# An urn model from learning theory

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[summary by Stéphane Boucheron]

**Abstract**

The analysis of a learning problem motivates the definition of an urn model. In this model, two kinds of balls representing bad and good data are allocated at random in a collection of urns. This is a variation on the classical occupancy model where one is concerned with allocation of one kind of balls in a family of urns. In this model, the relevant quantities are the number of urns that contain more bad than good balls or as many good as bad balls. We describe the law of those two quantities in the static and dynamic framework. The investigation rely both on complex analysis techniques (generating functions) and probabilistic tools (exchangeability, and finite De Finetti theorems). Using proper normalization, the limiting phenomena are Gaussian random variables. Most interesting is the fact that the moments of the laws are described using modified Bessel functions.

## 1. The modified urn problem

The modified urn problem was initially motivated by the analysis of the learning curve of symmetric functions under classification noise in the field of computational learning theory [6]. As in the classical random allocation problem, $k$ balls are thrown at random into those $n$ urns. Balls are allocated independently, and the probability to fall into some urn is $1/n$. But here, balls are not only allocated, they are also labelled independently at random as good (with probability $1-\mu > 1/2$) or bad. The balance of one urn is the difference between the number of good balls and the number of bad balls in that urn. All the issues tackled in this investigation have the following flavor: what is the law of linear combinations of the numbers of urns with positive, negative and null balances? This question can be answered in a static context, where $k/n$ remains equal to a positive constant $\alpha$ when $n$ tends to infinity, or in a dynamic context, where urns are allocated one at a time, and where we try to monitor the evolution of the fraction of urns with positive, negative and null balances at different normalized times $\alpha_1, \ldots, \alpha_i, \ldots$ with $\alpha_i = k_i/n$.

The goal of this analysis is to extend results stated in [7] on the empty urn problem. The empty urn problem can be treated by diffusion approximation techniques, or, using implicitly the Markov property, by generating functions. The problem examined here does not share this property. Moreover, the plausible enhancements of the state space that would make the fraction of urns with positive balance a function of a Markov chain, lead to consider processes which take values in infinite-dimensional spaces. The analysis presented in [1] relies both on generating functions and simple principles.

## 2. Generating functions

The generating functions manipulated here are of exponential type.

**2.1. Generating function describing the behavior of one urn.** Let $y$ mark the number of balls in that urn. Because balls are indistinguishable, the generating function describing the number of ways of allocating balls in one urn is $e^y$. To reflect the fact that balls are of two kinds, this is rewritten as $e^{\mu y+(1-\mu)/y}$. Using a second variable $z$ and expanding $e^{y(\mu z+(1-\mu)/z)}$, one notes that the coefficient of $y^k z^p$ is proportional to the probability that the urn has balance $p$ when $k$ balls are thrown into it. We get:

$$e^{y((1-\mu)z+\mu/z)} = \sum_{p\in\mathbb{Z}} a_p(y) z^p.$$

The exponent of $z$ is the balance of the urn. This expression stresses the importance of Bessel functions. Modified Bessel functions of the first kind at order $p \in \mathbb{Z}$ can be defined by:

$$I_p(x) = \sum_{r\geq \max(0,-p)} \frac{(x/2)^{2r+p}}{r!(r+p)!}.$$

Bessel functions obey the following identity: $e^{\frac{y}{2}(u+\frac{1}{u})} = \sum_{p\in\mathbb{Z}} u^p I_p(y)$. Then letting $\sigma = \sqrt{\mu(1-\mu)}$, $e^{y((1-\mu)z+\mu/z)} = e^{\frac{2\sigma y}{2}(\sigma z/\mu+\frac{1}{\sigma z/\mu})} = \sum_{p\in\mathbb{Z}} \left(\frac{\sigma z}{\mu}\right)^p I_p(2\sigma y)$. Marking urns with positive balance by $w$, null balance by $v$ and negative balance by $u$ , and letting

$$\phi(y) = \sum_{p<0} \left(\frac{\sigma}{\mu}\right)^p I_p(2\sigma y) = \sum_{p>0} \left(\frac{\mu}{\sigma}\right)^p I_p(2\sigma y); \qquad \psi(y) = \sum_{p>0} \left(\frac{\sigma}{\mu}\right)^p I_p(2\sigma y) = e^y - I_0(2\sigma y) - \phi(y),$$

the generating function describing the sign of the balance in one urn is:

$$f(u,v,w,y) = u\phi(y) + vI_0(2\sigma y) + w\psi(y).$$

**2.2. Generating function for a sequence of urns.** Because urns are exchangeable, the generating function describing the states of a sequence of $n$ urns is:

$$F(u,v,w,y) = f(u,v,w,y)^n = \left(u\phi(y) + vI_0(2\sigma y) + w\psi(y)\right)^n.$$

## 3. Exchangeability

The balances of different urns follow identical, non-independent but *exchangeable* laws: all permutations of a tuple of balances indexed by different urns have the same probability. Recall that the variation distance between two laws $D$ and $D'$ is defined by:

$$\|D - D'\|_{\text{var}} = \max_{\|\text{Test}\|_\infty \leq 1} |\,\mathrm{E}_D(\text{Test}) - \mathrm{E}_{D'}(\text{Test})|.$$

The following lemma shows that small sets of urns behave almost independently. Let $P_i$ be the law of a tuple of $i$ independent random variables that are distributed as the difference between two independent Poisson random variables with means $\mu k/n$ and $(1-\mu)k/n$.

Proposition 1. *The vector of balances in urns* 1 *to* $i$ *after throwing* $k$ *balls in* $n$ *urns is distributed according to a law* $Q_i$ *that is within variation distance* $2i/n$ *from* $P_i$.

The proof relies on the fact that conditionally on the number of balls allocated in urns $1,\ldots,i$, the balances of the $i$ urns are independent and on theorem (5.1) in [2].

## 4. Static analysis

The cost of an experiment (throwing $k$ balls into $n$ urns) is the sum of the costs of the urns. The cost of an urn with null (resp. negative, positive) balance is $C_0$ (resp. $C_1$, $C_2$). We let $d_0 = C_0 - C_2$ and $d_1 = C_1 - C_0$. For costs relevant to learning theory applications, we have $d_0 = d_1$.

Using either the generating function approach or Proposition 1, one may derive the following equivalents for the expectation and variance of the cost:

$$\mathrm{E}(\text{cost}) \sim n\left[C_2 + d_1 e^{-\alpha}\mathrm{I}_0(2\sigma\alpha) + (d_0+d_1)e^{-\alpha}\phi(\alpha)\right];$$

$$\mathrm{Var}(\text{cost}) \sim n d_1^2 e^{-\alpha}\left[4\phi(\alpha) + I_0(2\sigma\alpha) - e^{-\alpha}\left(\big(2\phi(\alpha)+I_0(2\sigma\alpha)\big)^2 + \alpha\big((1-2\mu)I_0(2\sigma\alpha)\big)^2\right)\right].$$

Using the generating function approach and a theorem in [4], one may also conclude that the normalized and centered variable defined by: $(\text{cost} - \mathrm{E}(\text{cost}))/\sqrt{n}$ is asymptotically Gaussian with variance $\mathrm{Var}(\text{cost})/n$.

## 5. Dynamic analysis

In the dynamic context, balls are allocated one at a time. If balls are allocated in $n$ urns, the $k = \alpha n$th ball is allocated at time $\alpha$. The cost is a random function of time. The average function when $n \to \infty$ is given by the above-stated expression for the average cost. The aim of this investigation is to characterize the limiting behavior of the normalized centered processes. To prove (weak) convergence of the processes to a limiting process, one needs to check that the sequence of processes is relatively compact, and that the finite dimensional distributions of the processes converge to the finite dimensional distributions of the limiting process.

Finite dimensional distributions are analyzed using both multivariate generating functions and elementary arguments building on exchangeability of urns.

Balls are assumed to be thrown in two groups. The first group is marked by $y_1$ and thrown at time $\alpha_1$; we use variable $z_1$ to distinguish good balls from bad balls, (a good ball is marked as $(1-\mu)y_1 z_1$ and a bad ball as $\mu y_1/z_1$). Similarly, the second group is thrown at time $\alpha_2$ and marked by $y_2$ and $z_2$. Variables $u_i$, $v_i$ and $w_i$ indicate the state of the urn after throwing the first group ($i=1$) and the second group ($i=2$).

Letting

$$\begin{aligned}
\Delta I(y_1,y_2) &= \sum_{n>0} I_n(2\sigma y_1)I_{-n}(2\sigma y_2) = [I_0(2\sigma(y_1+y_2)) - I_0(2\sigma y_1)I_0(2\sigma y_2)]/2,\\
S(y_1,y_2) &= \sum_{n>0,n+p>0} I_n(2\sigma y_1)I_p(2\sigma y_2)\left(\frac{\sigma}{\mu}\right)^{n+p},\\
T(y_1,y_2) &= \psi(y_1+y_2) - S(y_1,y_2) - I_0(y_1)\psi(y_2),
\end{aligned}$$

the following is derived:

Proposition 2. *The multivariate generating function describing the behavior of a single urn at the times $\alpha_1$ and $\alpha_2$ is*

$$\begin{aligned}
& w_1w_2S(y_1,y_2) + w_1v_2\Delta I(y_1,y_2) + w_1u_2(\psi(y_1)e^{y_2} - S(y_1,y_2) - \Delta I(y_1,y_2))\\
&\quad + v_1w_2I_0(2\sigma y_1)\psi(y_2) + v_1v_2I_0(2\sigma y_1)I_0(2\sigma y_2) + v_1u_2I_0(2\sigma y_1)\phi(y_2)\\
&\quad + u_1w_2T(y_1,y_2) + u_1v_2\Delta I(y_1,y_2) + u_1u_2(\phi(y_1)e^{y_2} - T(y_1,y_2) - \Delta I(y_1,y_2)).
\end{aligned}$$

The single-urn generating function is used to compute the generating function of a sequence of urns at different instants. Then the limiting value of the characteristic function of the cost at a finite number of instants can be computed using saddle-point approximation methods as in [4, 5].

This allows to conclude that the finite dimensional distributions of the centered normalized processes converge to the finite dimensional distributions of a non-Markov Gaussian process with covariance between times $\alpha_1$ and $\alpha_2$:

$$d_0^2 e^{-\alpha_2}\Bigg(\Big(\mathrm{I}_0(2\sigma\alpha_2) + 2\mathrm{I}_0(2\sigma\alpha_1)\phi(\alpha_2-\alpha_1) + 4\sum_{i>0,j>0}\left(\frac{\mu}{\sigma}\right)^j \mathrm{I}_i(2\sigma\alpha_1)\mathrm{I}_{j-i}(2\sigma(\alpha_2-\alpha_1))\Big)$$
$$- e^{-\alpha_1}\Big(\big(\mathrm{I}_0(2\sigma\alpha_1)+2\phi(\alpha_1)\big)\big(\mathrm{I}_0(2\sigma\alpha_2)+2\phi(\alpha_2)\big) + \alpha_1(1-2\mu)^2\mathrm{I}_0(2\sigma\alpha_1)\mathrm{I}_0(2\sigma\alpha_2)\Big)\Bigg).$$

Proving the weak convergence of the processes to the above-stated Gaussian process requires the proof of the relative compactness of the sequence of processes. This has not been done although the verification of the Kolmogorov-Centsov criterion raises more cumbersome computations than theoretical difficulties.

## 6. Questions

A plausible contribution of [1] is the presentation of a new kind of admissible construction: the majority phenomenon that comes from building a combinatorial structure on two types of objects (good and bad in this paper), then deciding on the type of the structure according to the type of the majority of the basic objects. For example, we can have two types of basic objects, build cycles on theses objects and combine these cycles into a set, then ask for the number of cycles of the set that have a majority of elements of one type, or an equal number of elements of each type. It should be possible to extend the distribution results on the number of components presented by Flajolet and Soria [3] to study the number of components of a given type (good, bad or neutral) for various combinatorial constructs.

## Bibliography


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