Algebraic Computation of Matrix-like Padé Approximants

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[summary by Bruno Salvy]

Abstract

Padé approximants are rational approximants to functions represented as power series. There are many classes and generalizations of Padé approximants, with various kinds of applications. After reviewing some of these approximants and their use, this work presents a unified way of computing them.

1. A gallery of Padé approximants

Given a formal power series $A(z)$, a *Padé approximant* of type (m, n) is a pair of polynomials $(u(z), v(z))$ of degrees at most m and n respectively, such that $A(z) - u(z)/v(z) = O(z^{m+n+1})$.

Hermite-Padé approximants constitute a natural generalization of Padé approximants. Instead of *one* power series, the input consists in ℓ power series $A_1(z), \ldots, A_\ell(z)$ and ℓ integers $n_1, \ldots, n_\ell$. The approximant is then an ℓ-tuple of polynomials $(p_1(z), \ldots, p_\ell(z))$, with $p_i(z)$ of degree at most $n_i - 1$, such that

$$p_1(z)A_1(z) + \cdots + p_\ell(z)A_\ell(z) = O(z^{N-1}),$$

where $N = \sum n_i$.

The extended Euclidean algorithm can be seen as the calculation of a Hermite-Padé approximant. Given two polynomials $P(z)$ and $Q(z)$, the extended Euclidean algorithm computes three polynomials $U(z)$, $V(z)$ and $G(z)$, such that $G(z)$ is the gcd of $P(z)$ and $Q(z)$, and the Bézout identity holds

$$U(z)P(z) + V(z)Q(z) = G(z).$$

This is the same as computing a Hermite-Padé approximant for the reciprocal polynomials of $P(z)$ and $Q(z)$.

Hermite-Padé approximants are used in `gfun` [3] to guess linear differential equations or algebraic equations satisfied by a formal power series $A(z)$. In this context, one starts with $A_i(z) = A^{(i-1)}(z)$ or $A_i(z) = A^{i-1}(z)$.

A generalization of these approximants is obtained by considering *vectors* or *matrices* of power series, leading to vector and matrix Hermite-Padé approximants. Vector Hermite-Padé approximants are used in algorithms factoring linear differential operators [4].

Another kind of generalization called *simultaneous Padé approximants* was introduced by Hermite in 1873 in order to prove the transcendence of e. As in the case of Hermite-Padé approximants one starts with ℓ power series $A_1(z), \ldots, A_\ell(z)$. Given $\ell + 1$ integers $(n_0, n_1, \ldots, n_\ell)$, the aim is to find $\ell + 1$ polynomials $q(z), p_1(z), \ldots, p_\ell(z)$ such that $A_j(z) = p_j(z)/q(z) + O(z^K)$.

Again, vector and matrix versions are of interest.

2. Computation

All these approximants can be computed by linear algebra algorithms, since they correspond to solving an equation of the type $AX = B$, where X is a vector of the unknown coefficients of the approximants, A encodes the product $\bmod z^N$ by the initial data in the basis $1, z, z^2, \ldots$ and B represents the desired right-hand side $\bmod z^N$. Thus efficient algorithms for Gaussian elimination and fraction free versions of these can be used. The solution set has the structure of a module. In many cases, this module has dimension one, so that any approximant generates all of them. In other cases, it might be useful to compute a basis of this module.

Example. This example helped discover a nice generating function [3]. The coefficients of the series

$$y(z) = 3 + 19z + 193z^2 + 2721z^3 + 49171z^4 + 1084483z^5 \\ + 28245729z^6 + 848456353z^7 + 28875761731z^8 + O(z^9)$$

are the numerators of convergents to $e = \exp(1)$ of index $3k+1$. We are looking for a Hermite-Padé approximant of $(1, y, y')$ with degree constraints $(1, 2, 2)$. The matrix version of this problem is

$$\begin{bmatrix} 1 & 0 & 3 & 0 & 0 & 19 & 0 & 0 \\ 0 & 1 & 19 & 3 & 0 & 386 & 19 & 0 \\ 0 & 0 & 193 & 19 & 3 & 8163 & 386 & 19 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \cdot X = 0.$$

A basis of the kernel is readily found to be ${}^t(-3, -1, 1, -6, -1, 0, 0, -4)$, so that $y(z)$ satisfies the following differential equation up to $O(z^8)$:

$$4z^2 y'(z) - (1 - 6z - z^2)y + 3 + z = 0.$$

Another way of viewing the same computation, which preserves sparseness, is as a standard basis computation. For instance, in the case of Hermite-Padé approximants, one introduces new variables $t, a_1, \ldots, a_\ell$ and computes a standard basis for the set of series

$$a_1 - tA_1(z), \ldots, a_\ell - tA_\ell(z), z^{N-1},$$

with respect to any ordering such that $t > z$ and $z > z^2 > \cdots$ are smaller than the a_i's. The polynomials of the basis are linear in the a_i's, those which do not contain t generate the module of approximants.

Bibliography

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