Biased Random Walks, Lyapunov Functions, and Stochastic Analysis of Best Fit Bin Packing

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[summary by Philippe Robert]

This talk considers the average case behavior of the best fit algorithm for on-line bin packing in the case where the item sizes are uniformly distributed in $\{1/k, \ldots, j/k\}$. The best fit algorithm works as follows: the items are packed on-line, each item goes to the bin for which the wasted space is minimized. The bins are of size 1. We focus here on the average wasted space of the algorithm. It is known that this quantity is bounded when j is small compared to k ($j < \sqrt{2k+2.25} - 1.5$) or when j is sufficiently close to 1 or k. In the cases where it is known to be unbounded it appears to grow linearly (see [1]). The motivation of this study is to analyze the sensitivity of the performances of best fit algorithm with respect to the probability distribution of the sizes of the items. The main result is the following theorem.

Theorem 1. *For the uniform distribution on $\{1/k, 2/k, \ldots, (k-2)/k\}$, the average wasted space is bounded.*

We sketch the main ideas of the proof. The main variable of interest is the the multi-dimensional Markov chain $S(t) = (s_1(t), \ldots, s_{k-1}(t))$, where $s_i(t)$ is the number of bins at time t with a residual space of size i/k. The transitions of this Markov chain are described as follows:
If the $(t+1)$th item is of size x/k,

- If $s_x(t) \neq 0$ then a bin is completely full with this item and so $s_x(t+1) = s_x(t) - 1$;
- if not and $\{s_i(t) \neq 0\}$ is not empty and $\nu = \inf\{i > x / s_i(t) \neq 0\}$ then $s_\nu(t+1) = s_\nu(t) - 1$ and $s_{\nu-x}(t+1) = s_{\nu-x}(t) + 1$;
- Otherwise $s_{k-x}(t+1) = s_{k-x}(t) + 1$.

If $W(t) = \sum_1^{k-1} i s_i(t)$ is the wasted space at time t, the theorem is that $\limsup_{t\to\infty} E(W(t)) < +\infty$. Because of the Markovian context, the first thing to check is whether the Markov chain $(S(t))$ is ergodic or not (i.e. has an equilibrium measure). A classical idea in this domain is to try to construct a Lyapunov function which is decreasing at infinity if the Markov chain is ergodic. The following result (see [2]) gives a useful criterion for our problem.

Theorem 2. *If $X(t)$ is an irreducible homogeneous Markov chain on a countable state space $\mathcal{S} \subset \mathbb{N}$, and if there exists an integer $b \in \mathbb{N}$ and a function $f : \mathcal{S} \to \mathbb{R}_+$ such that*

(1) *$f(s) > C_1 s^\mu$, for some constants C_1, μ;*
(2) *$P(X(b) = b \mid X(0) = a) = 0$ if $|f(b) - f(a)| > C_2$;*
(3) *there exists a finite subset B of $\mathcal{S}$ such that $E_s(f(X(b)) - f(s)) < -\varepsilon$ if $s \notin B$.*

Then the Markov chain is ergodic with the invariant probability π satisfying

$$\pi(s) \leq C e^{-\delta f(s)},$$

for some constants C and δ.

The function f is usually called a Lyapunov function. The main assumption is condition (3) which expresses that the trajectory of $(f(X(t)))$ goes back ultimately (after b steps) towards the origin in average when it is far away. In our case the Lyapunov function is the wasted space $f(s) = \sum_1^{[j/2]} is_i$. Using stochastic comparisons with simple random walks on the integers, it is proved that the above assumptions are satisfied for this function. The result on the tail of the invariant distribution shows that the wasted space converges in distribution and also in average. Hence the wasted space is bounded.

Bibliography

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