A Zero-One Law for Maps

Kevin Compton
University of Michigan, Ann Arbor, U.S.A.

June 10, 1996

[summary by Frédéric Chyzak]

Abstract

A class of structures has a 0–1 law when any property expressible in a certain logic has limiting probability 0 or 1 as the size of the structures tends to infinity. We prove 0–1 laws for classes of maps of a given genus. This is a joint work with E. Bender and B. Richmond [1].

1. Definition of the problem

Let S be a set of primitive elements called *sorts*. A *vocabulary* Σ consists of a collection of constant and relation symbols, together with a mapping from each constant symbol to a sort, and a mapping from each relation symbol to a sequence of sorts, the *arity* of the relation (see [4] for an introduction to model theory). A *multi-sorted structure* $\mathcal{A}$ over Σ then consists of

- a collection of disjoints sets (or *universes*) A_s, one for each sort s;
- elements $c^{\mathcal{A}} \in A_s$, one for each constant symbol c of sort s;
- relations $R^{\mathcal{A}} \subset A_{s_1} \times \cdots \times A_{s_p}$, one for each relation symbol R of arity $(s_1, \ldots, s_p)$.

A *class* of structures is a set of structures defined on the same vocabulary. In the study of random structures, one says that a class of finite structures *has a 0–1 law* when any property expressible in a certain logic has limiting probability 0 or 1 as the size of the structures tends to infinity. The *relational signature* of a class of structures over Σ is the common set of relation symbols in the vocabulary Σ, together with their arities. A famous theorem by Glebskiĭ, Kogan, Liogon'kiĭ and Talanov [9], and proved independently by Fagin [7], states that if $\mathcal{C}$ is the class of all structures for a given relational signature, then $\mathcal{C}$ has a first-order 0–1 law. However, deciding the limiting probability of a given property is a difficult problem, as formalized by a theorem by Grandjean: when a class $\mathcal{C}$ has a 0–1 law, the set of first-order sentences of limiting probability 1 is PSPACE-complete.

A *map* $\mathcal{M}$ is an embedding of a connected graph $\mathcal{G}$ into a closed surface $\mathcal{S}$ such that all connected components of $\mathcal{S} \setminus \mathcal{G}$, the *faces* of $\mathcal{M}$, are homeomorphic to a disc. Let $t = 1 - (v - e + f)/2$ be the *genus* of $\mathcal{M}$, with v, e and f its number of vertices, edges and faces respectively. When t is an integer, the map is called *orientable*. The *size* $|\mathcal{M}|$ of a map is e. The purpose of this exposition is to provide similar results to the theorems mentioned above for maps, even in the non-orientable case. Our main result is the following theorem [1].

THEOREM 1. *The class of all maps on surfaces of fixed genus has a 0–1 law. The set of first-order sentences of limiting probability 1 for this class has lower bound complexity of* $\mathrm{DTIME}(\exp_\infty(cn))$, *for some* $c > 0$.

(Recall that $\exp_\infty(n) = 2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}$, with n nested exponentiations.)

The 0–1 law theorem for structures cannot be applied to maps, since the latter do not form a full class of structures of a given relational since. Besides, we have to explain how maps can be represented as structures.

2. Representation of maps as structures

Any naive attempt of representing a map $\mathcal{M}$ on a surface $\mathcal{S}$ by its graph, i.e., by its set of edges, is bound to fail. Indeed, this representation would not encapsulate any information about the embedding of $\mathcal{M}$ into $\mathcal{S}$: easy examples show that isomorphic graphs need not correspond to homeomorphic maps, and that the order of edges around a vertex has to be taken into account. However, on a non-orientable surface there is no consistent way to choose an edge order around each vertex.

A solution stems from an idea of Edmonds [5], later elaborated by Tutte [10] as a basis for a combinatorial theory of maps: to each edge, one associates a pair of *darts*, pointing in opposite directions. On orientable surfaces, a possible representation of maps is then given by an involution α on the set of darts, mapping a dart to its opposite dart, together with a permutation β whose cycles consist of all darts out of a vertex, listed clockwise. Then, $\alpha\beta$ is a permutation whose cycles consist of all darts around a face, listed counter-clockwise. One is thus able to determine the embedding using α and β. In the context of possibly non-orientable surfaces, a map is analogously described as a structure by the sets U_v, U_d and U_f of its vertices, darts and faces, together with incidence relations $I(x_v, x_d)$ and $J(x_f, x_d)$ of darts with vertices and faces, a co-dart relation $C(x_d, x_{d'})$ and a dart adjacency relation $A(x_d, x_{d'}, x_f)$. The co-dart relation is an analogue for α, while the dart adjacency relation encapsulates the information formerly supplied by β, specifying a face to supply the orientable information.

3. Ehrenfeucht-Fraïssé games

The 0–1 law theorem for structures still does not apply to maps: not all structures of signature (I, J, C, A) are maps. We overcome this difficulty in the case of the class of all maps on surfaces of a fixed genus by determining subclasses of limiting probability 1.

The sentences of first-order logic under consideration for our 0–1 laws can all be written in the form $S = \theta_1 x_1 \ldots \theta_r x_r \phi(x_1, \ldots, x_r)$, where the θ_i's are quantifiers, either $\forall$ or $\exists$, the x_i's are variables and ϕ is a boolean expression free from quantifiers built on the x_i's using conjunctions and disjunctions. The *rank* of the sentence S is the integer r. Let $\mathcal{A}$ and $\mathcal{B}$ be two structures with same relational signature. We write $\mathcal{A} \equiv_m \mathcal{B}$ when both structures satisfy exactly the same sentences of rank m. This defines an equivalence relation between structures. The next paragraph describes this equivalence relation by a game-theoretic approach.

The Ehrenfeucht-Fraïssé game is an m-round game between two players called *Spoiler* and *Duplicator* and played on a pair of structures $\mathcal{A}$ and $\mathcal{B}$ of same relational signature. In each round, Spoiler picks any element from either structure and Duplicator responds by picking any element from the other structure. This yields two substructures $\mathcal{A}' = \{a_1, \ldots, a_m\} \subset \mathcal{A}$ and $\mathcal{B}' = \{b_1, \ldots, b_m\} \subset \mathcal{B}$, with relations induced in a natural way. Duplicator wins if he is able to choose his responses so as to make $\mathcal{A}'$ and $\mathcal{B}'$ isomorphic; if not, Spoiler wins. Duplicator *has a winning strategy* if and only if he is capable of winning for any choices made by Spoiler. A fundamental result used in the sequel is the Ehrenfeucht-Fraïssé theorem [6, 8] which states that Duplicator has a winning strategy in the m-round first-order game played on two structures $\mathcal{A}$ and $\mathcal{B}$ if and only if $\mathcal{A} \equiv_m \mathcal{B}$.

Now, the relation $\equiv_m$ defines a finite number of (possibly infinite) equivalence classes on the ambient class. It can be proved that one of these classes has limiting probability 1, and this suffices to prove our theorem. For the sake of clarity, we present the idea of the proof on a simplified example only.

4. A 0–1 law by a 3^{r-k} strategy for a simplified problem

For this example, the class of structures under consideration is the set of square toroidal grids with a unary relation (we simply tag some vertices). We play r-round Ehrenfeucht-Fraïssé games on pairs of grids. The crucial fact we use is that any fixed square plane grid with vertices tagged at random appears in a toroidal grid with limiting probability 1.

It follows that Duplicator has a strategy to win *almost surely*, i.e., with limiting probability 1. Define a *distance* between two vertices of a grid by the minimum number of edges in a connecting path. The *ball* $N(c_1, \ldots, c_p; d)$ is the set of vertices at distance at most d from any c_i. Let $\mathcal{A}$ and $\mathcal{B}$ be two structures. Assume we are in round $k+1$ and that $a_1, \ldots, a_k$ have already been picked out of $\mathcal{A}$, $b_1, \ldots, b_k$ out of $\mathcal{B}$ in a way such that $N(a_1, \ldots, a_k; 3^{r-k})$ and $N(b_1, \ldots, b_k; 3^{r-k})$ are isomorphic, when viewed as substructures with naturally induced relations. Now, Spoiler picks an element out of either structure, say a_{k+1} out of $\mathcal{A}$—the case b_{k+1} out of $\mathcal{B}$ is symmetric. If $N(a_1, \ldots, a_{k+1}; 3^{r-k-1}) \subset N(a_1, \ldots, a_k; 3^{r-k})$, then Duplicator can trivially choose b_{k+1} in $N(b_1, \ldots, b_k; 3^{r-k})$ so that $N(a_1, \ldots, a_{k+1}; 3^{r-k-1})$ and $N(b_1, \ldots, b_{k+1}; 3^{r-k-1})$ are isomorphic. Otherwise, there is almost surely a ball in the complement of $N(b_1, \ldots, b_k; 3^{r-k-1})$ in $\mathcal{B}$ which is isomorphic to $N(a_{k+1}; 3^{r-k-1})$. Duplicator then chooses b_{k+1} to be its center. After r rounds, the balls $N(a_1, \ldots, a_r; 1)$ and $N(b_1, \ldots, b_r; 1)$ are almost surely isomorphic. Thus, Duplicator wins almost surely by following the strategy that we have just described. By the Ehrenfeucht-Fraïssé theorem, $\mathcal{A} \equiv_r \mathcal{B}$ almost surely. Therefore, one of the (finitely many) equivalence classes of $\equiv_r$ has limiting probability 1. Call it $\mathcal{E}_r$.

Consider now a first-order sentence S of rank r on toroidal grids. By the Ehrenfeucht-Fraïssé theorem, the set of all grids satisfying S is either contained in $\mathcal{E}_r$, or disjoint from $\mathcal{E}_r$. In the former case S has limiting probability 1, in the latter 0. We have thus proved a 0–1 law for the class of toroidal grids with a unary relation.

5. A 0–1 law for maps of a given genus

We first recall two difficult results on maps.

The first result [2, Sec. 5] plays the rôle of the crucial fact we used in the previous section, namely the limiting probability 1 of the appearance of a fixed plane grid in a toroidal grid. It states that for a class $\mathcal{C}$ of maps of fixed genus, there is a $c > 0$ such that for any given planar map $\mathcal{P}$, the property for maps in $\mathcal{C}$ to contain more than cn disjoint copies of $\mathcal{P}$ has limiting probability 1.

The second result [3] is about *representativity* of maps. The representativity of a map $\mathcal{M}$ on a surface $\mathcal{S}$ is the smallest number of intersections a non-contractible curve in $\mathcal{S}$ has with $\mathcal{M}$. The result is that for a class of maps of fixed genus, there is a $c > 0$ such that the property for maps to have representativity more than $c \ln n$ has limiting probability 1. This result is used in the proof of Theorem 1 to ensure the planarity of certain submaps built on balls playing a rôle similar to the $N(a_1, \ldots, a_k; 3^{r-k})$ of the previous section.

Next, the proof of Theorem 1 runs as for the example of the previous section: we prove a first-order 0–1 law for the class of all maps of a given genus by showing that for each r, Duplicator has an almost surely winning strategy in r-round Ehrenfeucht-Fraïssé games. More specifically, this strategy is a 3^{r-k} strategy using balls around elements picked by Spoiler and Duplicator. However, the notion of distance used is not that of the previous section. The proper distance to prove the result is by means of *quadrangulations* of maps. For a given map $\mathcal{M}$ on a surface, add a new point on each edge and a point in each face. Next add new edges from the new points on the edges to the new points in the faces. The quadrangulation of $\mathcal{M}$ is then the new map on the same surface built in this way. This construction induces a natural mapping from a map $\mathcal{M}$ to its quadrangulation $\mathcal{Q}$. We extend this map to the dart representation of $\mathcal{M}$ by mapping both co-darts defined by an edge to the image of this edge in $\mathcal{Q}$. A distance is then defined on the set $U_v \cup U_d \cup U_f$ of all vertices, darts and faces of the dart representation, as the distance between the images in $\mathcal{Q}$. This distance is not a metric, since two co-darts are at distance 0 for each other. However, the concept of balls it induces is sufficient for the proof of Theorem 1.

6. Conclusions

Theorem 1 has been refined for several classes of maps on a surface of fixed genus [1] (see this reference for missing definitions): the class of all maps; the class of smooth maps; the class of k-connected maps where k is 2 or 3; the class of k-connected triangulations where k is 1, 2 or 3. However, the question of a 0–1 law for planar graphs remains open, though we believe it should be true.

As for complexity results, we proved an $\exp_\infty(cn)$ *lower* bound for the complexity of the set of first-order sentences of limiting probability 1 in the case of the dart representation. Another result holds for an *extended* dart representation (see [1] for the definition): in this extended representation, we proved undecidability. What we have not been able to prove is an *upper* bound in the case of the dart representation, though we feel $\exp_\infty(dn)$ is a good candidate for such an upper bound.

Finally, all results presented here concern sentences of first-order logic. An extension to other logics seems reasonable, in particular to MSO (monadic second-order) logic, with application to the theory of databases.

Bibliography

[1] Bender (Edward A.), Compton (Kevin J.), and Richmond (L. Bruce). – Zero-one laws for maps. – In preparation, 1996.

[2] Bender (Edward A.), Gao (Zhi-Cheng), McCuaig (William D.), and Richmond (L. Bruce). – Submaps of maps I: General 0-1 laws. *Journal of Combinatorial Theory*, vol. 55, n° B, 1992, pp. 104–117.

[3] Bender (Edward A.), Gao (Zhi-Cheng), and Richmond (L. Bruce). – Almost all rooted maps have large representativity. *Journal of Graph Theory*, vol. 18, 1994, pp. 545–555.

[4] Chang (Chen Chung) and Keisler (H. Jerome). – *Model theory*. – North-Holland, Amsterdam, 1990, third edition, *Studies in Logic and the Foundations of Mathematics*, vol. 73.

[5] Edmonds (Jack R.). – A combinatorial representation for polyhedral surfaces. *Notices of the American Mathematical Society*, vol. 7, 1960, p. 646.

[6] Ehrenfeucht (A.). – An application of games to the completeness problem for formalized theories. *Fundamenta Mathematicae*, vol. 49, 1961, pp. 129–141.

[7] Fagin (Ronald). – Probabilities on finite models. *Journal of Symbolic Logic*, vol. 41, 1976, pp. 50–58.

[8] Fraïssé (Roland). – *Sur quelques classifications des systèmes de relations*. – Technical report, Université d'Alger, 1954. English summary.

[9] Glebskiĭ (Y. V.), Kogan (D. I.), Liogon'kiĭ (M. I.), and Talanov (V. A.). – Range and degree of realizability of formulas in the restricted predicate calculus. *Cybernetics*, vol. 5, 1969, pp. 142–154. – English translation.

[10] Tutte (William T.). – Combinatorial oriented maps. *Canadian Journal of Mathematics*, vol. 31, 1979, pp. 986–1004.