Wiener-Hopf Factorization: Probabilistic methods

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[summary by Jean-François Dantzer]

1. Introduction

We consider a discrete random walk on $\mathbb{Z}$, defined by

$$S_0 = 0 \qquad \text{and} \qquad S_n = \sum_{i=1}^{n} X_i, \quad n > 0,$$

where $(X_i)_{i\geq 1}$ is an independent identically distributed (i.i.d.) sequence of random variables. We define two hitting times ν_+ and ν_-,

$$\nu_+ = \inf\{k > 0 / S_k > 0\}, \qquad \nu_- = \inf\{k > 0 / S_k \leq 0\},$$

with the convention $\inf(\emptyset) = +\infty$.

We also define M and L two variables indicating respectively the maximum of the random walk and the hit moments at which it is attained

$$M = \sup_{n\geq 0}\{S_n\}, \qquad L = \inf_{n\geq 0}\{S_n = M\}.$$

This talk presents the classical probabilistic methods to derive the join distributions of (ν_+, S_{ν_+}), (ν_-, S_{ν_-}) and (M, L).

Applications of these results have been found in biology [3, 4] or in queueing theory [2]. For more details on this subject, see [1, 5].

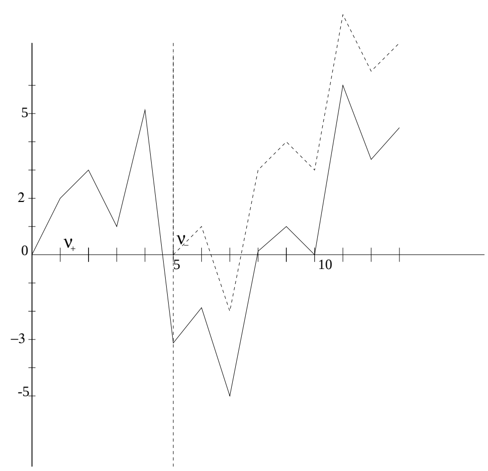

FIGURE 1. Random walks $(S_n)_{n\geq 0}$ and $(S_{n+\nu_-})_{n\geq 0}$ (in dotted line)

2. Distribution of (ν_+, S_{ν_+}) and (ν_-, S_{ν_-})

The distributions of the pairs (ν_+, S_{ν_+}) and (ν_-, S_{ν_-}) will be expressed through their generating functions, i.e., $E(u^{\nu_+} z^{S_{\nu_+}})$ and $E(u^{\nu_-} z^{S_{\nu_-}})$.

We consider three variables

$$\Psi_+(u,z) = \frac{1}{1 - E(u^{\nu_+} z^{S_{\nu_+}})} \qquad \text{on} \qquad \{|u| < 1, |z| \le 1\},$$

$$\Psi_-(u,z) = \frac{1}{1 - E(u^{\nu_-} z^{S_{\nu_-}})} \qquad \text{on} \qquad \{|u| < 1, |z| \ge 1\},$$

$$\Psi(u,z) = \sum_{n \ge 0} E(u^n z^{S_n}) \qquad \text{on} \qquad \{|u| < 1, |z| = 1\}.$$

The main result, the factorization of Wiener-Hopf described in the following proposition, gives an analytic characterization of Ψ_+ and Ψ_-.

Proposition 1. *Ψ can be uniquely decomposed on $\{|z| = 1\}$ as:*

$$\Psi(u,z) = \Psi_+(u,z)\Psi_-(u,z),$$

where Ψ_+ and Ψ_- have the following properties:

- *Ψ_+ is analytic on $\{|z| < 1\}$;*
- *Ψ_+ and $1/\Psi_+$ are continuous and bounded on $\{|z| \le 1\}$;*
- *$\Psi_+(u, 0) = 1$;*

and

- *Ψ_- is analytic on $\{|z| > 1\}$;*
- *Ψ_- and $1/\Psi_-$ are continuous and bounded on $\{|z| \ge 1\}$.*

Proof. The proof is based on the following arguments:

- The function Ψ can be expressed with the distribution of X_1, the sequence $(X_i)_{n \ge 1}$ is independent identically distributed, then $E(Z^{S_n}) = E(Z^{\sum_{i=1}^n X_i}) = E(Z^{X_1})^n$, thus

$$\Psi(u,z) = \sum_{n \ge 0} u^n E(z^{X_1})^n = \frac{1}{1 - uE(z^{X_1})};$$

- $(S_n)_{n \ge 0}$ and $(S_{n+\nu_-})_{n \ge 0}$ have same distribution, $(S_{n+\nu_-})_{n \ge 0}$ is the random walk beginning at the time ν^-;
- The independence between the pair (ν_-, S_{ν_-}) and the sequence $(S_{n+\nu_-})_{n \ge 0}$;
- the same properties are also valid for $(S_{n+\nu_+})_{n \ge 0}$.

□

3. Examples

The factorization is easy when Ψ has a finite number of poles and zeros. We consider two such examples.

3.1. Random walk $\pm$ 1.

We suppose $X_i = 1$ with probability p and $X_i = -1$ with probability $(1 - p)$. In that case,

$$\Psi(u,z) = \frac{z}{-upz^2 + z - u(1-p)}$$

Ψ has only two poles

$$\alpha_1(u) = \frac{1-\sqrt{1-4u^2p(1-p)}}{2up}, \qquad \alpha_2(u) = \frac{1+\sqrt{1-4u^2p(1-p)}}{2up},$$

$$\text{with} \qquad 0 \le \alpha_1(u) \le 1 \le \alpha_2(u).$$

The decomposition gives:

$$\Psi_+(u,z) = \frac{\alpha_2(u)}{\alpha_2(u)-z} \qquad \text{and} \qquad \Psi_-(u,z) = \frac{z}{\alpha_2(u)up(z-\alpha_1(u))}.$$

Here, we obtain the generating functions:

$$E(u^{\nu_+}z^{S_{\nu_+}}) = \frac{z}{\alpha_2(u)}, \qquad E(u^{\nu_-}z^{S_{\nu_-}}) = (1-\alpha_2(u)up) + \frac{u(1-p)}{z}.$$

3.2. **Random walk left bounded.** We suppose $\Pr(X_i < -1) = 0$.

$$\Psi(u,z) = \frac{1}{1-uE(z^X)} = \frac{z}{z-uE(z^{X+1})}.$$

In that case, the factorization is easy because by Rouché's theorem the function $z \mapsto z - uE(z^{X+1})$ has one only root which belongs to $\{|z|<1\}$, which we denote by $\alpha(u)$. One proves that $\alpha(u) \in [0,1]$ and the decomposition of Ψ is the following:

$$\Psi_+(u,z) = \frac{z-\alpha(u)}{z-uE(z^{X+1})}\frac{u\Pr(X=-1)}{\alpha(u)}, \qquad \Psi_-(u,z) = \frac{z}{z-\alpha(u)}\frac{\alpha(u)}{\Pr(X=-1)},$$

and the generating functions are:

$$E(u^{\nu_-}z^{S_{\nu_-}}) = 1 - \frac{u\Pr(X=-1)}{\alpha(u)} + \frac{u\Pr(X=-1)}{z},$$
$$E(u^{\nu_+}z^{S_{\nu_+}}) = 1 - \frac{\alpha(u)}{u\Pr(X=-1)} + \frac{z-uE(z^{X+1})}{z-\alpha(u)}.$$

4. **Distribution of the Maximum and its first Hitting time** (M,L)

The distribution of the pair (M,L) is expressed through its generating function $E(x^Lz^M)$.

Proposition 2.

$$E(x^Lz^M) = \lim_{u\to 1}\frac{\Psi_+(ux,z)}{\Psi_+(u,1)}.$$

Proof. We define the variables M_n and L_n, as

$$M_n = \max_{0\le k\le n} S_k, \qquad L_n = \inf\{k/S_k = M_n\},$$

and the function H on $\{|u|<1, |x|<1, |z|<1\}$ as

$$H(u,x,z) = E(\sum_{n\ge 0} u^n x^{L_n} z^{M_n}).$$

Using the same arguments as in proposition 1, it can be proved that:

$$H(u,x,z) = \frac{\Psi_+(ux,z)}{\Psi_+(u,1)(1-u)}.$$

We conclude applying

$$\lim_{u\to 1}(1-u)H(u,x,z) = \lim_{u\to 1}(1-u)E(\sum_{n\geq 0} u^n x^{L_n} z^{M_n}) = \lim_{n\to+\infty} E(x^{L_n} z^{M_n}) = E(x^L z^M).$$

□

For the case of the random walk of subsection 3.1, it gives for $p < \frac{1}{2}$:

$$E(x^L z^M) = \frac{\alpha_2(u)}{\alpha_2(u) - z}\frac{1-2p}{1-p},$$

then

$$\Pr(L < +\infty, M < +\infty) = 1,$$

and the distribution of M is geometric with parameter $\frac{p}{1-p}$, for $p > \frac{1}{2}$:

$$E(x^L z^M) = 0 \qquad \text{and} \qquad \Pr(L = M = +\infty) = 1.$$

References

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