New Algorithms for Definite Summation and Integration

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[summary by Bruno Salvy]

Abstract

In 1978, W. Gosper gave an algorithm to compute the indefinite sum of an hypergeometric sequence. This algorithm has been incorporated in most computer algebra systems as the basis of their summation routines. Then, in the early 1990's D. Zeilberger applied a version of Gosper's algorithm in a clever way to the efficient calculation of definite sums of hypergeometric sequences. Zeilberger also gave a very general but slow algorithm for the general case of holonomic functions. F. Chyzak shows how Zeilberger's fast algorithm can be extended to a much more general context, including summation and integration of holonomic functions and sequences.

Introduction

F. Chyzak's algorithms [7] aim at computing automatically the right-hand side of identities like the following ones, given their left-hand side:

$$\sum_{k=1}^{n}\binom{k}{m}H_k=\binom{n+1}{m+1}\left(H_{n+1}-\frac{1}{m+1}\right),$$

$$\sum_{k=0}^{n}\left(\sum_{j=0}^{k}\binom{n}{k}\right)^3=n2^{3n-1}+2^{3n}-3n2^{n-2}\binom{2n}{n},$$

$$\sum_{n=0}^{\infty}H_n(x)H_n(y)\frac{u^n}{n!}=\frac{\exp\left(\frac{4u(xy-u(x^2+y^2))}{1-4u^2}\right)}{\sqrt{1-u^2}},$$

$$\int_{-1}^{+1}\frac{e^{-px}T_n(x)}{\sqrt{1-x^2}}dx=(-1)^n\pi I_n(p),$$

$$\int_0^{+\infty}xe^{-px^2}J_n(bx)I_n(cx)dx=\frac{1}{2p}\exp\left(\frac{c^2-b^2}{4p}\right)J_n\left(\frac{bc}{2p}\right),$$

$$\int_0^{+\infty}xJ_1(ax)I_1(ax)Y_0(x)K_0(x)dx=-\frac{\ln(1-a^4)}{2\pi a^2},$$

$$\sum_{n=0}^{\infty}\frac{r_n(a,b)z^n}{(q;q)_n}=\frac{1}{(az;q)_\infty(bz;q)_\infty}.$$

More precisely, for indefinite summations or integrations, the output of the algorithm is an expression for the right-hand side in terms of the left-hand side, while in the definite case, these new algorithms will produce a linear operator or a system of linear operators annihilating the right-hand side. From there, the solution can be found by Petkovšek's algorithm [9] in the recurrence case, by its q-version [4, 5] in the q-case, or by algorithms on differential equations [10, 11] in the differential case.

These examples are treated in a unified manner by considering algebras of linear operators $\partial_1, \dots, \partial_r$ with coefficients in a suitable field $\mathbb{K}$. In most applications, these operators are partial differentiations, shifts or q-shifts. Then one considers *∂-finite terms* t which are such that the

$$\boldsymbol{\partial}^{\boldsymbol{\alpha}} \cdot t = \partial_1^{\alpha_1} \cdots \partial_r^{\alpha_r} \cdot t, \qquad \alpha_i \in \mathbb{N}, \tag{1}$$

span a finite-dimensional vector space over $\mathbb{K}$.
Examples are:

- $\exp(z^2)$ for D_z (derivation) over $\mathbb{Q}(z)$ (with dimension 1);
- the binomial coefficient $\binom{n}{k}$ for S_n and S_k (shifts) over $\mathbb{Q}(n, k)$ (with dimension 1);
- the Bessel $J_n(z)$ functions, the Tchebychev polynomials $T_n(z)$, the Legendre polynomials $P_n(z)$ for S_n (shift) and D_z (derivation) over $\mathbb{Q}(n, z)$ (with dimension 2);
- the product of Bessel functions $J_1(ax)I_1(ax)Y_0(x)K_0(x)$ for D_a and D_x (derivations) over $\mathbb{Q}(a, x)$ (with dimension 16).

In the case of shifts, ∂-finite terms corresponding to 1-dimensional vector spaces are exactly the hypergeometric sequences. In the differential case, Zeilberger's holonomic functions [12] are also ∂-finite. The closure properties under sum, product and the ∂_i's generalize to this context [8].

1. Indefinite ∂-finite summation and integration

Given a ∂-finite term and a basis $b_1, \dots, b_N$ of the vector space generated by its $\boldsymbol{\partial}^{\boldsymbol{\alpha}}$ as in (1), Chyzak's first algorithm finds solutions in this vector space for equations of the form

$$P(\partial_1, \dots, \partial_r) \cdot X = B,$$

where P is a polynomial in $\mathbb{K}\langle\partial_1, \dots, \partial_r\rangle$, X is the unknown and B is a given element of the vector space. The first step of the algorithm consists in setting

$$X = \phi_1 b_1 + \cdots + \phi_N b_N$$

with undeterminate coefficients ϕ_i's in $\mathbb{K}$. The left-hand side of the equation is then expressed in the basis of the b_i's. Identifying coefficients then yields a system of linear equations satisfied by the ϕ_i's. It then suffices to find *rational* solutions of this system, i.e., solutions in $\mathbb{K}$. Several algorithms developed by S. Abramov and coauthors are available for this purpose, depending on the ∂_i's [1, 2, 3, 6].

An important special case of this algorithm is when $P = S_n - 1$, which correspond to indefinite summation. In this setting the problem solved by Gosper's algorithm corresponds to vector spaces of dimension 1, and then a single rational coefficient has to be found.

For instance, consider computing the primitive of the integral cosine,

$$\mathrm{Ci}(z) = \int_0^z \frac{\cos(t)}{t}\, dt.$$

From the differential equation satisfied by $\cos(t)$, one readily computes a third order linear differential equation satisfied by Ci. In the algebra $\mathcal{A} = \mathbb{Q}(z)\langle D_z\rangle$, we thus work in the vector space generated by $\mathrm{Ci}, \mathrm{Ci}', \mathrm{Ci}''$. Thus we look for

$$T = \phi_0 \,\mathrm{Ci} + \phi_1 \,\mathrm{Ci}' + \phi_2 \,\mathrm{Ci}'', \qquad \text{such that} \qquad D_z \cdot T = \mathrm{Ci}\,.$$

This leads to a simple system whose only rational solution is

$$\phi_0(z) = z, \qquad \phi_1(z) = 1, \qquad \phi_2(z) = z,$$

or in other words

$$\int \mathrm{Ci}(z)\,dz = z\,\mathrm{Ci}(z) - \sin(z).$$

2. Definite ∂-finite summation and integration

Zeilberger's algorithms for definite summation and integration are based on *creative telescoping.* To simplify, we describe this method in the case of summation, i.e. we consider a ∂-finite term t_n when $\partial_1 = S_n$ is a shift with respect to a variable n, and we want to compute

$$\sum_{n=a}^{b} t_n,$$

or more precisely an operator annihilating this sum.

If we can find two polynomials P and Q in the algebra such that P commutes with $\sum_n$ and

$$P \cdot t_n = (S_n - 1)Q \cdot t_n, \tag{2}$$

then interchanging P and $\sum$ we get

$$P \cdot \sum_{n=1}^{b} t_n = [Q \cdot t_n]_a^b.$$

When moreover the bounds a and b are such that t_n and all its $\boldsymbol{\partial}^{\boldsymbol{\alpha}}$'s are zero there (the so-called case of *natural* boundaries), then this computation has for consequence that the right-hand side telescopes, whence the name of the method.

In the hypergeometric case, Zeilberger's fast algorithm [13] to find P and Q relies on the observation that (2) is equivalent to $P \cdot t_n$ being Gosper-summable. He then shows that Gosper's algorithm can be extended to handle undeterminate coefficients in P and find conditions on these coefficients for the sum to be hypergeometric. The algorithm then increases the order of P (and consequently the number of indeterminate coefficients) till the extended Gosper algorithm finds a sum. Termination is guaranteed via Bernstein's theory of holonomic functions.

Chyzak's second algorithm is an extension of Zeilberger's algorithm to the more general context of ∂-finite terms. Gosper's algorithm is replaced by the algorithm of the previous section, which is first extended to handle undeterminate coefficients and find conditions on these coefficients for an indefinite ∂^{-1} to exist in the vector space. Termination is guaranteed only when Bernstein's theory can be invoked.

As an example, consider Neumann's addition theorem on Bessel functions:

$$J_0(z)^2 + 2\sum_{k=1}^{\infty} J_k(z)^2 = 1. \tag{3}$$

In the algebra $\mathcal{A} = \mathbb{Q}(z,k)\langle D_z, S_k\rangle$, the sequence of functions $J_k(z)^2$ satisfies the system

$$\begin{cases} zD_z^2 + (-2k+1)D_z - 2S_kz + 2z, \\ zD_zS_k + zD_z + (2k+2)S_k - 2k, \\ z^2S_k^2 - 4(k+1)^2S_k - 2z(k+1)D_z + 4k(k+1) - z^2, \end{cases}$$

from which follows that $J_k(z)^2$ is ∂-finite and generates a vector space of dimension 3 with basis $\{J_k(z)^2, S_k \cdot J_k(z)^2, D_z \cdot J_k(z)^2\}$. The output of the algorithm is

$$P = D_z, \qquad Q = \frac{k}{z} + \frac{1}{2} D_z,$$

which means that

$$D_z \cdot \sum_{k=0}^{\infty} J_k(z)^2 + \left[Q \cdot J_k(z)^2\right]_{k=0}^{\infty} = 0.$$

After some rewriting and considering the initial conditions, this is indeed equivalent to (3).

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