Determinants, Catalan numbers and Macdonald’s symmetric functions

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[summary by Bruno Salvy]

Abstract

A famous conjecture in the theory of symmetric functions states that the coefficients of Macdonald’s polynomials in the basis of Schur’s symmetric functions are positive. F. Bergeron, A. M. Garsia and M. Haiman have introduced a linear operator $∇$ whose eigenvalues are related to Macdonald’s polynomials. Properties of this operator in a special case are related to combinatorial determinants which can be evaluated by the Gessel-Viennot technique relating them to non-intersecting paths.

1. Introduction to symmetric functions

This section and the following one are based on [4].

Partitions and symmetric functions are strongly related. A partition is an infinite decreasing sequence of positive integers $λ = (λ_1, λ_2, \ldots)$, with finitely many non-zero elements. The index of the last non-zero element in the partition is called its length and is denoted $\ell(λ)$; the sum of the $λ_i$’s is called the weight of the partition and is denoted $|λ|$. For $n \geq \ell(λ)$, $λ$ is identified with the $n$-tuple of its first elements. Then if $x = (x_1, \ldots, x_n)$ is a $n$-tuple of indeterminates, $x^λ$ denotes the monomial $x_1^{λ_1} \cdots x_n^{λ_n}$ and $S_n^λ$ denotes a maximal set of distinct permutations of $λ$.

A fundamental basis of symmetric functions is constituted by the monomial symmetric functions, indexed by the partitions: for $n \geq \ell(λ)$,

$$m_λ(x_1, \ldots, x_n) = \sum_{σ \in S_n^λ} x^{σ(λ)}.$$ 

Clearly, the set of $m_λ$’s, when $λ$ runs through all partitions of length at most $n$ is a basis of the symmetric polynomials in $n$ variables. The set $Λ$ of symmetric functions is defined as the vector space generated by the $m_λ$’s.

Three important sets of symmetric functions, $e_r = m_{(1^r)}$ (elementary), $h_r = \sum_{|λ|=r} m_λ$ (complete) and $p_r = m_{(r^1)}$ (power sum), have simple generating functions:

\[
E(t) = \sum_{r \geq 0} e_r t^r = 1 + t \sum_{i} x_i + t^2 \sum_{i < j} x_i x_j + \cdots = \prod_{i>0} \frac{1}{1 - x_i t},
\]

\[
H(t) = \sum_{r \geq 0} h_r t^r = 1 + t \sum_{i} x_i + t^2 \sum_{i \leq j} x_i x_j + \cdots = \prod_{i>0} \frac{1}{1 - x_i t},
\]

\[
P(t) = \sum_{r \geq 0} p_r t^r = \sum_{i} x_i + t \sum_{i} x_i^2 + \cdots = \sum_{i>0} \frac{x_i}{1 - x_i t}.
\]
Each of these three sets of symmetric functions generates $\Lambda$ as a ring. In all three cases, defining for a partition $\lambda$ a function $f_\lambda = f_{\lambda_1} f_{\lambda_2} \cdots$, where $f$ is $e$, $h$ or $p$ yields a basis of $\Lambda$ as a vector space, when $\lambda$ runs through the set of partitions.

Formulas giving the coordinates of one of these functions in terms of the other families are obtained by extracting the coefficient of $t^n$ in the following straightforward relations between the generating functions:

\begin{align}
(1) \quad & E(t)H(-t) = 1, \quad P(t) = \frac{H'(t)}{H(t)}, \quad P(-t) = \frac{E''(t)}{E(t)},
\end{align}

The last two equations yield the classical Newton formulas between power sums and elementary symmetric functions. Integrating these equation also yields

\begin{align}
H(t) &= \exp \sum_{r>0} p_r \frac{t^r}{r!} = \sum_{\lambda} p_{\lambda} \frac{t^{\lambda}}{z_{\lambda}}, \quad E(t) = \sum_{\lambda} (-1)^{|\lambda| - q(\lambda)} p_{\lambda} \frac{t^{\lambda}}{z_{\lambda}}, \quad \text{with} \quad z_{\lambda} = \prod_{i>0} i^{m_i} m_i!,
\end{align}

where $m_i$ is the number of occurrences of the part $i$ in $\lambda$.

Another family of symmetric functions, the Schur functions, is defined for $n \geq \ell(\lambda)$ by

\begin{align}
 s_\lambda(x_1, \ldots, x_n) &= \frac{\det(x_i^{j+n-j})_{1 \leq i,j \leq n}}{\det(x_i^{j})_{1 \leq i,j \leq n}},
\end{align}

The $s_\lambda$'s are indeed polynomials, since the numerator is a polynomial in the $x_i$'s which vanishes whenever $x_i = x_j$ with $i \neq j$, and thus is a multiple of the Vandermonde determinant in the denominator. The $s_\lambda$'s form another basis of $\Lambda$. They are related to the complete and elementary symmetric functions by the Jacobi-Trudi identities:

\begin{align}
(2) \quad & s_\lambda = \det(h_{\lambda_i-i+j})_{1 \leq i,j \leq n}, \quad s_\lambda = \det(e_{\lambda_i-i+j})_{1 \leq i,j \leq m},
\end{align}

where $\lambda'$ is the conjugate of $\lambda$, i.e. the partition whose Ferrers diagram is the reflexion of that of $\lambda$ with respect to the diagonal.

Recall that a Young tableau of shape $\lambda$ is a Ferrers diagram of shape $\lambda$ with squares numbered by consecutive positive integers $1, 2, \ldots, r$, the numbers increasing strictly in each column and weakly along each row. The weight $w(T)$ of a tableau $T$ is the $r$-tuple $(m_1, \ldots, m_r)$, $m_i$ counting the number of occurrences of $i$. The tableau is called standard when it contains each number $1, 2, \ldots, |\lambda|$ exactly once, i.e. its weight is $(1^{\lambda})$. The Schur functions are related to tableaux by

\begin{align}
 s_\lambda &= \sum_T x^{w(T)},
\end{align}

summed over all tableaux $T$ of shape $\lambda$. From this follows that the coordinates $K_\lambda$ of $s_\lambda$ in the basis $m$ are positive integers counting the number of tableaux of shape $\lambda$ and weight $\mu$ and thus are positive integers. Macdonald's conjecture is a generalization of this property.

All these symmetric functions can also be related by expanding in several ways the doubly infinite product $P(x,y) = \prod(1 - x_i y_j)^{-1}$. Thus one gets

\begin{align}
(3) \quad & \prod_{i,j} \frac{1}{1 - x_i y_j} = \sum_\lambda z_{\lambda}^{-1} p_\lambda(x) p_\lambda(y) = \sum_\lambda h_\lambda(x) m_\lambda(y) = \sum_\lambda m_\lambda(x) h_\lambda(y) = \sum_\lambda s_\lambda(x) s_\lambda(y).
\end{align}

This motivates the definition of a scalar product by $\langle h_\lambda, m \rangle = \delta_\lambda$ for all partitions $\lambda, \mu$, where $\delta_\lambda$ is the Kronecker delta. The relations (3) show that the $p_\lambda$'s form an orthogonal basis, while the $s_\lambda$'s form an orthonormal basis of $\Lambda$. This property characterizes the Schur functions.
The next step is to consider the Hall-Littlewood symmetric functions with one parameter

\[ P_\lambda(x_1, \ldots, x_n; t) = \sum_{\sigma \in S_n} \sigma \left( x^\lambda \prod_{i \neq j} \frac{x_i - t x_j}{x_i - x_j} \right). \]

These functions interpolate between the monomial symmetric functions—obtained when \( t = 1 \)—and the Schur symmetric functions—obtained when \( t = 0 \). They form a \( \mathbb{Z}[t] \)-basis of \( \Lambda[t] \). Therefore, one may consider the polynomials \( K_\lambda(t) \) defined by

\[ s_\lambda(x) = \sum K_\lambda(t) P(x; t). \]

The polynomials \( K_\lambda(t) \) turn out to have positive coefficients, and this has been proved by Lascoux and Schützenberger who gave an expression of the form

\[ K_\lambda(t) = \sum_T t^{c(T)}, \]

summed over all tableaux \( T \) of shape \( \lambda \) and weight \( \mu \), where \( c(T) \) is a certain combinatorial function of the tableau (its charge). Several expansions of the product \( P(x, y; t) = \prod_{i,j} (1 - t x_i y_j)/(1 - x_i y_j) \) lead to results very similar to those obtained above and to the definition of a scalar product on \( \Lambda[t] \) with values in \( \mathbb{Q}(t) \) with respect to which the \( P_\lambda(x; t) \) are orthogonal. Also \( \langle P_\lambda, m \rangle = 0 \) when \( \mu \not\subseteq \lambda \) (the Ferrers diagram of \( \mu \) is not included in that of \( \lambda \)), and together with their orthogonality this characterizes the \( P_\lambda \). The basis which is dual to the Schur functions \( s_\lambda(x) \) with respect to this scalar product is denoted \( S_\lambda(x; t) \), i.e., \( \langle S_\lambda(x; t), s_\lambda(x) \rangle = \delta_\lambda \).

2. Macdonald’s conjecture

Macdonald’s conjecture concerns the Macdonald symmetric functions, which have two parameters. The doubly infinite product

\[ \Pi(x, y; q, t) = \prod_{i,j} \frac{(t x_i y_j; q)_\infty}{(x_i y_j; q)_\infty}, \]

where \( (a; q)_\infty = \prod_{r=0}^{\infty} (1 - a q^r) \),

can be expanded as

\[ \Pi(x, y; q, t) = \sum_\lambda \frac{1}{z_\lambda(q, t)} P_\lambda(x) P_\lambda(y), \quad \text{with} \quad z_\lambda(q, t) = z_\lambda \prod_{i=1}^{e(\lambda)} \frac{1 - q^{\lambda_i}}{1 - t^{\lambda_i}}. \]

This motivates the definition of a scalar product by

\[ \langle p_\lambda, p_\mu \rangle_{q, t} = \delta_\lambda z_\lambda(q, t). \]

The Macdonald symmetric functions are defined uniquely by two properties: they are orthogonal with respect to this scalar product and they decompose in the basis of the monomial symmetric functions as

\[ P_\lambda(x; q, t) = m_\lambda + \sum_\lambda u_\lambda m_\lambda. \]

When \( q = t \), they reduce to the Schur functions \( s_\lambda \), and when \( q = 0 \) to the Hall-Littlewood functions \( P_\lambda(x; t) \).

For a partition \( \lambda \) and a cell \( c = (i, j) \) of its Ferrers diagram, one defines the arm of \( c \) to be \( a(c) = \lambda_i - j \) and its leg to be \( l(c) = \lambda_j - i \). Now we can state Macdonald’s conjecture.
Conjecture 1 (Macdonald). The coefficients $K_\lambda(q,t)$ of the following decomposition are polynomials with positive coefficients:

$$
(4) \quad \tilde{H}_\lambda(x;t) := c_\lambda(q,t)P_\lambda(x;q,t) = \sum_{\lambda} K_\lambda(q,t)S_\lambda(x;t), \quad \text{where} \quad c_\lambda(q,t) = \prod_{x \in \lambda} (1 - q^{\sigma(x)}t^{\sigma(x)+1}).
$$

These coefficients possess a lot of structure. For instance, for $\lambda = (3,1)$, Eq. (4) becomes

$$
\tilde{H}_{(3,1)} = S_{(3)} + (q^2 + t + q)S_{(3,1)} + (t + q)qS_{(2,2)} + (tq + q^2 + t)qS_{(2,1,1)} + tq^2S_{(1,1,1,1)}.\]

Only special cases of Macdonald’s conjecture have been proved.

3. Combinatorial properties of $\nabla$ when $t = 1$

In order to study the polynomials $\tilde{H}_\lambda$, Bergeron, Garsia and Haiman have introduced a linear operator $\nabla$ which is diagonal in the basis $\tilde{H}_\lambda$, with eigenvalues $T_{\lambda}(q,t) = q^{\sigma(\lambda)}t^{\sigma(\lambda)}$, where $\sigma(\lambda) = \sum (i - 1)\lambda_i$. The matrix of $\nabla$ in the Schur basis turns out to have a fascinating structure of which much is still only conjectured [2].

The aim of [1] is to study this operator in more detail in the special case $t = 1$. Then the basis $\tilde{H}_\lambda(x;q) := \tilde{H}_\lambda(x;q,1)$ becomes multiplicative: $\tilde{H}_\lambda(x;q) = \tilde{H}_{(\lambda_1)}(x;q)\tilde{H}_{(\lambda_2)}(x;q)\cdots$ and $\nabla$ becomes multiplicative too. Thus any identity involving symmetric functions gives rise to a similar identity for its image by $\nabla$. In particular, from (2) follows $\nabla(s_{\lambda}) = \det(\nabla e_{\lambda \cup_{i+j-1} \lambda})_{1 \leq i,j \leq m}$. Moreover, still when $t = 1$, the coordinate $\nabla(e_n)|_e_{n}$ of $\nabla(e_n)$ on $e_n$ is a $q$-Catalan number $C_n$, with generating function $C(x)$ defined by $C(x) = 1 + zC(x)C(xq)$. Hence $D(\lambda) := \nabla(s_{\lambda})|_{e_n} = \det(C_{\lambda \cup_{i+j-1} \lambda})_{1 \leq i,j \leq m}$, and the idea of [1] is to use the Gessel-Viennot technique [3] to evaluate determinants of this type for various classes of partitions $\lambda$. Typical results are summarised in the following theorem.

**Theorem 1.**

$$
D((k^b)) = (-1)^{\binom{k}{2}} q^{\frac{\binom{k}{2}}{6}(k+1)}, \quad D((k^b+1)) = (-1)^{\binom{k+1}{2}} q^{\frac{\binom{k+1}{2}}{6}(k+1)},
$$

$$
D((k^2+2)) = (-1)^{\binom{k+2}{2}+1} q^{\frac{\binom{k+2}{2}+1}{6}k+1}, \quad D((k+1)^k) = (-1)^{\binom{k}{2}} q^{\frac{\binom{k}{2}}{6}k+1} (k+1),
$$

$$
D((k+2)^k) = (-1)^{\binom{k}{2}} q^{\frac{\binom{k}{2}}{6}(k+1)} \frac{[k+1][k+2][k+1+q][k+2]}{[2][3]},
$$

where $[k] = 1 + q + q^2 + \cdots + q^{k-1}$.

Another linear operator diagonal in the basis $\tilde{H}_\lambda$ is also studied in [1]. Similar techniques apply and results of a similar kind are obtained.

**Bibliography**

[1] Bergeron (François), Bousquet-Mézou (Mireille), and Gouyou-Beauchamps (Dominique). - Preprint, 1996.

[2] Bergeron (François), Garsia (Adriano), and Haiman (Mark). - New identities and conjectures for Macdonald’s $H_\lambda(x,q)$. - Preprint, 1995.
