The (max,+) semiring. An introduction

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[summary by Marianne Akian]

Abstract

Endowing real (or natural) numbers with max and + laws leads to an idempotent semiring which has been reinvented in many domains: graph optimization, language theory, statistical physics, quantum mechanics, discrete event systems, etc. The talk presents applications together with basic results of the so-called (max,+) algebra.

Introduction

We say that $(\mathbb{S},\oplus,\otimes)$ is an idempotent semiring or dioid [19, 2] if $\oplus$ and $\otimes$ are associative laws on $\mathbb{S}$ with neutral elements $\mathbf{0}$ and $\mathbf{1}$ respectively, $\oplus$ is commutative and idempotent, that is $a \oplus a = a$, $\otimes$ is distributive with respect to the $\oplus$ law and $\mathbf{0}$ is absorbing with respect to the $\otimes$ law. By the idempotency property, $a \oplus b = \mathbf{0}$ implies $a = \mathbf{0}$. Then, the $\oplus$ law is not symmetrizable (and not simplifiable). However, idempotency leads to "simplifications" that partially compensate the non simplifiability. An idempotent semiring is said commutative when $\otimes$ is commutative and it is a semifield if the $\otimes$ law is invertible. Examples of commutative idempotent semifields are $\mathbb{R}_{\max} = (\mathbb{R} \cup \{-\infty\}, \max, +)$ with $\mathbf{0} = -\infty$ and $\mathbf{1} = 0$, $\mathbb{R}_{\min} = (\mathbb{R} \cup \{+\infty\}, \min, +)$, $(\mathbb{R}^+, \max, \times)$ which are isomorphic. They are called respectively $(\max, +)$, $(\min, +)$ and $(\max, \times)$ algebra and are used in operations research [7], graph theory [19], discrete event systems [2, 14, 13], dynamic programming, Hamilton-Jacobi-Bellman equations [28, 1, 8], exponential asymptotics [29, 23, 5, 4]. The subsemiring $\mathbb{N}_{\min} = (\mathbb{N} \cup \{+\infty\}, \min, +)$ of $\mathbb{R}_{\min}$, called tropical semiring, is used in language theory [21, 22, 33, 34, 25, 24]. Concerning theoretical results on $(\max, +)$ algebra, an historical reference is [7]. More recent accounts can be found in [2, 28], collections of survey papers will be presented in [20] and a general and complete bibliography can be found in [26].

1. Some applications

1.1. Shortest path problem. The traditional application of the $(\min, +)$ algebra concerns the shortest path problem in a graph [19]. Let G be a graph with nodes denoted $\{1, \ldots, n\}$ representing towns and arcs representing roads. Let A_{ij} denote the time to go from i to j (or the length of arc (i, j)) with $A_{ij} = +\infty$ when there is no arc. If $A = (A_{ij})$ is considered as a $(\min, +)$ matrix,

$$(A^k)_{ij} = \bigoplus_{i_1,\ldots,i_{k-1}} A_{ii_1} \otimes \cdots \otimes A_{i_{k-1}j} = \min_{i_1,\ldots,i_{k-1}} A_{ii_1} + \cdots + A_{i_{k-1}j}$$

represents the minimal time from i to j (or the minimal distance between i and j) in k steps. If $A^* = \oplus_{k=0}^{\infty} A^k$, then $(A^*)_{ij}$ represents the minimal time from i to j.

A similar problem arises in discrete deterministic optimal control. Let now A_{ij} represent the cost of i to j transition, b_i the final cost in state i at time N and let v_i^n denote the minimal cost of a trajectory starting in i at time $n \leq N$. The value function v^n satisfies the backward dynamic programming (or Hamilton-Jacobi-Bellman) equation

$$v_i^n = \min_j A_{ij} + v_j^{n+1}, \qquad v_i^N = b_i$$

that is $v^n = A \otimes v^{n+1}$ with $v^N = b$, which is the $(\min,+)$ analogue of the Kolmogorov or backward Fokker-Planck equation, (final or transition) costs replacing probabilities [1, 8]. More generally, dynamic programming equations with continuous time and state are solved using $(\min,+)$ algebra in [28, 23].

1.2. Synchronization problems. Let us consider a manufacturing system where 2 types of parts are assembled, taking a fixed duration τ. Let $u_i(t)$ denote the number of parts of type $i = 1,2$ arrived at time t and $y(t)$ the number of parts assembled. Then

$$y(t) = \min(u_1(t-\tau), u_2(t-\tau)) = u_1(t-\tau) \oplus u_2(t-\tau)$$

in the $(\min,+)$ algebra. If now $u_i(n)$ (resp. $y(n)$) denotes the date of the n-th arriving of part i (resp. of the n-th assemblage of parts), we obtain

$$y(n) = \tau + \max(u_1(n) + u_2(n)) = \tau \otimes (u_1(n) \oplus u_2(n))$$

in $(\max,+)$ algebra. More generally, any problem that can be modelled by a timed event graph (a subclass of timed Petri nets modelled synchronization features) can also be represented by a $(\min,+)$-linear dynamical system (for counter variables)

$$\begin{cases} x(t) = A \otimes x(t-1) \oplus B \otimes u(t), \\ y(t) = C \otimes x(t) \end{cases}$$

or by a $(\max,+)$-linear dynamical system (for dater variables $y(n)$, $x(n)$ and $u(n)$). A linear system theory in $(\min,+)$ and $(\max,+)$ algebras analogous to the classical linear control theory is developed in [2].

1.3. Exponential asymptotics. Let us consider a one-dimensional system of n atoms with energy $H_n(q_1,\dots,q_n) = V(q_1) + \sum_{k=2}^n K(q_{k-1}, q_k)$, where q_n is the position (state) of the n-th atom with $q_1 < \cdots < q_n$ and $K(q,q') = V(q') + W(q'-q)$ is the sum of the potential V in position q and the potential energy W linking nearest neighbours. The Gibbs distribution of this system has density $\exp(-\beta H_n(q_1,\dots,q_n))/Z_n$, where β is the inverse of the temperature and $Z_n = \sum_{q_1,\dots,q_n} \exp(-\beta H_n(q_1,\dots,q_n))$ is the partition function. Let T be the transfer matrix

$$T_{qq'} = \exp\left(-\beta K(q,q')\right),$$

Q be the row vector with entries $Q_q = \exp(-\beta V(q))$ and e the vector with entries 1. Then $Z_n = QT^{n-1}e$ and the probability for the first atom to be in position q is $P(q) = Q_q(T^{n-1}e)_q/Z_n$. For good matrices T, $P_n(q)$ tends to $P(q) = Q_q R_q$ when n goes to infinity, where R is a right eigenvector of the transfer matrix such that $Q \cdot R = 1$. Similarly, the probability of the n-th atom tends to L_q, where L is a left eigenvector of the transfer matrix such that $L \cdot e = 1$. Moreover, for any transfer matrix, $\log Z_n/n$ tends to $\log \rho$, where ρ is the Perron root of T. The free energy by atom is then $\lambda = \log \rho/\beta$.

If now the temperature is zero ($\beta = +\infty$), either the previous results have to be obtained passing to the limit in β using the property that the $(\min, +)$ algebra is the limit of the $(\mathbb{R}^+, +, \times)$ semifield:

$$\lim_{\beta \to +\infty} \frac{-1}{\beta} \log(e^{-\beta a} + e^{-\beta b}) = \min(a, b), \qquad \frac{-1}{\beta} \log(e^{-\beta a} \cdot e^{-\beta b}) = a + b;$$

or a similar reasoning has to be done directly in the $(\min, +)$ algebra. In this last case, the transfer matrix method is replaced by the effective potential method [5, 4]. Let us consider the $(\min, +)$-matrix K in place of T. The effective potential of the extremal atom of a semi-infinite chain of atoms extending to the right (resp. left) is equal to $F(q) = V(q) + R_q$ (resp. $F(q) = L_q$), where R and L are right and left $(\min, +)$-eigenvectors of K such that $\min_q V(q) + R_q = \min_q L_q = 0$. The energy by atom for a minimum-energy configuration is then the $(\min, +)$-eigenvalue λ of K: $K \otimes R = \lambda \otimes R = \lambda + R$, $L \otimes K = \lambda \otimes L = \lambda + L$. Exponential asymptotics also occur in large deviations and asymptotics of Schrödinger equations (WKB method) [29, 23].

1.4. Language theory. A finite automaton with cost or distance is an automaton with multiplicity over the tropical semiring $\mathbb{N}_{\min}$. For any rational language L over the finite alphabet Σ, a finite automaton with cost A can be constructed, recognizing $L^* = \cup_{n=0}^{\infty} L^n$ (where product means concatenation) and counting for each word $w \in L^*$ the least n such that $w \in L^n$. This has been used by Simon and Hashiguchi [21, 22, 33] to solve positively a long standing problem of J. A. Brzozowski, the decidability for a rational language of the finite power property (FPP) (a language L has the FPP iff there exists N such that $L^* = \cup_{n=0}^{N} L^n$). Indeed, the automaton A has only one initial state and one terminal state and since the language L has the FPP iff A is limited (that is costs of recognized words are bounded), the FPP is equivalent to the finite section property of a finitely generated subsemigroup of matrices of $\mathbb{N}_{\min}^{n \times n}$. Following this first application, other decidability properties for finitely generated subsemigroups of matrices over the tropical semiring and/or automata with cost have been studied [21, 22, 33, 34, 25, 24].

Similarly to cost automata, $(\max, +)$ automata can be also constructed. They allow to represent heaps of pieces and parallel (multitask, multiresource) discrete event systems [17, 16, 27].

2. $(\max, +)$ linear algebra

2.1. Solutions of linear equations and subsemimodules. Since the $\oplus$ law is not symmetrizable in a dioid, general linear equations are of the form $A \otimes x \oplus b = C \otimes x \oplus d$. Important particular cases are $A \otimes x = b$ and $x = A \otimes x \oplus b$. The following result is classical [7] and shows that the first particular equation is not easy to solve.

THEOREM 1. *$A \in \mathbb{R}_{\max}^{n \times n}$ is invertible iff $A = DS$, where D and S are diagonal and permutation matrices.*

THEOREM 2 ([30, 36]). *Any finitely generated subsemimodule of $\mathbb{R}_{\max}^n$ has a base (minimal generating family) which is unique up to invertible linear operations.*

THEOREM 3 ([3, 14]). *For any matrices $A, B \in \mathbb{R}_{\max}^{m \times n}$, the set of solutions of $A \otimes x = B \otimes x$ is a finitely generated semimodule.*

Let us solve $x = A \otimes x \oplus b$. To any dioid is associated a partial order: $a \preceq b \Leftrightarrow a \oplus b = b$. In $\mathbb{R}_{\max}$ it is the classical order $\leq$, in $\mathbb{R}_{\min}$ it is the opposite order $\geq$. The dioid $(\mathbb{S}, \oplus, \otimes)$ is complete if any set (even empty) has a least upper bound and if $\otimes$ is distributive with respect to infinite sums. $\mathbb{R}_{\max}$ is not complete but it may be completed in the complete dioid $\overline{\mathbb{R}}_{\max} = (\mathbb{R} \cup \{+\infty, -\infty\}, \max, +)$ with the convention $+\infty + -\infty = -\infty$ ($\mathbf{0}$ is absorbing).

THEOREM 4. *In a complete dioid* $\mathbb{S}$, *the least solution of* $x = a \otimes x \oplus b$ *is* $a^* \otimes b$, *where* $a^* = \oplus_{n\in\mathbb{N}} a^n = \sup_{n\in\mathbb{N}} a^n$. *Similarly, the least solution of* $x = A \otimes x \oplus b$ *in* $\mathbb{S}^n$ *is* $x = A^* b$. *It can be computed by Gauss algorithm.*

In order to solve the general equation $A \otimes x \oplus b = C \otimes x \oplus d$, a symmetrization of $\mathbb{R}_{\max}$ seems necessary. Although no idempotent field or ring containing $\mathbb{R}_{\max}$ exists, a symmetrized idempotent semiring $\mathbb{S}_{\max}$ has been constructed. It contains positive numbers $x \in \mathbb{R}_{\max}$, negative numbers $\ominus x$, but also doted numbers $\dot{x} = x \ominus x$ which are not invertible. Symmetrizing linear equations in $\mathbb{R}_{\max}$, we obtain balance equations in $\mathbb{S}_{\max}$, where x balances y iff $x \ominus y$ is doted. In $\mathbb{S}_{\max}$, determinants can be calculated and linear balance equations can be solved using Cramer formula or Gauss-Seidel and Jacobi algorithms [2, 14, 31].

2.2. Subsolutions of linear equations: residuation.

DEFINITION 1. Let $f : (E, \leq) \to (F, \leq)$ be a nondecreasing application between lattices. f is residuable iff $\{x \in E, f(x) \leq b\}$ has a maximal element for any $b \in F$.

THEOREM 5. *If* $f : \mathbb{S} \to \mathbb{S}'$ *is an application between complete dioids such that* $f(\mathbf{0}) = \mathbf{0}$ *and* $f(\sup_{x\in X} x) = \sup_{x\in X} f(x)$ *for any subset* X *of* $\mathbb{S}$, *then* f *is residuable.*

As a corollary, any multiplication operation (by a scalar or a matrix) is residuable. Let us denote by $a\backslash b = \max\{x, a \otimes x \preceq b\}$ and $b/a = \max\{x, x \otimes a \preceq b\}$ the residuations of multiplications by the scalar a in any complete dioid. The residuation of the multiplication by a matrix in $\mathbb{R}_{\max}$, $A\backslash b = \max\{x \in \mathbb{R}^n_{\max}, A \otimes x \preceq b\}$ gives the vector with entries $(A\backslash b)_i = \inf_j A_{ji}\backslash b_j = \min_j -A_{ji} + b_j$, that is the $\mathbb{R}_{\min}$ product of the matrix $-A^T$ by b. Applications to system theory can be found in [2]. While linear operators represent the earliest behaviour of a system, the latest behaviour can be represented by a dynamical equation involving residuation.

2.3. Spectral theory. The most useful result of $(\max, +)$ linear algebra is perhaps the following analogue of Perron-Frobenius theorem.

THEOREM 6 ([7, 35, 32, 18, 6, 10]). *Any irreducible matrix* $A \in \mathbb{R}^{n\times n}_{\max}$ *has a unique eigenvalue* $\rho(A)$ *and*

$$\rho(A) = \oplus_{k=1}^n \operatorname{tr}(A^k)^{\frac{1}{k}} = \max_{k=1,\ldots,n} \max_{i_1,\ldots,i_k} \frac{A_{i_1 i_2} + \cdots + A_{i_k i_1}}{k}$$

If A *is reducible, the previous formula gives the maximal eigenvalue.*

The $(\min, +)$ eigenvalue is then the minimal mean cost (ergodic cost) of a control problem or the asymptotic production rate of a manufacturing system, etc. As in the statistical physics application of section 1.3, it can be obtained as the limit of the Perron root of a matrix.

THEOREM 7 ([12, 11]). *Let* A *be any* $n \times n$ *matrix with entries in* $\mathbb{R}^+$. *If* $\rho_{PF}(A)$ *is the Perron-Frobenius root of* A *and* $\rho_{(\max,\times)}(A) = \exp(\rho((\log A_{ij})))$ *its* $(\max, \times)$*-eigenvalue, we have*

$$\rho_{(\max,\times)}(A) \leq \rho_{PF}(A) \leq n\rho_{(\max,\times)}(A).$$

COROLLARY 1. *Let* $A^{\circ r} = (A^r_{ij})$ *and* $e^{\circ\beta A} = (\exp(\beta A_{ij})$ *denote the* r*-th power of* A *and the exponential of* βA *for the Hadamard product. For any matrix with positive entries*

$$\rho_{(\max,\times)}(A) = \lim_{r\to+\infty} (\rho_{PF}(A^{\circ r}))^{\frac{1}{r}}$$

and for any matrix with entries in $\mathbb{R}_{\max}$

$$\rho(A) = \lim_{\beta \to +\infty} \frac{1}{\beta} \log \rho_{PF}(e^{\circ \beta A}).$$

THEOREM 8 ([6, 9]). *For any irreducible matrix* $A \in \mathbb{R}_{\max}^{n \times n}$*, there exists* c *and* $N \geq 1$ *such that* $A^{n+c} = \rho(A)^c A^n$ *for* $n \geq N$.

In the context of timed event graphs, this means that the system reaches after a finite transient behaviour (of length N) a periodic regime of period c in which the production rate is equal to the eigenvalue.

These periodicity results can also be dealt with using rational generating series over the $(\max, +)$ semiring [2, 15].

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