Differential Equations, Nested Forms and Star Products

John Shackell
University of Kent at Canterbury, U.K.

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Abstract

This presentation is in two parts. First, we recall the definition of two types of asymptotic expansions known as *nested form* and *nested expansions*. This theory makes it possible to adapt the asymptotic scale to the function under expansion and is based on the theory of Hardy fields [1]. Next, we suggest a reformulation of nested forms in terms of generalized products called *star products*, and a prospective theory of multivariate Hardy fields called *partial Hardy fields*.

PART I: ASYMPTOTIC NESTED EXPANSIONS

1. Hardy fields

From the asymptotic viewpoint, $\mathcal{C}^\infty$ real-valued functions do not behave as nicely as holomorphic functions. In particular, asymptotic comparisons of functions that involve the symbols O, o and $\sim$ cannot be termwise differentiated. A simple example is provided by $f(x) = x + \cos x \sim x$, whereas $f'(x)$ is not asymptotic to 1. In rough terms, this defect is due to allowing the functions to oscillate. A remedy to this problem is the use of *Hardy fields* instead of rings of functions. A construction is as follows. In order to deal with fields of functions, one first considers the ring $\mathcal{G}$ of *germs* of $\mathcal{C}^\infty$ real-valued functions at $+\infty$, where two functions are identified when they agree on a neighbourhood of $+\infty$. Then, a *Hardy field* $\mathcal{H}$ is a subring of $\mathcal{G}$ which is also a differential field. An example is $\mathbb{R}(x)$ viewed as a ring of germs with the usual derivation.

Considering fields of germs has nice consequences. A non-zero function f of a Hardy field $\mathcal{H}$ is invertible with $\mathcal{C}^\infty$ inverse, so that asymptotically it never vanishes, and is therefore of asymptotically constant sign. This defines a total order on $\mathcal{H}$ by $f < g$ if and only if $g - f$ has asymptotically positive sign. The derivative f' is also of asymptotically constant sign, so that f is monotonic and tends to a limit in $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ when x goes to $+\infty$. Applying this property to the ratio f/g of two functions in $\mathcal{H}$, we obtain that either $f = o(g)$ or $f \sim cg$ for a non-zero real constant c or $g = o(f)$.

For f in a Hardy field $\mathcal{H}$, an elementary result is that both differential ring extensions $\mathcal{H}(\exp f)$ and $\mathcal{H}(\ln f)$ are again Hardy fields. Let $\ell_0(x) = x$ and for $n \geq 0$, $\ell_{n+1}(x) = \ln|\ell_n(x)|$. Similarly, let $e_0(x) = x$ and for $n \geq 0$, $e_{n+1}(x) = \exp e_n(x)$. For $n < 0$, define $e_n = \ell_{-n}$ and $\ell_n = e_{-n}$. Then, for a Hardy field $\mathcal{H}$, there is a Hardy field extension containing each of the $\ell_n(f)$ and $e_n(f)$. Again for a function f in a Hardy field $\mathcal{H}$, it follows from a theorem by Rosenlicht that there exists a smallest Hardy field containing $\mathbb{R}$, f, and each g^d for $g > 0$ and $d \in \mathbb{R}$. This field is denoted $\mathbb{R}\langle\langle f\rangle\rangle$. A computationally interesting fact is that $f^\Delta = (\ln|f|)' = f'/f \in \mathbb{R}\langle\langle f\rangle\rangle$ even when $\ln f \notin \mathbb{R}\langle\langle f\rangle\rangle$.

2. Comparability classes

In view of various types of expansions, we need to extend asymptotic equivalence $\sim$ to coarser and coarser equivalences. In particular, beside asymptotic equivalence (x is equivalent to $x + \ln x$ but not to $2x$), we need to consider asymptotic equivalence up to a non-zero constant factor (x is equivalent to $2x$ but not to x^2), asymptotic equivalence up to powers (x is equivalent to x^2 but not to $\exp x$). For two functions f and g in a Hardy field $\mathcal{H}$ and with limiting values 0, $-\infty$ or $+\infty$, we define $f \approx_n g$ to mean that there exists a non-zero constant $c \in \mathbb{R}$ such that $\ell_n(f) \sim c\ell_n(g)$. Furthermore, we agree that $f \approx_n g$ when both functions tend to finite non-zero limits. For each $n \geq 0$, this defines an equivalence relation on $\mathcal{H} \setminus \{0\}$. Call $\gamma_n(f)$ the class of $f \neq 0$ and $\Upsilon_n(\mathcal{H})$ the set of equivalence classes.

Beside this, we need to measure the accuracy of an expansion (a series in $\ln x$ is finer than a series in x). This is done by defining an order between equivalence classes. Set $\gamma_n(f) < \gamma_n(g)$ to mean $\ell_n(f) = o(\ell_n(g))$. For each n, this defines a total order on $\Upsilon_n(\mathcal{H})$: $\ell_n(f)/\ell_n(g)$ is in a suitable Hardy field extension of $\mathcal{H}$, so that it has a limit a in $\overline{\mathbb{R}}$ (independent of the extension). Either $a = 0$ and $\gamma_n(f) < \gamma_n(g)$; or $a = \pm\infty$ and $\gamma_n(g) < \gamma_n(f)$; or $\gamma_n(f) = \gamma_n(g)$.

Here are a few examples:

$$\gamma_0(\ln x) < \gamma_0\left(\exp\left(\ell_2(x)^2\right)\right) < \gamma_0(x) = \gamma_0(x + \ln x) < \gamma_0\left(x^2\right) < \gamma_0\left(\exp\left(\ln(x)^2\right)\right) < \gamma_0(\exp(x)),$$

$$\gamma_1(\ln x) < \gamma_1\left(\exp\left(\ell_2(x)^2\right)\right) < \gamma_1(x) = \gamma_1(x + \ln x) = \gamma_1\left(x^2\right) < \gamma_1\left(\exp\left(\ln(x)^2\right)\right) < \gamma_1(\exp(x)),$$

$$\gamma_2(\ln x) = \gamma_2\left(\exp\left(\ell_2(x)^2\right)\right) < \gamma_2(x) = \gamma_2(x + \ln x) = \gamma_2\left(x^2\right) = \gamma_1\left(\exp\left(\ln(x)^2\right)\right) < \gamma_2(\exp(x)).$$

The previous equivalence relations extend known cases: γ_0 is the valuation map with the usual ordering reversed; the elements of $\Upsilon_1(\mathcal{H})$ are Rosenlicht's comparability classes.

3. From γ_n to γ_{n+1}

In view of the examples above, one proves that the $\sim_n$ are coarser and coarser relations: if $\gamma_n(f) = \gamma_n(g)$, then $\gamma_{n+1}(f) = \gamma_{n+1}(g)$. Thus for $n \geq 0$, the map γ_{n+1} factors through $\Upsilon_n(\mathcal{H})$: $\gamma_{n+1} = \eta_n \circ \gamma_n$ for a surjection η_n from $\Upsilon_n(\mathcal{H})$ to $\Upsilon_{n+1}(\mathcal{H})$. Taking the direct limit, we get $\Upsilon_\infty(\mathcal{H})$ and γ_∞ such that for any $k \geq 0$ and any non-zero $d \in \mathbb{R}$,

$$\gamma_\infty(\ln x) < \gamma_\infty(x) = \gamma_\infty\left(x^d\right) = \gamma_\infty\left(e_k\left(\ell_k^d(x)\right)\right) < \gamma_\infty(\exp x).$$

On the contrary, inequalities are not always preserved by the η_n's: when $\gamma_n(f) < \gamma_n(g)$, i.e., $\ell_n(f) = o(\ell_n(g))$, it is not always true that $\gamma_{n+1}(f) \leq \gamma_{n+1}(g)$. For instance,

$$\gamma_0\left(x^{-1}\right) < \gamma_0(\ln x) < \gamma_0(x) \qquad \text{but} \qquad \gamma_1(\ln x) < \gamma_1(x) = \gamma_1\left(x^{-1}\right).$$

However, the property is valid when comparing functions that are infinite at $+\infty$, as shown by the examples of the previous section.

On the other hand, comparing $\gamma_2(f)$ to $\gamma_2(\ell_{p-1}(x))$ yields information on $\gamma_1\left(L_p(x)f^\Delta(x)\right)$ where $L_p(x) = 1/\ell_p'(x) = x\ell_1(x)\cdots\ell_{p-1}(x)$. The key idea is to restrict to functions f with infinite or zero limit at $+\infty$ and to use *L'Hôpital's rule*: if $f, g \to 0$ or $\pm\infty$, $\frac{f(x)}{g(x)} \sim \frac{f'(x)}{g'(x)}$. Applying this rule to $\ell_2(f)$ and $\ell_{p+1}(x)$, we obtain

$$\frac{\ell_2(f)}{\ell_{p+1}(x)} \sim \frac{f^\Delta}{\ln|f|}\frac{\ell_p}{\ell_p'}, \qquad \text{and hence} \qquad L_p(x)f^\Delta(x) \sim \frac{1}{\ell_p(x)\ell_{p+1}(x)}\frac{\ln|f|}{\ln|\ln|f||}.$$

Taking logarithms whenever appropriate, we then get:

1. if $\gamma_2(f) > \gamma_2(\ell_{p-1}(x))$ then $\gamma_1\left(L_p f^\Delta\right) = \gamma_1(\ln f)$;

2. if $\gamma_2(f) < \gamma_2(\ell_{p-1}(x))$ then $\ln\left(L_p f^{\Delta}\right) \sim -\ell_{p+1}(x)$;
3. if $\gamma_2(f) = \gamma_2(\ell_{p-1}(x))$ and $\ell_2(f) \not\sim \ell_{p+1}(x)$ then $\gamma_1\left(L_p f^{\Delta}\right) = \gamma_1(\ell_p(x))$;
4. if $\ell_2(f) \sim \ell_{p+1}(x)$ then $\gamma_1\left(L_p f^{\Delta}\right) < \gamma_1(\ell_p(x)) = \gamma_1(\ln f)$.

4. Nested forms

For an integer $m \geq 0$, reals $d \geq 0$ and $\epsilon > 0$ and a function ϕ such that $\gamma_1(\phi) < \gamma_1(\ell_m(x))$, consider $f = \ell_m^d(x)\phi(x)$ and $g = \ell_m^{d+\epsilon}(x)$. Then $\ln(f/g) = \ln\phi(x) - \epsilon\ell_{m+1}(x) = -\epsilon\ell_{m+1}(x) + o(\ell_{m+1}(x))$ is negative infinite at $+\infty$. Thus $f = o(g)$. If similarly $\epsilon < 0$, $g = o(f)$. In view of the previous relations, we define a *partial nested form* for a function f that is infinite at $+\infty$ as an expression of the form

$$f = e_s\left(\ell_m^d(x)\phi\right) \qquad \text{where } s, m \geq 0,\ d \in \mathbb{R}^+ \text{ and } \gamma_1(\phi) < \gamma_1(\ell_m(x)).$$

Not every function f in a Hardy field admits a nested form. In fact, if $f \to +\infty$, then either $\gamma_2(f) > \gamma_2(e_s(x))$ for all $s \geq 0$; or for all $m \geq 0$, there exists $f_m \in \mathbb{R}\langle\langle f\rangle\rangle$ such that $f_m \sim \ell_m(x)$; or else f admits a partial nested form $f = e_s\left(\ell_m^d(x)\phi\right)$. The first two cases are strange situations in which, for example, f cannot satisfy an algebraic differential equation over $\mathbb{R}$. In the third case, $\mathbb{R}\langle\langle f\rangle\rangle$ contains an element asymptotic to ϕ. So the previous result applies to one of $\pm\phi^{\pm}$ and the function ϕ may in turn admit a partial nested form. Continuing in this way, we produce a sequence of ϕ_i's which either stops on a ϕ_i for which the second case above holds, or is an infinite sequence with decreasing γ_1, or contains a ϕ_i asymptotic to a non-zero constant A.

This gives a recursive definition of a *nested form*: a function f has a nested form if there exists a finite sequence of ϕ_i's, $i = 0, \ldots, n$, with $\phi_0 = f$, such that each ϕ_{i+1} is the function ϕ which appears in a partial nested form of ϕ_i and $\phi_n = A + o(1)$ for a non-zero real A. (A few other technical constraints are added to ensure the uniqueness of the nested form of a function, in the case of existence.) For example, the following are two nested forms:

$$e_1\left(\ell_2^2(x)e_2\left(\ell_5^{1/3}(x)(2+o(1))\right)\right), \qquad \text{and} \qquad -e_1^{-1}\left(x^{\pi}\ell_1(x)e_2\left(\ell_5^{\sqrt{2}}(x)(13+o(1))\right)\right).$$

As a consequence to the previous results, if f belongs to a Hardy field and $\Upsilon_1(\mathbb{R}\langle\langle f\rangle\rangle)$ is well ordered, then f has a nested form. In particular, when f satisfies an algebraic differential equation over $\mathbb{R}$ and belongs to a Hardy field, then $\Upsilon_1(\mathbb{R}\langle\langle f\rangle\rangle)$ is finite, with cardinality bounded by the order of the differential equation. It follows that only finitely many nested forms are possible for such solutions, and that those possible forms can be listed.

5. Further expansions

Nested forms only give a certain amount of asymptotic information. In particular, nothing is known about the $o(1)$. In certain cases, a possibility is to give a nested form representation for this $o(1)$, and continue recursively. We then get *nested expansions*, which extend asymptotic expansions, with no presumption of convergence. Nested expansions are guaranteed to exist for solutions of algebraic differential equations in a Hardy field and yield a unique representation. In particular, nested expansions can be calculated for exp-ln functions (modulo an oracle for constants); and for Liouvillian functions, although there is difficulty specifying *which* integral or algebraic function is under consideration. There are also known algorithms to add and multiply nested expansions, but this may be awkward. On the other hand the functional inverse of a nested expansion can be easily computed by an algorithm due to Salvy and Shackell.

PART II: PROSPECTIVE RESULTS

6. Star products

In view of the identities $ab = \exp(\ln a + \ln b)$ and $ab = \ln\left(e^a e^b\right)$, we define the *star products* $*_k$ by

$$a *_k b = e_k(\ell_k(a) + \ell_k(b)) = e_{k-1}(\ell_{k-1}(a)\ell_{k-1}(b)), \qquad \text{for } k \in \mathbb{Z}.$$

We have $a *_0 b = a + b$, $a *_1 b = ab$. For $r \in \mathbb{R}$, we also define $a^{*_k^r} = e_k(r\ell_k(a))$. These definitions yield properties which give star products their name: each $*_k$ is commutative and associative; $*_{k+1}$ is left and right distributive over $*_k$; $*_{k+1}$ admits $e_{k+1}(0)$ as a neutral element and $e_k(0)$ as a zero.

Sample expressions that involve star products are:

$$x *_2 \ell_1(x) = \exp(\ell_1(x)\ell_2(x)), \qquad (\exp x)^{*_2^2} = \exp\left(x^2\right), \qquad e^x *_{-1} x = \ln\left(e_2(x) + e^x\right),$$

$$x^{*_{-1}^2} = \ln\left(2e^x\right) = x + \ln 2, \qquad (\exp x)^{*_3^2} *_2 \ell_1(x) = e_2\left(\ell_1^3(x)\right) *_2 \ell_1(x) = e_1\left(e_1\left(\ell_1^3(x)\right)\ell(2(x)\right).$$

Of course, any exp-ln function can be written as an expression in the real constants and the functions e_n for $n \in \mathbb{Z}$ using star products $*_k$ for $k \in \mathbb{Z}$ only.

The advantage of star products is their nice behaviour with the γ_k's: if a and b have infinite limit at $+\infty$, then $\gamma_{k-1}(a *_k b) > \max\{\gamma_{k-1}(a), \gamma_{k-1}(b)\}$ and $\gamma_k(a *_k b) = \max\{\gamma_k(a), \gamma_k(b)\}$. Furthermore, taking $*_k$ powers does not affect asymptotic relations between $e_s(x)$'s. In view of these results, we expect to find a better presentation of nested expansions in terms of star products and hope for simpler algorithms to deal with nested expansions.

7. Partial Hardy fields

We say that a ring $\mathcal{H}$ of functions of two variables x and y is a *partial Hardy field* if

1. for sufficiently large y_0, $\{f(x, y_0) \mid f \in \mathcal{H}\}$ is a Hardy field;
2. for sufficiently large x_0, $\{f(x_0, y) \mid f \in \mathcal{H}\}$ is a Hardy field;
3. for $f \in \mathcal{H}$, $\lim_{y \to +\infty} f(x, y)$ is either identically $\pm\infty$ or else an element of a Hardy field;
4. for $f \in \mathcal{H}$, $\lim_{x \to +\infty} f(x, y)$ is either identically $\pm\infty$ or else an element of a Hardy field.

The point of this definition is to allow for *multivariate nested expansions.*

Let $f(x, y)$ be a function in a partial Hardy field. Provided $\mathbb{R}\langle\langle f\rangle\rangle$ is a partial Hardy field, the sequence of $\phi_i(x)$ obtained by taking the nested expansion of $f(x, y_0)$ for fixed $y = y_0$ becomes independent of y_0 for sufficiently large y_0. Thus with an assumption of well-orderedness, we obtain a first nested expansion,

$$f(x, y) \underset{x \to +\infty}{\sim} e_{s_0}\left(\ell_{m_0}^{d_0}(x) e_{s_1}^{\pm 1}\left(\cdots e_{s_n}^{\pm 1}\left(\ell_{m_n}^{d_n}(x)(\phi(y) + o(1))\right)\cdots\right)\right),$$

where by point (4) of the definition, ϕ belongs to a Hardy field. Now, with reasonable assumptions, ϕ admits a nested form.

Once again, if f satisfies an algebraic partial differential equation, there is a limitation on the nested forms allowed to occur. However, a complication is that solutions of PDE's contain arbitrary functions, which can be set by specifying the limiting behaviour of f in one direction.

References

[1] Bourbaki (N.). – *Éléments de Mathématiques,* Chapter V: Fonctions d'une variable réelle (appendice), pp. 36–55. – Hermann, Paris, 1961, second edition.