# Graph colouring via the probabilistic method


*Bruce Reed*

Équipe Combinatoire CNRS
Université Pierre et Marie Curie, Paris, France



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[summary by F. Morain and P. Robert]


## 1. Introduction

Colouring a graph with the minimum number of colours is a classical problem in graph theory and has many applications. For instance, think of a cellular phone network on which each vertex (phone) must use a different frequency with its neighbours. This problem is also known to be a difficult one (see for instance [3]).

The purpose of the talk is to present a naive algorithm for colouring a certain type of graphs and explain how to analyze it with elementary probabilistic tools that we will describe first.

## 2. Probabilistic tools

Throughout this section we denote by $\Pr(A)$ the probability of the event $A$, $E(X)$ the expected value of the random variable $X$, and $E(X/A_1, \ldots, A_n)$ the conditional expectation of $X$ relative to the events $A_1, \ldots, A_n$.

2.1. **The Lovász Local lemma.** Suppose that on some probability space $\Omega$, there are $n$ events $A_1, \ldots, A_n$ that are undesirable. We wish to estimate if there is a positive probability to avoid any of them, i.e., if there is a positive lower bound for the quantity

$$\Delta = \Pr\left(\cap_{i=1}^{n} A_i^c\right),$$

where $A_i^c = \Omega - A_i$. If the events are independent, that is for any $k$-tuple $1 \leq i_1 < \cdots < i_k \leq n$,

$$\Pr(\cap_{j=1}^{k} A_{i_j}) = \prod_{j=1}^{k} \Pr(A_{i_j}),$$

then

$$\Delta = \prod_{1=1}^{n} (1 - \Pr(A_i)).$$

The problem is that in practice, the events are not always completely independent but weakly independent, in the sense that for each $i$ there exists a subset $V_i \subset \{1, \ldots, n\}$ such that $A_i$ is independent of the events $A_j, j \in V_j^c$. In other words, $A_i$ is possibly dependent of $A_j$ with $j$ in the "neighbourhood" $V_i$ of $i$. If the cardinality of the $V_i$'s is small, one might expect an estimate close to the one we saw for the independent case. This is the conclusion of Lovács's lemma, see [1].

**Lemma 1.** *If the events are such that for all* $1 \leq i \leq n$*,*

1. $\Pr(A_i) \le p$,
2. $A_i$ *is independent of* $(A_j)_{j \notin V_i}$,
3. $|V_i| \le d$,

*and if* $ep(d+1) < 1$ *then none of the events* $A_i$, $i = 1, \ldots, n$, *occurs with positive probability.*

2.2. **Azuma's inequality.** If $(Y_i)$ be a sequence of independent random variables with the same distribution on $\{0,1\}$, $p = \Pr(Y_i = 1)$; The result of successive coin tossings is a good model for this sequence of random variables. It is well known that the time averages $\frac{1}{n}\sum_{i=1}^n Y_i$ converges exponentially fast to $p$ as $n \to +\infty$. Rigourously, this is Chernoff's bound

$$\Pr\left(\left|\frac{1}{n}\sum_{i=1}^{n} Y_i - p\right| > a\right) < 2e^{-a^2/3np},$$

it says basically that with high probability $[p/a^2]$ coin tossings are sufficient to get an estimate of $p$ with an accuracy of the order of $a$. This kind of result has been extended, for independent variables, to the case of arbitrary distributions, i.e., not only with values in $\{0,1\}$, as long as they have an exponential moment. This is a part of large deviations theory, see [2].

Another possible generalization is to consider the case where instead of the sum of independent variables, one looks at some functional $X$ of some arbitrary random variables $Y_1, \ldots, Y_n$ with values in $\{0,1\}$. Azuma's inequality says that if the conditional exceptions of $X$ with respect to $Y_1, \ldots, Y_i$ do not jump sharply as $i$ goes from 1 to $n$, then $X$ is concentrated around its average value, formally,

**Proposition 1.** *If for each* $i \le n$,

$$\max_{y_1,\ldots,y_{i+1}\in\{0,1\}} |E(X/Y_1 = y_1, \ldots, Y_i = y_i, Y_{i+1} = y_{i+1}) - E(X/Y_1 = y_1, \ldots, Y_i = y_i)| \le c_i, \tag{1}$$

*then*

$$\Pr(|X - E(X)| > a) \le 2e^{-\frac{a^2}{2\sum_1^n c_i^2}}.$$

Azuma's inequality is surprisingly sharp considering the weak hypotheses of the proposition. In the independent case, for $X = \sum_1^n Y_i$, condition (1) is satisfied with $c_i = 1$, hence the inequality is in this case,

$$\Pr\left(\left|\frac{1}{n}\sum_{i=1}^{n} Y_i - p\right| > a\right) < 2e^{-\frac{-a^2}{2n}},$$

which is very close to Chernoff's bound.

## 3. Graph colouring

We *colour* a graph $G$ such that every pair of adjacent vertices receive different colours. The *chromatic number* of $G$, noted $\chi(G)$ is the minimum number of colours required to colour $G$. It is easy to see that, if $\Delta(G)$ denotes the maximal degree of $G$, then $\chi(G) \le \Delta(G) + 1$.

We can obtain good bounds for $\chi(G)$ for certain types of graphs, as explained in [4]. For fixed $\varepsilon > 0$, we saw that a vertex $v$ is $\varepsilon$*-sparse* if the subgraph induced by $N_v$, the neighbourhood of $v$, has at most $(1-\varepsilon)\frac{\Delta(\Delta-1)}{2}$ edges. A graph is $\varepsilon$*-sparse* if each of its vertices is $\varepsilon$-sparse.

**Theorem 1.** *For* $\Delta$ *sufficiently large, if* $G$ *has maximum degree* $\Delta$ *and* $G$ *is* $\varepsilon$*-sparse, then* $\chi(G) \le (1 - \varepsilon/2e^6)\Delta$.

Let us indicate a rough proof of this theorem. In a first step, we construct a partial colouring $\mathcal{C}$ of $G$ such for each vertex $v$, the number of neighbours of $v$ which are coloured exceeds the number of colours appearing on $N_v$ by at least $\frac{\varepsilon}{2e^6}\Delta + 1$.

From this, we complete the colouring of $\mathcal{C}$ to a $\left(1 - \frac{\varepsilon}{2e^6}\right)\Delta$-colouring of $G$ in a greedy manner: We colour the remaining vertices one at a time. When we come to colour $v$, there must be an available colour: Since $v$ has at most $\Delta$ neighbours (this is where the sparseness comes in), the number of colours appearing in $N_v$ is bounded by

$$\Delta - \left(\frac{\varepsilon}{2e^6}\Delta + 1\right).$$

Hence fewer than $\left(1 - \frac{\varepsilon}{2e^6}\right)\Delta$ colours appear in its neighbourhood.

Let us come back to the construction of $\mathcal{C}$. We first assign each vertex of $G$ a uniformly random colour from $\{1, 2, \ldots, \lceil \Delta/2 \rceil\}$. If two adjacent vertices have the same colour, we uncolour them. The resulting partial colouring yields $\mathcal{C}$.

The first thing to show is that $\mathcal{C}$ is not too small, which is rather easy. Then we must study, for vertex $v$, the random variable $Z_v$ which counts the number of pairs of vertices in $N_v$ which have the same colour in $\mathcal{C}$. It can be shown that since $G$ is sparse, the expectation of $Z_v$ is greater than $\varepsilon\Delta/e^4$.

Now that we have proved that many vertices in $N_v$ are coloured, we must show that $Z_v$ does not differ too much from its expected value. Once this is done, we use the Local Lemma to prove that *every* vertex will have such a property, thus proving the property on $\mathcal{C}$. By a technical argument replacing $Z_v$ with a more amenable quantity, Azuma's Inequality is used to prove the assumption on $Z_v$. Roughly speaking, the idea is that a colouring of $v$ should not influence the colouring of the other parts of $\mathcal{C}$, since $G$ is sparse.

### References


[1] Alon (Noga) and Spencer (Joel H.). – *The probabilistic method.* – John Wiley & Sons Inc., New York, 1992, *Wiley-Interscience Series in Discrete Mathematics and Optimization*, xvi+254p.

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[3] Karp (Richard M.). – Reducibility among combinatorial problems. In *Complexity of Computer Computations.* pp. 85–103. – New York, 1972. Proceedings of a Symposium held at IBM Thomas J. Watson Research Center, Yorktown Heights, N.Y., 1972.

[4] Molloy (M.) and Reed (B.). – Graph colouring via the probabilistic method. – April 1997. Preprint.