Determinants, Catalan numbers and Macdonald's symmetric functions

Dominique Gouyou-Beauchamps
LRI, Orsay

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[summary by Bruno Salvy]

Abstract

A famous conjecture in the theory of symmetric functions states that the coefficients of Macdonald's polynomials in the basis of Schur's symmetric functions are positive. F. Bergeron, A. M. Garsia and M. Haiman have introduced a linear operator ∇ whose eigenvalues are related to Macdonald's polynomials. Properties of this operator in a special case are related to combinatorial determinants which can be evaluated by the Gessel-Viennot technique relating them to non-intersecting paths.

1. Introduction to symmetric functions

This section and the following one are based on [4].

Partitions and symmetric functions are strongly related. A *partition* is an infinite decreasing sequence of positive integers $\lambda = (\lambda_1, \lambda_2, \dots)$, with finitely many non-zero elements. The index of the last non-zero element in the partition is called its *length* and is denoted $\ell(\lambda)$; the sum of the λ_i's is called the *weight* of the partition and is denoted $|\lambda|$. For $n \geq \ell(\lambda)$, λ is identified with the n-tuple of its first elements. Then if $x = (x_1, \dots, x_n)$ is a n-tuple of indeterminates, x^λ denotes the monomial $x_1^{\lambda_1} \cdots x_n^{\lambda_n}$ and S_n^λ denotes a maximal set of distinct permutations of λ.

A fundamental basis of symmetric functions is constituted by the *monomial* symmetric functions, indexed by the partitions: for $n \geq \ell(\lambda)$,

$$m_\lambda(x_1, \dots, x_n) = \sum_{\sigma \in S_n^\lambda} x^{\sigma(\lambda)}.$$

Clearly, the set of m_λ's, when λ runs through all partitions of length at most n is a basis of the symmetric polynomials in n variables. The set Λ of symmetric functions is *defined* as the vector space generated by the m_λ's.

Three important sets of symmetric functions, $e_r = m_{(1^r)}$ (elementary), $h_r = \sum_{|\lambda|=r} m_\lambda$ (complete) and $p_r = m_{(r)}$ (power sum), have simple generating functions:

$$\begin{aligned}
E(t) &= \sum_{r\geq 0} e_r t^r = 1 + t\sum_i x_i + t^2 \sum_{i<j} x_i x_j + \cdots = \prod_{i>0}(1 + x_i t),\\
H(t) &= \sum_{r\geq 0} h_r t^r = 1 + t\sum_i x_i + t^2 \sum_{i\leq j} x_i x_j + \cdots = \prod_{i>0}\frac{1}{1 - x_i t},\\
P(t) &= \sum_{r\geq 0} p_r t^r = \sum_i x_i + t\sum_i x_i^2 + \cdots \qquad = \sum_{i>0}\frac{x_i}{1 - x_i t}.
\end{aligned}$$

Each of these three sets of symmetric functions generates Λ as a ring. In all three cases, defining for a partition λ a function $f_\lambda = f_{\lambda_1} f_{\lambda_2} \cdots$, where f is e, h or p yields a basis of Λ as a vector space, when λ runs through the set of partitions.

Formulæ giving the coordinates of one of these functions in terms of the other families are obtained by extracting the coefficient of t^n in the following straightforward relations between the generating functions:

$$(1) \qquad E(t)H(-t) = 1, \qquad P(t) = \frac{H'(t)}{H(t)}, \qquad P(-t) = \frac{E'(t)}{E(t)}.$$

The last two equations yield the classical Newton formulæ between power sums and elementary symmetric functions. Integrating these equation also yields

$$H(t) = \exp \sum_{r>0} p_r \frac{t^r}{r} = \sum_\lambda p_\lambda \frac{t^{|\lambda|}}{z_\lambda}, \qquad E(t) = \sum_\lambda (-1)^{|\lambda| - \ell(\lambda)} p_\lambda \frac{t^{|\lambda|}}{z_\lambda}, \qquad \text{with} \qquad z_\lambda = \prod_{i>0} i^{m_i} m_i!,$$

where m_i is the number of occurrences of the part i in λ.

Another family of symmetric functions, the *Schur functions*, is defined for $n \geq \ell(\lambda)$ by

$$s_\lambda(x_1, \ldots, x_n) = \frac{\det(x_i^{\lambda_j + n - j})_{1 \leq i,j \leq n}}{\det(x_i^{n-j})_{1 \leq i,j \leq n}}.$$

The s_λ's are indeed polynomials, since the numerator is a polynomial in the x_i's which vanishes whenever $x_i = x_j$ with $i \neq j$, and thus is a multiple of the Vandermonde determinant in the denominator. The s_λ form another basis of Λ. They are related to the complete and elementary symmetric functions by the Jacobi-Trudi identities:

$$(2) \qquad s_\lambda = \det(h_{\lambda_i - i + j})_{1 \leq i,j \leq n}, \qquad s_\lambda = \det(e_{\lambda'_i - i + j})_{1 \leq i,j \leq m},$$

where λ' is the *conjugate* of λ, i.e. the partition whose Ferrers diagram is the reflexion of that of λ with respect to the diagonal.

Recall that a *Young tableau* of shape λ is a Ferrers diagram of shape λ with squares numbered by consecutive positive integers $1, 2, \ldots, r$, the numbers increasing strictly in each column and weakly along each row. The *weight* $w(T)$ of a tableau T is the r-tuple $(m_1, \ldots, m_r)$, m_i counting the number of occurrences of i. The tableau is called *standard* when it contains each number $1, 2, \ldots, |\lambda|$ exactly once, i.e. its weight is $(1^{|\lambda|})$. The Schur functions are related to tableaux by

$$s_\lambda = \sum_T x^{w(T)},$$

summed over all tableaux T of shape λ. From this follows that the coordinates $K_{\lambda\mu}$ of s_λ in the basis m_μ are positive integers counting the number of tableaux of shape λ and weight μ and thus are positive integers. Macdonald's conjecture is a generalization of this property.

All these symmetric functions can also be related by expanding in several ways the doubly infinite product $P(x, y) = \prod (1 - x_i y_j)^{-1}$. Thus one gets

$$(3) \qquad \prod_{i,j} \frac{1}{1 - x_i y_j} = \sum_\lambda z_\lambda^{-1} p_\lambda(x) p_\lambda(y) = \sum_\lambda h_\lambda(x) m_\lambda(y) = \sum_\lambda m_\lambda(x) h_\lambda(y) = \sum_\lambda s_\lambda(x) s_\lambda(y).$$

This motivates the definition of a scalar product by $\langle h_\lambda, m_\mu \rangle = \delta_{\lambda\mu}$ for all partitions λ, μ, where $\delta_{\lambda\mu}$ is the Kronecker delta. The relations (3) show that the p_λ's form an orthogonal basis, while the s_λ's form an orthonormal basis of Λ. This property characterizes the Schur functions.

The next step is to consider the Hall-Littlewood symmetric functions with one parameter

$$P_\lambda(x_1,\ldots,x_n;t)=\sum_{\sigma\in S_n^\lambda}\sigma\left(x^\lambda\prod_{i<j}\frac{x_i-tx_j}{x_i-x_j}\right).$$

These functions interpolate between the monomial symmetric functions—obtained when $t=1$—and the Schur symmetric functions—obtained when $t=0$. They form a $\mathbb{Z}[t]$-basis of $\Lambda[t]$. Therefore, one may consider the polynomials $K_{\lambda\mu}(t)$ defined by

$$s_\lambda(x)=\sum_\mu K_{\lambda\mu}(t)P_\mu(x;t).$$

The polynomials $K_{\lambda\mu}(t)$ turn out to have positive coefficients, and this has been proved by Lascoux and Schützenberger who gave an expression of the form

$$K_{\lambda\mu}(t)=\sum_T t^{c(T)},$$

summed over all tableaux T of shape λ and weight μ, where $c(T)$ is a certain combinatorial function of the tableau (its *charge*). Several expansions of the product $P(x,y;t)=\prod_{i,j}(1-tx_iy_j)/(1-x_iy_j)$ lead to results very similar to those obtained above and to the definition of a scalar product on $\Lambda[t]$ with values in $\mathbb{Q}(t)$ with respect to which the $P_\lambda(x;t)$ are orthogonal. Also $\langle P_\lambda,m_\mu\rangle=0$ when $\mu\not\leq\lambda$ (the Ferrers diagram of μ is not included in that of λ), and together with their orthogonality this characterizes the P_λ. The basis which is dual to the Schur functions $s_\lambda(x)$ with respect to this scalar product is denoted $S_\lambda(x;t)$, i.e., $\langle S_\lambda(x;t),s_\mu(x)\rangle=\delta_{\lambda\mu}$.

2. Macdonald's conjecture

Macdonald's conjecture concerns the Macdonald symmetric functions, which have two parameters. The doubly infinite product

$$\Pi(x,y;q,t)=\prod_{i,j}\frac{(tx_iy_j;q)_\infty}{(x_iy_j;q)_\infty},\qquad\text{where}\qquad (a;q)_\infty=\prod_{r=0}^\infty(1-aq^r),$$

can be expanded as

$$\Pi(x,y;q,t)=\sum_\lambda\frac{1}{z_\lambda(q,t)}p_\lambda(x)p_\lambda(y),\qquad\text{with}\qquad z_\lambda(q,t)=z_\lambda\prod_{i=1}^{\ell(\lambda)}\frac{1-q^{\lambda_i}}{1-t^{\lambda_i}}.$$

This motivates the definition of a scalar product by

$$\langle p_\lambda,p_\mu\rangle_{q,t}=\delta_{\lambda\mu}z_\lambda(q,t).$$

The Macdonald symmetric functions are defined uniquely by two properties: they are orthogonal with respect to this scalar product and they decompose in the basis of the monomial symmetric functions as

$$P_\lambda(x;q,t)=m_\lambda+\sum_{\mu<\lambda}u_{\lambda\mu}m_\mu.$$

When $q=t$, they reduce to the Schur functions s_λ, and when $q=0$ to the Hall-Littlewood functions $P_\lambda(x;t)$.

For a partition λ and a cell $c=(i,j)$ of its Ferrers diagram, one defines the *arm* of c to be $a(c)=\lambda_i-j$ and its *leg* to be $l(c)=\lambda'_j-i$. Now we can state Macdonald's conjecture.

CONJECTURE 1 (MACDONALD). *The coefficients $K_{\lambda\mu}(q,t)$ of the following decomposition are polynomials with positive coefficients:*

$$(4)\ \tilde{H}_\lambda(x;t) := c_\lambda(q,t)P_\lambda(x;q,t) = \sum_\lambda K_{\lambda\mu}(q,t)S_\lambda(x;t), \quad \textit{where} \quad c_\lambda(q,t) = \prod_{c\in\lambda}(1-q^{a(c)}t^{l(c)+1}).$$

These coefficients possess a lot of structure. For instance, for $\lambda = (3,1)$, Eq. (4) becomes

$$\tilde{H}_{(3,1)} = S_{(4)} + (q^2+t+q)S_{(3,1)} + (t+q)qS_{(2,2)} + (tq+q^2+t)qS_{(2,1,1)} + tq^3S_{(1,1,1,1)}.$$

Only special cases of Macdonald's conjecture have been proved.

3. Combinatorial properties of ∇ when $t=1$

In order to study the polynomials $\tilde{H}_\lambda$, Bergeron, Garsia and Haiman have introduced a linear operator ∇ which is diagonal in the basis $\tilde{H}_\lambda$, with eigenvalues $T_\lambda(q,t) = q^{n(\lambda')}t^{n(\lambda)}$, where $n(\lambda) = \sum(i-1)\lambda_i$. The matrix of ∇ in the Schur basis turns out to have a fascinating structure of which much is still only conjectured [2].

The aim of [1] is to study this operator in more detail in the special case $t=1$. Then the basis $\tilde{H}_\lambda(x;q) := \tilde{H}_\lambda(x;q,1)$ becomes multiplicative: $\tilde{H}_\lambda(x;q) = \tilde{H}_{(\lambda_1)}(x;q)\tilde{H}_{(\lambda_2)}(x;q)\cdots$ and ∇ becomes multiplicative too. Thus any identity involving symmetric functions gives rise to a similar identity for its image by ∇. In particular, from (2) follows $\nabla(s_\lambda) = \det(\nabla e_{\lambda'_i+j-i})_{1\le i,j\le m}$. Moreover, still when $t=1$, the coordinate $\nabla(e_n)|_{e_n}$ of $\nabla(e_n)$ on e_n is a q-Catalan number C_n, with generating function $C(x)$ defined by $C(x) = 1 + xC(x)C(xq)$. Hence $D(\lambda) := \nabla(s_\lambda)|_{e_n} = \det(C_{\lambda'_i+j-i})_{1\le i,j\le m}$, and the idea of [1] is to use the Gessel-Viennot technique [3] to evaluate determinants of this type for various classes of partitions λ. Typical results are summarised in the following theorem.

THEOREM 1.

$$D((k^k)) = (-1)^{\binom{k}{2}}q^{\frac{k(k-1)(4k+1)}{6}}, \qquad D((k^{k+1})) = (-1)^{\binom{k+1}{2}}q^{\frac{k(k+1)(4k-1)}{6}},$$

$$D((k^{k+2})) = (-1)^{\binom{k+2}{2}+1}q^{\frac{k(k+1)(4k-1)}{6}+k^2}[k+1], \quad D((k+1)^k) = (-1)^{\binom{k}{2}}q^{\frac{k(k-1)(4k+7)}{6}}[k+1],$$

$$D((k+2)^k) = (-1)^{\binom{k}{2}}q^{\frac{k(k-1)(4k+13)}{6}}\frac{[k+1][k+2]([k+1]+q[k+2])}{[2][3]},$$

where $[k] = 1+q+q^2+\cdots+q^{k-1}$.

Another linear operator diagonal in the basis $\tilde{H}_\lambda$ is also studied in [1]. Similar techniques apply and results of a similar kind are obtained.

Bibliography

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