and $\sigma(k) \to \infty$ as $k \to \infty$,

$$\Rightarrow S_m = \infty$$

Proposition 3 *The continued fraction $\tilde{\theta}_m(z)$ defined by eq. (31) converges to an analytic function over the whole complex plane deprived of the negative real axis.*

Consequences

$\tilde{\Theta}_m(z)$ is the Stieltjes transform of some non-decreasing, bounded function $\hat{\theta}_m$, namely

$$\tilde{\Theta}_m(z) \equiv \tilde{\theta}_m(z) = \int_0^\infty \frac{1}{z+s} d\hat{\theta}_m(s). \tag{39}$$

The series $B_m = \infty$ and then, the measure $d\hat{\theta}_m(x)$ is unique. **(To be checked.)**

Let $\theta_n^-(z;m)$ be the nth denominator of $\tilde{\theta}_m(z)$.

$\{\tilde{\theta}_m(z)\}$ form an OPS w.r.t. $\tilde{\theta}_m(z)$

$\theta_{-1}^-(z;m) = 0$, $\theta_0^-(z;m) = 1$ and for $n \geq 0$

$$\lambda_{m+n}\left(\frac{\theta_{n+1}^-(z;m)}{\lambda_m \ldots \lambda_{m+n}}\right) - (z + \lambda_{m+n} + \mu_{m+n})\left(\frac{\theta_n^-(z;m)}{\lambda_m \ldots \lambda_{m+n-1}}\right) + \mu_{m+n}(1-\delta_{0,n})\left(\frac{\theta_{n-1}^-(z;m)}{\lambda_m \ldots \lambda_{m+n-2}}\right) = 0. \tag{40}$$

which entails that

$$\theta_n^-(z;m) = \lambda_m \ldots \lambda_{m+n-1} Q_n(-z;m); \tag{41}$$

Proposition 4 *The associated polynomials $\{Q_n(z;\gamma)\}$, $\gamma = 0,1,2,\ldots$ form an OPS.*

Numerators of the continued fractions

The numerators $\theta_n^+(z;m)$ of the continued fraction $\tilde{\theta}_m(z)$ satisfy the recursion $\theta_0^+(z;m) = 0$, $\theta_1^+(z;m) = \mu_m$ and for $n \geq 0$

$$\lambda_{m+n+1}\left(\frac{\theta_{n+2}^-(z;m)}{\lambda_{m+1}\ldots\lambda_{m+n+1}}\right) - (z + \lambda_{m+n+1} + \mu_{m+n+1})\left(\frac{\theta_{n+1}^-(z;m)}{\lambda_{m+1}\ldots\lambda_{m+n+1}}\right) + \mu_{m+n}\left(\frac{\theta_n^-(z;m)}{\lambda_{m+1}\ldots\lambda_{m+n-1}}\right) = 0. \quad (42)$$

By identification, we have

$$\theta_n^+(z;m) = \mu_m\lambda_{m+1}\ldots\lambda_{m+n-1}Q_{n-1}(-z, m+1) \quad (43)$$

It follows that the [n-1/n] Padé approximant $[\tilde{\theta}_m(z)]_n$ of the continued fraction $\tilde{\theta}_m(z)$ is

$$\frac{\mu_m}{\lambda_m}\frac{Q_{n-1}(-z;m+1)}{Q_n(-z;m)}.$$

Open issues:

1. Convergence of $\{Q_n(z;m)$ when $n \to \infty$.
2. Discrete spectrum for $\hat{\theta}_m(dx)$.
3. Relationship between $\mu(dx)$ and the measures $\hat{\theta}_m(dx)$.

Further Results on the Continued Fraction

Proposition 5 *For $m \geq 1$, $\tilde{\theta}_m(z)$ satisfies*

$$\tilde{\theta}_m(z) = \frac{Q_m(-z)}{Q_{m-1}(-z)} - \frac{\mu_1 \dots \mu_{m-1}}{\lambda_0 \dots \lambda_{m-1} Q_{m-1}^2(-z) \left(\tilde{p}_0(z) - [\tilde{p}_0(z)]_{m-1}\right)}. \qquad (44)$$

where $\{Q_m(x)\}$ is the fundamental OPS defined by the recurrence relations (4) and where $[\tilde{p}_0(z)]_{m-1}$ denotes the mth approximant of the continued fraction $\tilde{p}_0(z)$.

The zeros in z of $Q_m(-z)$ are negative and it follows that these zeros may potentially be singularities for the Laplace transform $\tilde{\theta}_m(z)$.

However, it can be shown that they are removable singularities.

The actual singularities of $\tilde{\theta}_m(z)$ are located in $(-\infty, 0]$ and are the negative roots of the equation

$$\tilde{p}_0(z) = [\tilde{p}_0(z)]_{m-1}.$$

Case study: the $M/M/\infty$ system

The fundamental OPS = Charlier polynomials satisfying

$$\text{for } n \geq 0,\ uQ_{n+1}(x;a) + (x-u-a)Q_n(x;a) + nQ_{n-1}(x;a) = 0,$$

with $Q_0(x;a) = 1$ and $Q_{-1}(x;a) = 0$.

Set $\theta = \theta_{C+1}$.

$$\tilde{\theta}(p) = -\frac{1}{u} \mathop{\Phi}_{m=1}^{\infty} \frac{-u(C+m)}{p+u+C+m} \text{ for } p \geq 0.$$

[m-1/m] Padé approximant

$$[\tilde{\theta}(z)]_m = \frac{C+1}{u} \frac{Q_{m-1}(-z;u;C+2)}{Q_m(-z;u;C+1)}$$

Associated Charlier polynomials:

$$\begin{aligned} & Q_m(x;u;C+1) \\ = \quad & (u)^{-m} \frac{\Gamma(C+1-x+m)}{\Gamma(C+1-x)} \Phi(C+1, C+1-x, -u) \Phi(-C-m-1, -C-m+x; u) \\ - \quad & u \frac{\Gamma(C+2+m)}{\Gamma(C+1)} \frac{\Gamma(C-x)}{\Gamma(C+m+2-x)} \Phi(-C, 1+x-C; u) \Phi(C+m+2, C+m+2-x; -u). \end{aligned} \tag{45}$$

($\Phi(\alpha, \gamma; z)$ denotes Kummer's function.)

Limiting value:

$$\tilde{\theta}(z) = \frac{C+1}{z+C+1} \frac{\Phi(z, z+C+2; u)}{\Phi(z, z+C+1; u)}.$$

The poles are computed by solving the equation:

$$\Phi(1, z; -u) = [\Phi(1, z; -u)]_C.$$

Volume of lost information

Definition

$$V = \int_0^\theta (\Lambda_t - C)dt \text{ with } \Lambda_0^- = C \text{ and} \Lambda_0 = C+1.$$

Proposition 6 *The Laplace transform $\tilde{V}$ can be represented by a continued fraction as*

$$\tilde{V}(z) = -\frac{1}{u} \mathop{\Phi}_{m=1}^{\infty} \frac{-u(C+m)}{mz+u+C+m} \textit{ for } \Re(p) \geq 0. \qquad (46)$$

[m-1/m] Padé approximant

$$[\tilde{V}(z)]_m = \frac{(C+1)(z+1)}{u} \frac{Q_{m-1}(r(z); s(z); C+2)}{Q_m(r(z); s(z); C+1)}, \qquad (47)$$

where $r(z) = \frac{Cz(z+1)-uz}{(z+1)^2}$, $s(z) = \frac{u}{(z+1)^2}$.

Limiting value

$$\tilde{V}(z) = \frac{C+1}{(z+1)(C+1-r(z))} \times \frac{\Phi(C+2, C+2-r(z); -s(z))}{\Phi(C+1, C+1-r(z); -s(z))}$$

The poles are the negative solutions to the equation

$$f(p) = [f(p)]_C \text{ with } f(p) = \frac{-1}{r(p)} \Phi(1, 1-r(p); -s(p)). \qquad (48)$$

The poles also satisfy

$$\mathcal{S}_C(p) \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} Q_C(n; s(p))^2 \frac{j_n(s(p))}{n-r(p)} = 0. \qquad (49)$$

Numerical observations

Numerical evidence shows that the poles accumulate at point -1. Specifically, it seems that the distance between a pole ρ_n and a solution p_n to the equation

$$r(p) = n \text{ for } n \geq 0,$$

satisfies

$$\rho_n - p_n = O\left(n^{-\frac{3}{2}}\right).$$

Proof ?

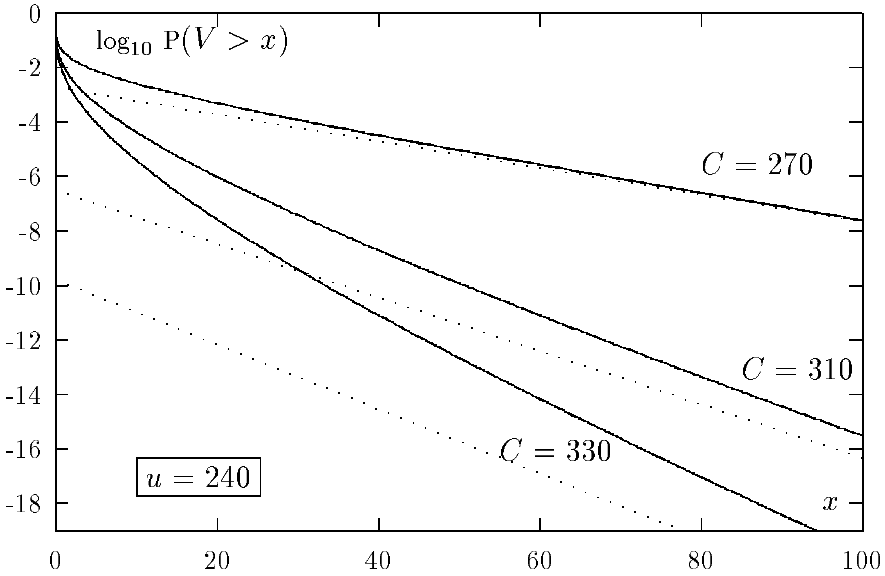

Figure 3: Survivor probability distribution function of the random variable V for $u = 240$ and $C = 270$, 310 and 330.

The tail of the distribution is exponential.

Area of an $M/M/1$

Consider an $M/M/1$ queue with input rate ρ and unit service rate.

Definition: $\mathcal{A} = \int_0^\beta n_t dt$, provided that a busy period starts at time 0; β is the duration of the busy period.

Continued fraction representation:

$$\tilde{\mathcal{A}}(s) = -\frac{1}{\rho} \mathop{\Phi}_{m=1}^{\infty} \frac{\dfrac{-\rho}{(m-1+\delta_{m,1})m}}{s + \dfrac{1+\rho}{m}}, \tag{50}$$

[m-1/m] approximant:

$$[\tilde{\mathcal{A}}(s)]_m = \frac{1}{s} \frac{g_{m-1,\nu(s)+1}\left(\frac{\rho}{s^2}\right)}{g_{m,\nu(s)}\left(\frac{\rho}{s^2}\right)}, \tag{51}$$

where $g_{m,\nu}(z)$ are modified Lommel polynomials satisfying the recursion

$$\begin{cases} g_{-1,\nu}(z) = 0, \; g_{0,\nu}(z) = 0, \\ \\ g_{m+1,\nu}(z) = (m+1+\nu) g_{m,\nu}(z) - z g_{m-1,\nu}(z) \text{ for } m \geq 0. \end{cases} \tag{52}$$

By Hurwitz limit,

$$\lim_{m\to\infty} [\tilde{\mathcal{A}}(s)]_m \stackrel{def}{=} \tilde{\mathcal{A}}(s) = \frac{1}{\sqrt{\rho}} \frac{J_{\nu(q)+1}\left(\dfrac{2\sqrt{\rho}}{q}\right)}{J_{\nu(q)}\left(\dfrac{2\sqrt{\rho}}{q}\right)} \tag{53}$$

for $q > 0$, where J_ν is the Bessel function of order ν, and with

$$\nu(q) = (1+\rho)/q. \tag{54}$$

Numerical considerations

Alternative representation

$$\tilde{\mathcal{A}}(s) = \frac{\alpha\nu(s)}{\sqrt{\rho}(\nu(s)+1)} \frac{{}_0F_1\left(\nu(s)+2; -\alpha^2\nu(s)^2\right)}{{}_0F_1\left(\nu(s)+1; -\alpha^2\nu(s)^2\right)}. \tag{55}$$

The non positive integers are simple poles for the function ${}_0F_1(x;y)$ of variable x.

The identity

$${}_0F_1(x;y) = 1 - \sum_{n=0}^{\infty} \frac{(-y)^{(n+1)}}{n!(n+1)!}\, {}_0F_1(n+2;y)\frac{1}{x+n}, \tag{56}$$

shows that $x = -n$ for integer $n \geq 0$ is a simple pole for the function ${}_0F_1(x;y)$ and that $\sigma_n = -\frac{1+\rho}{n}$ for $n \geq 1$ is not a pole for the function $\tilde{\mathcal{A}}(s)$.

The value of this function at σ_n is specifically given by

$$\tilde{\mathcal{A}}(\sigma_n) = \frac{1}{\sqrt{\rho}} \frac{J_{n-1}\left(\frac{2\sqrt{\rho}}{1+\rho}n\right)}{J_n\left(\frac{2\sqrt{\rho}}{1+\rho}n\right)}, \tag{57}$$

where we have used the identity $J_{-n}(-x) = J_n(x)$.

The relation (56) also shows that the function $s \to {}_0F_1\left(\nu(s)+1; -\alpha^2\nu(s)^2\right)$ has a unique zero s_n on each interval (σ_{n-1}, σ_n) for $n \geq 1$, which is a pole for the function $\tilde{\mathcal{A}}(s)$. (By convention, $\sigma_0 = -\infty$.)

Let r_m be the residue of $\tilde{\mathcal{A}}$ at pole s_m.

Asymptotic approximations

Conjecture 1 *Let M denote any positive integer. When $n \to \infty$, the dominating term in the expression of $\sum_{m=M}^{\infty} r_m/(\sigma_n - s_m)$ is $r_n/(\sigma_n - s_n)$.*

Conjecture 2 *As $n \to \infty$, the dominating term in the expression of $\frac{\partial \mathcal{A}^\star}{\partial s}(\sigma_n)$ is $r_n/(\sigma_n - s_n)^2$.*

These two conjectures allow us to determine the asymptotic behaviour of the poles and the residues.

Proposition 7 *The value of $\tilde{A}$ at point σ_n is such that*

$$\tilde{A}(\sigma_n) \to \frac{1}{\rho}. \tag{58}$$

when $n \to \infty$.

Proposition 8 *The poles and the residues of $\tilde{A}$ are such that*

$$\frac{r_n}{(\sigma_n - s_n)} \to \frac{1-\rho}{\rho} \tag{59}$$

when $n \to \infty$.

Proposition 9 *The poles and the residues of $\tilde{A}$ are such that*

$$\frac{r_n}{(s_n - \sigma_n)^2} \sim \frac{2\pi(1-\rho)n^2}{\rho(1+\rho)} e^{2n(\zeta - \tanh\zeta)} \tag{60}$$

as $n \to \infty$, where

$$\zeta = -\log\sqrt{\rho} \tag{61}$$

is such that $\zeta - \tanh\zeta > 0$.

Asymptotic approximations (contn'd)

Theorem 2 *The poles s_n and the residues r_n of the Laplace transform $\tilde{\mathcal{A}}$ are such that*

$$\sigma_n - s_n \sim \frac{1+\rho}{2\pi n^2} e^{-2n(\zeta - \tanh\zeta)} \quad \text{and} \quad -\frac{r_n}{s_n} \sim \frac{1-\rho}{2\pi\rho n} e^{-2n(\zeta - \tanh\zeta)} \tag{62}$$

when $n \to \infty$, where ζ is defined by eq. (61).

Theorem 3 *The tail of the distribution of the random variable $\mathcal{A}$ is given by*

$$\mathbb{P}\{\mathcal{A} > x\} \sim \frac{1-\rho}{\rho\sqrt{2\pi\gamma}} x^{-\frac{1}{4}} e^{-\gamma\sqrt{x}}. \tag{63}$$

as $x \to \infty$, where $\gamma = 2\sqrt{-2(1-\rho) - (1+\rho)\log\rho}$.

Table 1: Comparison between the survivor probability function of the random variable $\mathcal{A}$ and the approximation $G(x)$ for $\rho = 0.6$.

x	$-\log_{10} \mathbb{P}\{\mathcal{A} > x\}$	$-\log_{10} G(x)$
1	0.317347500	0.399653772
10	0.932404327	0.896833144
20	1.183226840	1.121826247
30	1.354022199	1.280745231
40	1.488799173	1.408841867
50	1.602446302	1.518406463
60	1.701973435	1.615352403
70	1.791291779	1.703036426
80	1.872827253	1.783570573
90	1.948198257	1.858381222
100	2.018543724	1.928482950
200	2.562992710	2.477246003
300	2.963536888	2.884602409
400	3.294765379	3.222141479
500	3.583356031	3.516228564
1000	4.699078480	4.650276670
2000	6.255457746	6.222890205

Conclusion

The random variable $\mathcal{A}$ is heavy tailed.

It is possible by using Laplace transforms (saddle point method) to prove Aldous' local linearization heuristic: For the $M/M/\infty$ queue with input rate u and unit service rate, if $u = \rho C$ with $\rho \in (0,1)$,

$$\{\Lambda^C_{Ct}\} \to \{n_t\} \text{ as } C \to \infty, \tag{64}$$

which entails

$$\begin{aligned} C\theta &\xrightarrow{d} \beta, \\ CV &\xrightarrow{d} \mathcal{A}, \\ N &\xrightarrow{d} n \end{aligned}$$

as $C \to \infty$.

Even if the probability that a congestion occurs is very low, the amount of information lost in a congestion period may take very large values.

This phenomenon should be taken into account to estimate the QoS offered to applications when REM is performed.