Asymptotics of Implicit Functions and Computer Algebra

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[summary by Joris Van der Hoeven]

Abstract

We describe several algorithms for the asymptotic inversion of functions, from the computer algebra viewpoint. We start with some classical results, such as the inversion theorem of Lagrange. We next consider functions which can only be expanded in more complicated asymptotic scales. Finally, we briefly discuss the problem of finding the asymptotic behaviour of implicit functions.

1. Introduction

Let f be a sufficiently regular function which admits a functional inverse f^{inv} in the neighbourhood of some point. One often encounters the problem of determining the asymptotic behaviour of f^{inv} in terms of the asymptotics of f. For instance, this problem arises systematically when computing integral transforms by the saddle-point method. If the function admits a power series expansion near the point of interest, then Lagrange's inversion theorem can be used, or Brent and Kung's fast algorithm for power series inversion (see section 2).

However, one often has to deal with logarithmic or exponential singularities, whence a more general device for asymptotic functional inversion is needed. A first approach would be to consider more general classes of singularities of a particular form. For instance, one may consider functions which admit an asymptotic expansion of the form

$$f = e^x x^{-\alpha} \sum_{n \geq 0} d_n x^{-n} \qquad (x \to \infty);$$

see section 3. More generally, one might consider the class of exp-log functions, i.e. functions built up from $\mathbb{Q}$ and x by $+, -, \times, /, \exp$ and $\log$. A first algorithm to obtain asymptotic information about functional inverses of such functions—in the form of a so-called *nested expansion*—was given by Salvy and Shackell [6]; see also section 4.

An interesting more general problem than the inversion problem is to find an algorithm to determine the asymptotic behaviour of implicit functions, say the solutions of exp-log functions in two variables. A partial algorithm to determine the potential nested expansions of such functions was proposed in [7] and will be discussed in section 5.

More generally, one might require complete asymptotic expansions of the inverses of exp-log functions and solutions of implicit equations. A nice algorithm to compute complete asymptotic expansions of exp-log functions themselves was given in [4]. Notice also that extending the techniques from [4], solutions for the inversion and implicit function problems for exp-log functions were given independently in [10] and [11].

2. Classical results

Let $f(y) = f_0 + f_1 y + f_2 y^2 + \cdots$ be a power series with $f_0 \neq 0$ and assume that $y(x)$ satisfies

$$y = x f(y).$$

Then Lagrange's inversion theorem gives a formula for $[x^n]y$:

$$[x^n]y = \frac{1}{n}[y^{n-1}]f^n.$$

The formula can be generalized to

$$[x^n]G = \frac{1}{n}[y^{n-1}]G' f^n,$$

for an arbitrary power series G. If f resp. G have an analytic meaning (i.e. the power series converge, or the power series are the asymptotic expansion of some functions), then the same holds for the above formulae.

Example. The generating function y of Cayley trees satisfies

$$y = x e^y.$$

Hence $[x^n]y = \frac{1}{n}[y^{n-1}]e^{ny} = \frac{n^{n-1}}{n!}$. Other trees can be treated in a similar way.

Assume now that we do not merely want the n-th coefficient of the inverse of a power series $x(y) = y + a_2 y^2 + a_3 y^3 + \cdots = F(y)$, but the first N coefficients of $y(x)$. This problem can be reduced to the problem of functional composition by the Newton method: define the sequence $y_n(x)$ by $y_0 = x$ and

$$y_{n+1} = y_n - \frac{F(y_n) - x}{F'(y_n)}.$$

Brent and Kung have shown that the number of correct terms doubles at each step [1]. Suitably truncating the Newton iterations then yields an inversion algorithm for power series of the same complexity as functional composition, i.e. $O(M(N)\sqrt{N \log N}) = O((N \log N)^{3/2})$ [1].

Example. How to find the n-th positive root x_n of $\tan x = x$? Looking at the graph, we see that $x_n = (2n+1)\frac{\pi}{2} - u_n$ with $u_n \to 0$. Applying tan we get

$$(2n+1)\frac{\pi}{2} - u_n = \tan\left(\frac{\pi}{2} - u_n\right),$$

whence

$$\frac{1}{u_n + \tan(\frac{\pi}{2} - u_n)} = \frac{2}{(2n+1)\pi} = t_n,$$

with $t_n \to 0$. Inverting the above power series, we get

$$x_n = \pi n + \frac{\pi}{2} - \frac{1}{\pi n} + \frac{1}{2\pi n^2} - \left(\frac{1}{4\pi} + \frac{2}{3\pi^3}\right)\frac{1}{n^3} + \cdots.$$

3. Inverses of more general functions

In this section we are interested in computing the asymptotic inverse of a function which admits

$$e^y y^{-\alpha} D(y^{-1}) \qquad (d_n \neq 0, y \to \infty)$$

as an asymptotic expansion, where $D(y^{-1}) = \sum_{n\geq 0} d_n y^{-n}$. In other words, we want the asymptotic solution to

$$e^y y^{-\alpha} D(y^{-1}) = x$$

in y. A first method [2] is to do this is to take logarithms

$$y = \log x + \alpha \log y - \log D(y^{-1}), \tag{1}$$

and to replace y by the right hand side in an iterative manner. This yields an expansion of the form

$$y \approx \log x + \sum_{n\geq 0} \frac{P_n(\log\log x)}{\log^n x},$$

where the P_n are polynomials.

A faster method to compute the P_n was proposed in [5]. Setting $\zeta = \log\log x$ and $t = (\log x)^{-1}$, we set

$$y - \log x = P(\zeta, t) = \sum_{n\geq 0} P_n(\zeta) t^n.$$

Taking logarithms, this yields

$$\log y = \log(\log x + P) = \zeta + \log(1 + tP).$$

From (1), we get

$$P = \alpha\zeta + \alpha \log(1 + tP) - \log D\left(\frac{t}{1+tP}\right).$$

In this equation ζ and t are decoupled. We get

$$P(0,t) = \alpha \log(1 + tP(0,t)) - \log D\left(\frac{t}{1+tP(0,t)}\right)$$

and

$$\left(\frac{1}{\alpha} - t\right)\frac{\partial P}{\partial \zeta} + t^2 \frac{\partial P}{\partial t} = 1.$$

These two relations enable us to compute the coefficients of P efficiently.

Example. The number of prime numbers $\pi(x)$ smaller than x satisfies

$$\pi(x) \approx \frac{x}{\log x}\left(1 + \frac{1!}{\log x} + \frac{2!}{\log^2 x} + \cdots\right).$$

Using the above method we find the asymptotic expansion

$$p(n) \approx n \log n \left(1 + \frac{\log\log n - 1}{\log n} + \frac{\log\log n - 2}{\log^2 n} + \cdots\right)$$

for the n-th prime number $p(n)$.

The above result can be extended to get asymptotic expansions for $\log y$ and $e^{\beta y} y^{\gamma} G(y^{-1})$ [5].

4. Inversion of exp-log functions

Let f be an exp-log function (i.e. a function built up from $\mathbb{Q}$ and x by $+, -, \times, /, \exp$ and $\log$), which tends to infinity for $x \to \infty$. We denote by $\exp_i$ resp. $\log_i$ the i-th iterate of exp resp. log. We write $f \lll g$, if f is of a smaller comparability class than g, i.e. $(\log|f|)/(\log|g|)$ tends to 0.

In this section, we present an algorithm to compute a "nested expansion" for the asymptotic functional inverse of f. Such a nested expansion often (although not always) yields an asymptotic expansion of the function. In any case, the limit behaviour of a function can be determined from its nested expansion.

One first defines a *nested form* as being a finite sequence $\{(s_i, \varepsilon_i, m_i, d_i, \phi_i)\}$ for $1 \leq i \leq n$. Here $s_i, m_i \in \mathbb{N}, d_i \in \mathbb{R}, \varepsilon_i = \pm 1$, ϕ_i lies in a Hardy field and

$$\phi_{i-1}(x) = \exp_{s_i}^{\varepsilon_i}(\log_{m_i}^{d_i}(x)\phi_i(x)),$$

with $\phi_i \lll \log_{m_i}$ for $2 \leq i \leq n$. We also require that

- ϕ_n has a finite limit l;
- $d_n \neq 1$ unless $s_n = 0$ or $m_n = 0$;
- $d_i > 0$ unless $s_i = 0$.

Continuing the process of taking nested forms with $\phi_n - l$ (if possible), we obtain a *nested expansion.*

Example. $\exp[\log^2 x \exp[\sqrt{\log\log x}(7 + \phi(x))]]$ is a nested form, if $\phi(x)$ tend [illegible].

John Sha[illegible] the first to prove in [8] that any exp-log function admits a nested form, a[illegible] even a nested expansion. Nested forms and expansions are particularly well sse [illegible] of functional inversion. The algorithm proceeds as follows :

Algorithm. [illegible] s given at input, and we compute a nested expansion of its functional inverse at infinity.

1. Rewrite f as a NF

$$f(y) = \exp_s(\log_m^d(y) f_1(y)) = x.$$

2. Invert

$$y = \exp_m(\log_s^{1/D} x g_1(x)). \qquad (E)$$

3. Iterate
   - Compute $NF(g_i) = \exp_{s_i}^{\varepsilon_i}(\log_{m_i}^{d_i} y G_i(y))$,
   - Substitute (E) in $\log_{m_i}^{d_i} y$,
   - g [illegible] $i+1$,

   until $NF(g_i) = c + \varepsilon(y)$, with $\varepsilon(y) \to 0$, yielding $NF(y(x))$.
4. Repeat the above procedure for $\varepsilon(y)$, whence $NE(y(x))$.

Example.

$$f(y) = y e^{\log^2 y e^{\sqrt{\log\log y}}} = x.$$

Step 1: rewrite f as NF

$$\exp\left[\log^2 y \exp[\sqrt{\log_2 y}(1 + W)]\right],$$

with

$$W = \log_2^{-1/2} y \log(1 + \log^{-1} y e^{-\sqrt{\log_2 y}}).$$

Step 2: Invert

$$y=\exp\left[\sqrt{\log x}\,\exp\left(-\frac{1}{2}\sqrt{\log_2 y}(1+W)\right)\right].$$

Step 3: Iterate

$$\sqrt{\log_2 y}=\frac{1}{\sqrt{2}}\sqrt{\log_2 x}\left(1+\frac{1+W}{2\sqrt{\log_2 y}}\right)^{-1/2},$$

whence the nested form for y:

$$y=\exp\left[\sqrt{\log x}\,\exp\left(-\frac{1}{2\sqrt{2}}\sqrt{\log_2 x}(1+\varepsilon(x))\right)\right].$$

Step 4: Repeat

$$y=\exp\left[\frac{\sqrt{\log x}}{e^{\frac{1}{2\sqrt{2}}\log_2 x}}\cdot e^{1/8}\cdot\left(1-\frac{1}{32\sqrt{2}}\log_2^{-1/2}x+\right.\right.$$
$$\left.\left.\frac{1}{4096}\log_2^{-1}x+\frac{383}{393216\sqrt{2}}\log_2^{-3/2}x+\cdots\right)\right].$$

5. Asymptotic expansion of implicit functions

Assume now that H is an exp-log function in two variables x and y. By theorems of Khovanskii and van den Dries [3, 9], if $y(x)$ is a real solution of $H(x,y)=0$ (for $x\to\infty$), then (the germ of) $y(x)$ belongs to a Hardy field. In particular, $y(x)$ does not present any oscillatory phenomena at infinity. As a corollary [7], $y(x)$ has a nested form and even a nested expansion. A question is how to compute all possible nested expansions of such solutions.

The idea from the algorithm in [7] is to compute the asymptotic behaviour for $H(x,y)$ for all possible asymptotic behaviours of x and y. In order to do so, one often has to distinguish several number of cases, but always finitely many, depending on the values of x and y. The strategy is best illustrated on an example.

Example.

$$H(x,y)=\frac{\exp(x^2+2x\log^2 x+y)}{\exp(x^2+x\log^2 x+y)}-1.$$

Three cases are distinguished:

$$H\sim\begin{cases}1, & \text{if } y\to-\infty \text{ and } x\lll y;\\ ?, & \text{if } \log|y|\sim 2\log x;\\ \exp(x\log^2 x), & \text{otherwise.}\end{cases}$$

Since $H=0$, we find that $\log|y|\sim 2\log x$, whence $y\asymp -x^2$. Continuing the process, we find

$$y=-x^2-2x\log^2 x+\frac{e^{-x\log^2 x}}{x^2}-\frac{e^{-2x\log^2 x}}{2x^2}+\cdots.$$

We remark that it is not claimed that the obtained nested expansions are indeed nested expansions of actual solutions. For a solution to this problem, we refer to [11].

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