Absolute Factorization of Differential Operators

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[summary by Frédéric Chyzak]

1. The Problem

Consider the linear ODE $y^{(n)}(x)+a_{n-1}(x)y^{(n-1)}(x)+\cdots+a_0(x)y(x)=0$, where the coefficients a_i are rational functions of $k=C(x)$ for an algebraic closure C of the rational number field $\mathbb{Q}$. Solving this equation is an easier task when the corresponding linear differential operator in $\partial = d/dx$,

$$L=\partial^n+a_{n-1}(x)\partial^{n-1}+\cdots+a_0(x),$$

admits a factorization $L=L_2L_1$ where the product denotes composition. The Leibniz rule

$$\partial\cdot ay=(ay)'=a'y+ay'=(a\partial+a')\cdot y \qquad (a\in k)$$

defines a degree on the non-commutative ring $\mathbb{A}=k[\partial]$, which makes it left and right Euclidean.

Consider the operator

$$L=\partial^4-\frac{1}{4}\partial^3+\frac{3}{4x^2}\partial^2-x.$$

It can be proved to be *irreducible* in $\mathbb{A}$, i.e., it admits no factorization L_2L_1 in $\mathbb{A}$. However, L factorizes over the extension ring $k(\sqrt{x})[\partial]$:

$$L=\left(\partial^2-\frac{1}{x}\partial+\frac{3}{4x^2}-\sqrt{x}\right)\left(\partial^2-\sqrt{x}\right)=\left(\partial^2-\frac{1}{x}\partial+\frac{3}{4x^2}+\sqrt{x}\right)\left(\partial^2+\sqrt{x}\right).$$

Note that since $\sqrt{x}$ and $-\sqrt{x}$ are algebraically and differentially indiscernable, the conjugates of a right factor of L are other right factors of L. In the example above, L is the *least common left multiple* of both conjugate right factors.

More generally, an operator $L\in\mathbb{A}$ is called *absolutely reducible* when there exists an algebraic extension k_{ext} of k such that L is reducible in $\mathbb{A}_{\text{ext}}=k_{\text{ext}}[\partial]$ (for a suitable extension of the action of ∂ on k_{ext}). For an absolutely reducible operator L with a right factor $L_1\in\mathbb{A}_{\text{ext}}$, let $\tilde{L}$ be the least common left multiple of the algebraic conjugates of L_1. As a simple result of differential Galois theory, $\tilde{L}$ is stable under the action of the differential Galois group of the extension $\mathbb{A}_{\text{ext}}$ over $\mathbb{A}$ (to be defined in the next section). This entails that $\tilde{L}\in\mathbb{A}$. Since $\tilde{L}$ divides L, we have that L is irreducible but absolutely reducible in $\mathbb{A}$ if and only if L is the least common left multiple of the conjugates of a right factor $L_1\in\mathbb{A}_{\text{ext}}$.

The example above motivates the following problems, sorted by increasing complexity:

1. find an algorithm to *decide* absolute reducibility;
2. find an algorithm to *compute* a factorization on an algebraic extension;
3. find an algorithm to compute a factorization on an algebraic extension with *absolutely irreducible factors*.

The algorithms to solve these problems, reduce to solving ODE's for solutions in special classes. A solution y such that $y \in k$ is called a *rational solution*, while a solution y such that $y'/y \in k$ is called an *exponential solution*[1] and a solution y such that y'/y is algebraic over k is called a *Liouvillian solution*. An early study on this topic dates back to Liouville [6, 7]. The first algorithm to solve for rational solutions was developed in [1]. It relies on the resolution for polynomial solutions, for which an optimized algorithm is presented in [2]. Next, algorithms for factorization as well as algorithms to solve for Liouvillian solutions rely on the resolution for rational or exponential solutions. Algorithms for factorization are given in [3, 4, 9, 12]. The first algorithm to solve for Liouvillian solutions of second-order ODE's is due to Kovacic [5] and was later elaborated in [11], again in the second-order case. A prototypical algorithm for higher-order equations is to be found in [8] and was highly improved on in [10] in the third order case.

In the remainder of this summary, we comment on an algorithm to solve the second problem.

2. Differential Galois Theory

In the suitable analytical framework, the solution space V of the equation $L \cdot y = 0$ is the C-vector space generated by n *linearly independent* solutions y_i. However, these solutions satisfy *algebraic differential relations*

$$P_i\left(y_1, y_1', \ldots, y_1^{(n-1)}, \ldots, y_n, y_n', \ldots, y_n^{(n-1)}\right) = 0$$

for polynomials P_i in n^2 variables and with coefficients in k. As an example, any solution y_1 of the equation $y'' + y = 0$ satisfies an algebraic equation $y_1^2 + {y_1'}^2 = c \in C$. For a given L, we would like to describe the ideal $\mathfrak{I}$ generated by all algebraic differential relations. A description is obtained by *differential Galois theory*.

For a *differential field extension* K of k, the group of automorphisms σ of K that induce the identity on k and such that $\sigma(f') = \sigma(f)'$ for $f \in K$ is called the *differential Galois group* of K over k and is denoted $\mathrm{Gal}(K/k)$. Let K be $k\left(y_1, \ldots, y_1^{(n-1)}, \ldots, y_n, \ldots, y_n^{(n-1)}\right)$, i.e., the smallest differential field extension of k which contains the y_i's and does not extend the field of constants C. This field is called the *Picard-Vessiot extension* of L. The group $\mathrm{Gal}(K/k)$ is called the *differential Galois group* of L and denoted $\mathrm{Gal}_k(L)$. A computational representation of G is obtained as follows. Assume y to satisfy $L \cdot y = 0$, then for any automorphism $\sigma \in G$, $L \cdot \sigma(y) = \sigma(L \cdot y) = 0$. In other words, each automorphism moves a solution of L to another solution. Consequently, $\sigma(y)$ is a linear C-combination of the y_i's with coefficients that are independent from y. This yields a matrix representation of G. Thus G is linear algebraic and the ideal $\mathfrak{I}$ is stable under the action of G.

We now proceed to introduce a lemma which is crucial to the algorithm discussed in the next section. Assume that L admits a right factor L_1 with solution space $V_1 \subset V$. For any $v_1 \in V_1$ and any automorphism $\sigma \in G$, $L_1 \cdot \sigma(v_1) = \sigma(L_1 \cdot v_1) = 0$, so that V_1 is stable under G. We want to prove a converse property.

For an r-tuple $(v_1, \ldots, v_r) \in K^r$, the Wronskian $\mathrm{Wr}(v_1, \ldots, v_r)$ is classically defined as the matrix $\left[v_i^{(j)}\right]$. The corresponding determinant induces an application from K^r to K. This application is an alternate r-linear form and satisfies

$$\sigma(\det(\mathrm{Wr}(v_1, \ldots, v_r))) = \det(\mathrm{Wr}(\sigma(v_1), \ldots, \sigma(v_r)))$$

for any $\sigma \in G$. Below, we more intrinsically use r-exterior products, i.e., formal alternate r-linear symbols $v_1 \wedge \cdots \wedge v_r$ that satisfy $\sigma(v_1 \wedge \cdots \wedge v_r) = \sigma(v_1) \wedge \cdots \wedge \sigma(v_r)$ for any $\sigma \in G$.

[1] Such a solution is also frequently referred to as a *hyperexponential solution*.

Let us assume V_1 to be a 2-dimensional C-vector subspace of V with basis (f_1, f_2) and stable under the action of G. More specifically, for each $\sigma \in G$ there exist $c_{i,j}^{(\sigma)} \in C \setminus \{0\}$ such that

$$\sigma(f_i) = c_{i,1}^{(\sigma)} f_1 + c_{i,2}^{(\sigma)} f_2.$$

Then in the exterior power $\Lambda^2(V_1)$ where $f_1 \wedge f_1 = f_2 \wedge f_2 = 0$,

$$\sigma(f_1 \wedge f_2) = \sigma(f_1) \wedge \sigma(f_2) = (c_{1,1}c_{2,2} - c_{1,2}c_{2,1})(f_1 \wedge f_2).$$

More generally, assume that V_1 is a C-subspace of V stable under G and with dimension $\dim V_1 = r < n = \dim V$. Then, the exterior r-power $\bigwedge^r(V_1)$ is a 1-dimensional vector space with basis $\omega = f_1 \wedge \cdots \wedge f_r$. For each $\sigma \in G$, there exists a non-zero $c_\sigma \in C$ such that $\sigma(\omega) = c_\sigma \omega$. In fact, $c_\sigma = \det \sigma$ when σ is viewed as a C-linear automorphism of V_1. Now, for $y \in V$, write

$$L_1 \cdot y = \frac{\det(\mathrm{Wr}(y, f_1, \ldots, f_r))}{\det(\mathrm{Wr}(f_1, \ldots, f_r))}.$$

This makes L_1 a linear operator of order r. For any $\sigma \in G$,

$$\sigma(L_1 \cdot y) = \frac{\sigma(\det(\mathrm{Wr}(y, f_1, \ldots, f_r)))}{\sigma(\det(\mathrm{Wr}(f_1, \ldots, f_r)))} = \frac{c_\sigma \sigma(\det(\mathrm{Wr}(y, f_1, \ldots, f_r)))}{c_\sigma \sigma(\det(\mathrm{Wr}(f_1, \ldots, f_r)))} = L_1 \cdot y.$$

The coefficients of L_1 are therefore left fixed by all elements of G, and $L_1 \in k[\partial]$.

Lemma 1. *An operator L with solution space V admits a right factor L_1 such that the solution space V_1 of L_1 is a subspace of V if and only if there exists a non-zero proper subspace of V which is stable under G.*

3. The Beke-Bronstein Algorithm

Wronskians relate the solutions of an ODE to its coefficients. In particular, the Wronskian $w = \det(\mathrm{Wr}(y_1, \ldots, y_n)) = \det\left[Y, Y', \ldots, Y^{(n-1)}\right]$ where Y is the column vector of the y_i's satisfies

$$\begin{aligned} w' &= \sum_{i=1}^{n-1} \det\left[Y, \ldots, Y^{(i-1)}, Y^{(i+1)}, Y^{(i+1)}, \ldots, Y^{(n-1)}\right] + \det\left[Y, \ldots, Y^{(n-2)}, Y^{(n)}\right] \\ &= -\sum_{i=0}^{n-1} a_i(x) \det\left[Y, \ldots, Y^{(n-2)}, Y^{(i)}\right] = -a_{n-1}(x) \det\left[Y, Y', \ldots, Y^{(n-1)}\right] = -a_{n-1}(x) w. \end{aligned}$$

In short $w' + a_{n-1}(x) w = 0$ (*Liouville relation*); the other coefficients of L satisfy similar relations.

The algorithm developed and implemented by Bronstein after Beke's work and described in [4] makes use of Wronskians in the following way. To obtain a right factor of the operator L:

1. solve $L \cdot y = 0$ for exponential solutions; if solutions are found, they yield first-order right factors of L;
2. similarly, find first-order left-hand factors by the method of adjoint operators [4]; if solutions are found, they yield right factors of L of order $n-1$;
3. if no solution was found, look for right factors of order r $(2 \le r \le n-2)$ as follows:
   (a) build an equation whose solution space is spanned by all Wronskians of order r;
   (b) solve for exponential solutions;
   (c) test which solutions are Wronskians, i.e., *pure* exterior products, and obtain a right factor.

As a comparison, Singer's method, which was implemented by Van Hoeij, relies on solving for rational solutions only.

4. An Example

Again, consider the operator $L = \partial^4 - \frac{1}{4}\partial^3 + \frac{3}{4x^2}\partial^2 - x$. Both first steps of the algorithm above fail, so that the only possible factorizations are of the form $L = L_2 L_1$ with factors of order 2. Write $w = y_1 y_2' - y_1' y_2$ for any two solutions of L. By computing its first derivatives, reducing them by L on the basis $\left(y_1^{(i)} y_2^{(j)}\right)_{i,j=0,\ldots,3}$, and looking for linear dependencies by Gaussian elimination, we obtain that w is annihilated by

$$P = \partial^5 - \frac{5}{2x}\partial^4 + \frac{21}{4x^2}\partial^3 - \frac{69}{8x^3}\partial^2 + \frac{8x^5+15}{2x^4}\partial.$$

The only exponential solutions are the constants $\lambda \in C$. This entails that $L_1 = \partial^2 - \lambda\partial + r(x)$ for an algebraic function r. By identification, one finds

$$L_2 = \partial^2 + \left(\lambda - \frac{1}{x}\right)\partial + \left(\lambda^2 - \frac{\lambda}{x} + \frac{3}{4x^2} - r(x)\right),$$

where $r(x) = \frac{1}{4x^2}\left(2\lambda^2x^2 - \lambda x \pm \sqrt{4\lambda^4x^4 - 8\lambda^3x^3 + 13\lambda^2x^2 - 15\lambda x + 16x^5}\right)$. Realizing that $\lambda = 0$, we get $r(x) = \pm\sqrt{x}$ and the factorizations of the first section.

References

[1] Abramov (S. A.). – Rational solutions of linear differential and difference equations with polynomial coefficients. *USSR Computational Mathematics and Mathematical Physics*, vol. 29, n° 11, 1989, pp. 1611–1620. – Translation of the Zhurnal vychislitel'noi matematiki i matematichesckoi fiziki.

[2] Abramov (Sergei A.), Bronstein (Manuel), and Petkovšek (Marko). – On polynomial solutions of linear operator equations. In Levelt (A.) (editor), *Symbolic and algebraic computation*. pp. 290–296. – New York, 1995.

[3] Beke (E.). – Die Irreducibilität des homogenen linearen Differentialgleichungen. *Mathematische Annalen*, vol. 45, 1884, pp. 278–294.

[4] Bronstein (M.) and Petkovšek (M.). – On Ore rings, linear operators and factorisation. *Programmirovanie*, n° 1, 1994, pp. 27–44. – Also available as Research Report 200, Informatik, ETH Zürich.

[5] Kovacic (Jerald J.). – An algorithm for solving second order linear homogeneous differential equations. *Journal of Symbolic Computation*, vol. 2, 1986, pp. 3–43.

[6] Liouville (J.). – Premier mémoire sur la détermination des intégrales dont la valeur est algébrique. *Journal de l'École polytechnique*, n° 14, 1833, pp. 124–148.

[7] Liouville (J.). – Second mémoire sur la détermination des intégrales dont la valeur est algébrique. *Journal de l'École polytechnique*, n° 14, 1833, pp. 149–193.

[8] Singer (Michael F.). – Liouvillian solutions of n-th order homogeneous linear differential equations. *American Journal of Mathematics*, vol. 103, n° 4, 1981, pp. 661–682.

[9] Singer (Michael F.). – Testing reducibility of linear differential operators: A group theoretic perspective. *Applicable Algebra in Engineering, Communication and Computing*, vol. 7, n° 2, 1996, pp. 77–104.

[10] Singer (Michael F.) and Ulmer (Felix). – Necessary conditions for Liouvillian solutions of (third order) linear differential equations. *Applicable Algebra in Engineering, Communication and Computing*, vol. 6, n° 1, 1995, pp. 1–22.

[11] Ulmer (Felix) and Weil (Jacques-Arthur). – *Note on Kovacic's algorithm*. – Prépublication n° 94-13, Institut de recherche mathématique de Rennes, Université de Rennes 1, France, July 1994.

[12] Van Hoeij (Mark). – Formal solutions and factorization of differential operators with power series coefficients. *Journal of Symbolic Computation*, vol. 24, n° 1, July 1997, pp. 1–30.