On a problem of Rubel

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[summary by Frédéric Chyzak]

Abstract

For a given function f, we study all the functions that satisfy every algebraic differential equation with constant coefficients which is satisfied by f. This question was suggested by Lee Rubel in [3, Problem 22]. Here the author characterizes this set of functions, first when f is a linear combination of exponential functions, next when f is a Liouvillian function. Finally, he applies these results to the computation of a series expansion of solutions of algebraic differential equations.

1. Exponential functions

For two functions f and g, define $g \ll f$ to mean that g satisfies every algebraic differential equation with constant coefficients which is satisfied by f. Let f be the following $\mathbb{C}$-linear combination of exponential functions

$$\sum_{k=1}^{n} a_k e^{\lambda_k x}.$$

Trivially, $g \ll f$ implies that $g = \sum_{k=1}^{n} A_k e^{\lambda_k x}$ with $A_k \in \mathbb{C}$, since the differential polynomial $L(y)$ defined by the linear operator $\prod_{k=1}^{n} \left(\frac{d}{dx} - \lambda_k\right)$ vanishes at f. (We refer the reader to [2] for an introduction to differential algebra.) This necessary condition for $g \ll f$ is not always sufficient. Two cases occur, according to the dimension d of the $\mathbb{Q}$-vector space generated by the λ_k. Note that this dimension is also the transcendence degree of $\mathbb{C}(e^{\lambda_1 x}, \dots, e^{\lambda_n x})$ over $\mathbb{C}$.

Transcendence degree $d = n$. In this case, no equation of order less than d is satisfied by f. If $P(y)$ is another differential polynomial of order d that vanishes at f, L must divide P. Otherwise, using L to rewrite $f^{(d)}$ as a polynomial in the derivatives of f of lower orders yields a differential polynomial of order less than d. This polynomial must then be zero, which gives a contradiction. Therefore, g satisfies any equation of order d satisfied by f. Next, let $Q(y)$ be a differential polynomial satisfied by f. Differentiating L sufficiently many times makes it possible to rewrite all the derivatives of f of order greater or equal to d that occur in Q as polynomials in derivatives of order less than d. Once again, L divides Q so that $Q(g) = Q(f) = 0$. Hence, $g \ll f$.

Transcendence degree $d \leq n$. In this case, assume that $\lambda_1, \dots, \lambda_d$ are linearly independent over $\mathbb{Q}$, whereas

$$\lambda_i = \sum_{j=1}^{d} c_{i,j} \lambda_j \qquad \text{for } c_{i,j} \in \mathbb{Q}, \text{ when } i = d+1, \dots, n. \tag{1}$$

Taking $n-1$ derivatives of the equation $f = \sum_{k=1}^{n} a_k e^{\lambda_k x}$ yields a linear system relating the $a_k e^{\lambda_k x}$'s and the derivatives of f. This system has a Vandermonde determinant, hence we obtain linear expressions

$$a_k e^{\lambda_k x} = R_k(f, \ldots, f^{(n-1)}) = R_k, \qquad \text{for } k = 1, \ldots, n. \tag{2}$$

Combining equations (1–2) so as to eliminate the λ_k's yields the equations

$$a_i^{b_i} \prod_{j=1}^{d} a_j^{-\gamma_{j,i}} = R_i^{b_i} \prod_{j=1}^{d} R_j^{-\gamma_{j,i}} = S_j(f, \ldots, f^{(n-1)}), \qquad i = d+1, \ldots, n, \tag{3}$$

where b_i is a least common multiple for the denominators of the $c_{i,j}$'s and each $\gamma_{i,j} = b_j c_{i,j}$ is an integer. Now, if $g \ll f$, the function g also satisfies the second equality in (3). In addition, it is of the form $g = \sum_{k=1}^{n} A_k e^{\lambda_k x}$ and therefore,

$$a_i^{b_i} \prod_{j=1}^{d} a_j^{-\gamma_{j,i}} = A_i^{b_i} \prod_{j=1}^{d} A_j^{-\gamma_{j,i}}, \qquad i = d+1, \ldots, n. \tag{4}$$

We have obtained necessary and sufficient conditions for $g \ll f$ when f is of the form $\sum_{k=1}^{n} a_k e^{\lambda_k x}$.

Another approach based on differential ring homomorphisms. We now give another derivation of these conditions. This second approach follows methods similar to methods of differential Galois theory and will prove very fruitful when generalizing to Liouvillian functions.

We have a tower of function rings

$$\Phi_0 = \mathbb{C} \subset \cdots \subset \Phi_k = \mathbb{C}[e^{\lambda_1 x}, \ldots, e^{\lambda_k x}] \subset \cdots \subset \Phi_n = \mathbb{C}[e^{\lambda_1 x}, \ldots, e^{\lambda_n x}].$$

Write $\hat{\Phi}_k$ for the quotient field of Φ_k. It follows from (1) that the field extensions $\hat{\Phi}_k : \hat{\Phi}_{k-1}$ are transcendental for $k = 1, \ldots, d$ and algebraic for $k = d+1, \ldots, n$, with minimal polynomials

$$m_k\left(e^{\lambda_k x}\right) = \left(e^{\lambda_k x}\right)^{b_k} - \prod_{i=1}^{d} \left(e^{\lambda_i x}\right)^{\gamma_{k,i}}. \tag{5}$$

For complex constants C_k, consider the ring homomorphism $T : \Phi_n \to \Phi_n$ given by $T\left(e^{\lambda_k x}\right) = C_k e^{\lambda_k x}$ for $k = 1, \ldots, n$. We want to constrain the C_k's so that T is also a differential ring homomorphism that maps $f = \sum_{k=1}^{n} a_k e^{\lambda_k x}$ to $g = \sum_{k=1}^{n} A_k e^{\lambda_k x}$. Necessarily, $A_k = C_k a_k$ and the minimal polynomials (5) are mapped to themselves, modulo non-zero multiplicative constants $\eta_k \in \mathbb{C}$, so that

$$T\left(m_k\left(e^{\lambda_k x}\right)\right) = \left(C_k e^{\lambda_k x}\right)^{b_k} - \prod_{i=1}^{d} \left(C_i e^{\lambda_i x}\right)^{\gamma_{k,i}} = \eta_k\left(m_k\left(e^{\lambda_k x}\right)\right) = \eta_k \left(\left(e^{\lambda_k x}\right)^{b_k} - \prod_{i=1}^{d} \left(e^{\lambda_i x}\right)^{\gamma_{k,i}}\right).$$

It follows that $\eta_k = C_k^{b_k} = \prod_{i=1}^{d} C_i^{\gamma_{k,i}}$, so that condition (4) is also a necessary and sufficient condition for T to be a differential ring isomorphism.

In the next section, we construct a set of differential ring homomorphisms and investigate its connection to the set $\{g \mid g \ll f\}$ when f is a Liouvillian function.

2. Liouvillian functions

We now turn to differential extension towers of the form

$$\Phi_0 = \mathbb{C} \subset \cdots \subset \Phi_k = \Phi_{k-1}[z_k] \subset \cdots \subset \Phi_n = \Phi_{n-1}[z_n], \tag{6}$$

where the extension $\Phi_k = \Phi_{k-1}[z_k]$ is either

(*i*) an algebraic extension given by the minimal polynomial $m_k(z_k) = 0$ with coefficients in Φ_{k-1};
(*ii*) an exponential extension given by $z_k = \exp(w_{k-1})$, for $w_{k-1} \in \hat{\Phi}_{k-1}$;
(*iii*) an integral extension given by $z_k = \int w_{k-1}$, for $w_{k-1} \in \hat{\Phi}_{k-1}$.

In cases (*ii*) and (*iii*), write $w_{k-1} = \zeta_{k-1}/\eta_{k-1}$ for coprime $\zeta_{k-1}, \eta_{k-1} \in \Phi_{k-1}$. An element of a field $\hat{\Phi}_k$ corresponding to a tower (6) is called a Liouvillian function.

We now proceed to define sets $\mathbf{G}_k$ of differential ring homomorphisms from Φ_k to rings of Liouvillian functions. This construction generalizes that of T in the previous section. We take $\mathbf{G}_0$ to be the singleton of the identity on $\mathbb{C}$ and define the $\mathbf{G}_k$'s by induction on k, considering the three cases above separately. For any differential polynomial $P \in \Phi_k\{y\}$ and any $\rho \in \mathbf{G}_k$, let $\tilde{\rho}(P)$ denote the differential polynomial in $\rho(\Phi_k)\{y\}$ obtained by applying ρ to each coefficient of P.

Algebraic extensions. For any $\rho \in \mathbf{G}_{k-1}$ and any choice of root s of $\tilde{\rho}(m_k)$, ρ extends to Φ_k as a differential ring homomorphism by mapping z_k to s. We define $\mathbf{G}_k$ to be the set of all these extensions.

Exponential extensions. For any $\rho \in \mathbf{G}_{k-1}$ such that $\rho(\eta_{k-1}) \neq 0$, $\rho(w_{k-1})$ is well-defined and ρ extends to Φ_k as a differential ring homomorphism by mapping z_k to $K \exp(\rho(w_{k-1}))$. We define $\mathbf{G}_k$ to be the set of all these extensions.

Integral extensions. For any $\rho \in \mathbf{G}_{k-1}$ such that $\rho(\eta_{k-1}) \neq 0$, $\rho(w_{k-1})$ is well-defined and ρ extends to Φ_k as a differential ring homomorphism by mapping z_k to $K \int \rho(w_{k-1})$. We define $\mathbf{G}_k$ to be the set of all these extensions.

The main theorem. The previous construction yields the following theorem. A proof is given in [5]. Similar results are also presented in [4, Proposition 2].

THEOREM 1. *Let the Liouvillian extension tower (6) and $\mathbf{G}_n$ be as above. Let $f = f_1/f_2 \in \hat{\Phi}_n$, with coprime $f_1, f_2 \in \Phi_n$. Then $g \ll f$ if and only if there exists an open dense subset W of $\mathbb{C}$ such that g belongs to the closure of the set*

$$\left\{\rho(f) \middle| \rho \in \mathbf{G}_n, \rho(f_2) \neq 0\right\}$$

in the topology of uniform $\mathcal{C}^\infty$ convergence on compact subsets of W.

3. An example

As an example, we compute the set of functions g such that $g \ll f$ with $f = (\exp(e^x) - 1)/e^x$. An algebraic differential equation satisfied by f is

$$ff'' - f'^2 - ff' - f' + f - f^2 = 0. \tag{7}$$

We have the tower of Liouvillian extensions $\mathbb{C} \subset \mathbb{C}[x] \subset \mathbb{C}[x, e^x] \subset \mathbb{C}[x, e^x, e^{e^x}] \ni f$. The first extension is given by $x = \int 1$; the latter two are exponential extensions. The differential ring homomorphisms T are defined such that:

(i) they are the identity on $\mathbb{C}$

$$T|_{\mathbb{C}} = T_0 : 1 \mapsto 1;$$

(ii) they extend to the integral extension $\mathbb{C}[x]$ by introducing a constant K_0

$$T|_{\mathbb{C}[x]} = T_1 : x \mapsto \int T_0(1) = x + K_0;$$

(iii) they extend to the first exponential extension $\mathbb{C}[x, e^x]$ by introducing a constant K_1

$$T|_{\mathbb{C}[x,e^x]} = T_2 : e^x \mapsto Ke^{T_1(x)} = K_1 e^x;$$

(iv) they extend to the second exponential extension $\mathbb{C}[x, e^x, e^{e^x}]$ by introducing a constant K_2

$$T = T_3 : e^{e^x} \mapsto K' e^{T_2(e^x)} = K_2 e^{K_1 e^x}.$$

Finally, the set of functions g such that $g \ll f$ is the closure of the set

$$\left\{ \frac{K_2 e^{K_1 e^x} - 1}{K_1 e^x} \middle| K_1, K_2 \in \mathbb{C}, K_1 \neq 0 \right\}.$$

Making $K_2 = 1$, next K_1 tend to 0 yields the function 1, which is indeed a solution of (7). We have thus proved that $1 \ll (\exp(e^x) - 1)/e^x$.

4. Series expansion

Theorem 1 can be used to help compute a series expansion for a solution of an algebraic differential equation belonging to a Hardy field [1]. It can be proved that the number of possible nested (asymptotic) forms f_0 for a solution is finite. This number grows exponentially with the order of the equation. Writing f in the form $f_0(1 + \epsilon)$, and substituting it into the equation yields an equation for the rest ϵ, of possibly doubled order. It follows that the exponential complexity of this first, naive method makes it impracticable.

Assume f can be written in the form $F + g$, where F is the sum of a finite number of first terms in an asymptotic expansion and g is the rest, of smaller asymptotic growth. If f does not belong to the closure under consideration in Theorem 1 applied to the Liouvillian function F, then there is a differential polynomial $P(y)$ that vanishes on F but not on f. From the equation defining f, the finitely many possible orders of growth of $P(f)$ can be computed. Next, each term in $P(f) = P(F + g)$ contains g or one of its derivatives. This yields a number of possible orders of growth for g, hopefully smaller than the one obtained by the general method.

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