∂-finite functions

Frédéric Chyzak
INRIA Rocquencourt

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[summary by Bruno Salvy]

Abstract

The algebra of ∂-finite functions and sequences enjoys several closure properties useful when computing a description suitable for creative telescoping. A simple description of ∂-finiteness can be given in the context of Ore algebras. In the special case of the Weyl algebra, a special property called holonomy plays a crucial role.

We consider an Ore algebra $\mathcal{A} = \mathbb{K}(x_1, \ldots, x_n)\langle\partial_1, \ldots, \partial_k\rangle$ (see previous summary). A function is ∂-finite with respect to $\mathcal{A}$ when its pseudo-derivatives $\boldsymbol{\partial}^{\boldsymbol{\alpha}} f = \partial_1^{\alpha_1} \cdots \partial_k^{\alpha_k} f$ with $\alpha_i \in \mathbb{N}$ for $i = 1, \ldots, k$ span a finite-dimensional vector space over $\mathbb{K}(x_1, \ldots, x_n)$. Examples of ∂-finite functions in the univariate case are: hypergeometric power series and sequences, solutions of linear recurrences and solutions of linear differential equations. In several variables, it becomes necessary to specify with respect to which operators one considers ∂-finiteness; for instance, all sequences of orthogonal polynomials are ∂-finite with respect to shift of the index and differentiation in the argument.

An equivalent definition is that f is ∂-finite when the module $\mathcal{M} = \mathcal{A} \cdot f$ is finitely generated: $\mathcal{M} = \oplus_{\alpha \in A} \mathbb{K}(\mathbf{x})\boldsymbol{\partial}^{\boldsymbol{\alpha}} f$, for a finite set of indices A. If $\operatorname{Ann} f$ denotes the ideal of the elements of $\mathcal{A}$ vanishing on f, then $\mathcal{A}/\operatorname{Ann} f$ is isomorphic to $\mathcal{A} \cdot f$, and this yields a purely ideal-theoretic definition of ∂-finiteness which avoids the introduction of functions. An ideal $\mathcal{I}$ of $\mathcal{A}$ is thus called ∂-finite when $\mathcal{A}/\mathcal{I}$ is finitely generated as a $\mathbb{K}(\mathbf{x})$-module.

1. Closure properties

What makes ∂-finite functions so useful is that it is possible to compute with these functions without reference to any sort of "closed-form". Many computations can be performed directly on sets of generators of their annihilating ideal. In particular, sum and product of ∂-finite functions can be obtained this way.

1.1. Rectangular systems. Before giving the algorithms for sum and product we note that a ∂-finite function f is always annihilated by a *rectangular* system of polynomials, which is such that each ∂_i of the algebra is involved in exactly one of the polynomials. Consequently, each of the polynomials involves only one ∂_i. That this is so follows from the finite dimension of $\sum_n \mathbb{K}(\mathbf{x})\partial_i^n f$, which implies the existence of a linear relation between a finite number of $\partial_i^n f$. Rectangular systems are useful to prove ∂-finiteness of various constructions, or in the case where Gröbner bases are not available. In other cases, they generally describe an ideal which is smaller than the one we would like to work with, and this leads to slower computations.

EXAMPLE. In $\mathcal{A} = \mathbb{Q}(x, y)\langle\partial_x, \partial_y\rangle$, the sum of the Bessel functions $J_\mu(x)$ and $J_\nu(y)$ is annihilated by the rectangular system $\mathcal{S} = \{\partial_x(x^2\partial_x^2 + x\partial_x + x^2 - \mu^2), \partial_y(y^2\partial_y^2 + y\partial_y + y^2 - \nu^2)\}$. If $\phi(x, y)$

is a solution of $\mathcal{S}$, and $\mathcal{I}$ is the ideal generated by $\mathcal{S}$ in $\mathcal{A}$, then it is easily checked that $\mathcal{A}/\mathcal{I}$ is generated by $\{\phi, \partial_x\phi, \partial_x^2\phi, \partial_y\phi, \partial_x\partial_y\phi, \partial_x^2\partial_y\phi, \partial_y^2\phi, \partial_x\partial_y^2\phi, \partial_x^2\partial_y^2\phi\}$ and thus is of dimension 9. However, the annihilating ideal of $f = J_\mu(x) + J_\nu(y)$ also contains $\partial_x\partial_y$. The ideal generated by the adjunction of this polynomial to the rectangular system above is $\operatorname{Ann} f$ and $\mathcal{A}/\operatorname{Ann} f$ is generated by $\{f, \partial_x f, \partial_x^2 f, \partial_y f, \partial_y^2 f\}$ and is only of dimension 5.

1.2. Sum. If f and g are two ∂-finite functions, then by linearity $\boldsymbol{\partial}^{\boldsymbol{\alpha}}(f+g) \in \mathcal{A}f + \mathcal{A}g$ which is finite-dimensional. Hence a sum of ∂-finite functions is ∂-finite.

Given a rectangular system for f and a rectangular system for g a rectangular system for $f+g$ is obtained by reducing $h_n = \partial_i^n f + \partial_i^n g$ for increasing values of n. These reductions use the initial rectangular systems and right Euclidean division, which works in any Ore algebra. All the h_n's are thus rewritten in a finite basis $\{f, \partial f, \dots, \partial^J f, g, \partial g, \dots, \partial^K g\}$. The value of n is increased until a linear relation between the h_n's is found by Gaussian elimination.

1.3. Product. We assume that for each ∂_i in the algebra, the morphisms σ_i and δ_i defined by the commutation rule

$$\partial_i p = \sigma_i(p)\partial_i + \delta_i(p)$$

are polynomials in ∂_i over $\mathbb{K}(x_1, \dots, x_n)$. This is not a severe restriction. Then by the same kind of argument as above, the product of two ∂-finite functions is ∂-finite. The algorithm to produce a rectangular system for the product out of two rectangular systems for the functions being multiplied is exactly the same as above.

1.4. Generalizations. Actually, the same algorithm extends to the direct computation of a rectangular system for any polynomial h in some $\boldsymbol{\partial}^{\alpha_{i,j}} f_i$'s given the rectangular systems defining the f_i's.

The FGLM algorithm [3] provides another generalization: given rectangular systems defining the f_i's and a term order T on the $\boldsymbol{\partial}^{\boldsymbol{\alpha}}$'s, this algorithm returns a Gröbner basis for T. Roughly speaking, this algorithm considers all the monomials $\boldsymbol{\partial}^{\boldsymbol{\alpha}} h$ in the order T and stops when it has found sufficiently many relations. More precisely we start with $F = \{h\}$, the resulting basis is set to $L = \{\}$ and the basis of $\mathcal{A}.h$ is set to $R = \{\}$. At each step the smallest element t of F with respect to T is selected and reduced by the rectangular systems defining the $f_i' s$. Gaussian elimination is then performed to detect a linear dependency between t and the elements of R. If no linear dependency is found, t is added to R, removed from F, and all the $\partial_i t$ are added to F. Otherwise, the dependency is added to L. The algorithm stops when F is empty, and returns L.

Note that the Gröbner basis returned by this method is not necessarily a basis of $\operatorname{Ann} f$ since, as we have already seen, the rectangular systems do not necessarily generate a sufficiently large ideal.

Yet another extension consists in using any Gröbner basis for the f_i's instead of a rectangular system. In the reduction step, the Euclidean division is replaced by a reduction using the Gröbner bases.

Once again, when it is available, the advantage of this approach over manipulating only rectangular systems is that it results in modules of a smaller dimension, and therefore lessens the complexity of further computations.

1.5. Example. The following identity between Apéry numbers and Franel numbers was proved by V. Strehl:

$$\sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} \sum_{j=0}^{k} \binom{k}{j}^3. \tag{1}$$

A system is easily found for $\binom{k}{j}^3$ which is hypergeometric:

$$(k+1-j)^3 S_k - (k+1)^3, \qquad (k-j)^3 S_j - (j+1)^3.$$

Then using creative telescoping (see previous summary), one gets an equation for the sum over j:

$$(k+2)^2 S_k^2 - (7k^2+21k+16)S_k - 8(k+1)^2.$$

Again, a system is easily found for $\binom{n}{k}\binom{n+k}{k}$ which is hypergeometric:

$$(n+1-k)S_n - (n+1+k), \qquad (k+1)^2 S_k - (n(n+1)-k(k+1)).$$

The product of this with the previous equation yields a system for the summand of the right-hand side of (1) whose first equation is the first one above (obviously!) and whose second equation is:

$$(k+2)^4 S_k^2 + (n-k-1)S_k + 8(n+k+2)(n+k+1)(n-k) - (7k^2+21k+6)(n+k+2).$$

Now, creative telescoping yields an equation for the right-hand side of (1):

$$(n+2)^3 S_n^2 - (2n+3)(17n^2+51n+39)S_n + (n+1)^3. \tag{2}$$

The same process is then applied to the left-hand side. First, $\binom{n}{k}^2\binom{n+k}{k}^2$ is hypergeometric and satisfies

$$(n+1-k)^2 S_n - (n+1+k)^2, \qquad (k+1)^4 S_k - (n(n+1)-k(k+1))^2.$$

Creative telescoping then yields (2) again. The identity is then proved by checking that two initial conditions coincide, which they do. The whole computation takes less that 10 seconds on a Dec Alpha.

2. Holonomy

The algorithms for creative telescoping which we have described in the previous summary depend on the existence of a polynomial free of one of several variables in the ideal we are working in. It is thus very important to be working in the proper ideal and to be able to check whether such a polynomial exists or not. In the Weyl algebra case, holonomy theory provides such a guarantee. We describe elements of this theory, and give some hints on what remains valid in the more general Ore algebra case.

2.1. Hilbert dimension. Let $\mathcal{A}$ be an Ore algebra: $\mathcal{A} = \mathbb{K}[\mathbf{x}]\langle\boldsymbol{\partial}\rangle$. Let deg denote the total degree with respect to $\mathbf{x}$ and $\boldsymbol{\partial}$. We consider the graduation F_n of $\mathcal{A}$ where F_n contains the elements of $\mathcal{A}$ of degree at most n. Finally, let $h_n = \dim_{\mathbb{K}}(F_n \cdot f)$.

Example. For $f=1$ in the algebra $K[x_1,\dots,x_p]\langle\partial_1,\dots,\partial_p\rangle$, one has $h_n = \binom{n+p}{p} \sim n^p/p!$.

For $f = \exp(x^2)$ in $\mathbb{K}[x]\langle\partial_x\rangle$, it is easy to compute the first few values and be convinced that $h_n = n+1$.

For $f = (s^3 - s^2 + sx)^{-1/2}$ in $\mathbb{K}[s,x]\langle\partial_x,\partial_y\rangle$, the first values indicate that $h_n = 3n^2+2$.

For $f = \exp(\sin(x))$ in $\mathbb{K}[x]\langle\partial_x\rangle$, one gets $h_n = n^2/2 + 3n/2 + 1$.

Finally, for $f = \binom{n}{k}$ in $\mathbb{K}[n,k]\langle S_n, S_k\rangle$, $h_n = 2n+1$.

A general theorem of Hilbert implies that asymptotically, $h_n \sim cn^d$ with d an integer which is called the Hilbert dimension of the ideal. The relevance of this notion to creative telescoping is of a combinatorial nature: if $\mathcal{B}$ is obtained by forming all the monomials in q of the variables $(\mathbf{x},\boldsymbol{\partial})$, then $F_n \cap \mathcal{B}$ contains $\binom{n+q}{q}$ monomials. As soon as this number grows faster than n^d where d is the Hilbert dimension of the annihilating ideal of some f, then a linear combination of elements of $\mathcal{B}$ has to vanish on f, which means that the ideal contains elements of $\mathcal{B}$.

2.2. Weyl algebra. The Weyl algebra is a special case of a polynomial Ore algebra $\mathcal{A}_p = \mathbb{K}[x_1, \dots, x_p]\langle \partial_1, \dots, \partial_p \rangle$ where ∂_i is the differentiation operator with respect to the corresponding x_i, for $i = 1, \dots, p$. A fundamental theorem of Bernstein states that in this case, the Hilbert dimension of an ideal is always larger than p. Those ideals for which the Hilbert dimension is exactly p are called *holonomic*. By extension, a function whose annihilating ideal in a Weyl algebra is holonomic will be called holonomic too. In the examples above, $\exp(x^2)$ and $(s^3 - s^2 + sx)^{-1/2}$ are holonomic functions, while $\exp(\sin x)$ is not.

Holonomy of functions is preserved under sum and product, algebraic functions are holonomic, algebraic substitution preserves holonomy, the diagonal of a holonomic function is holonomic [4, 5]. In addition, a result due to Kashiwara states that when an ideal $\mathcal{I}$ in the *rational* Ore algebra $\mathbb{K}(x_1, \dots, x_p)\langle \partial_1, \dots, \partial_p \rangle$ is ∂-finite, then $\mathcal{I} \cap \mathcal{A}$ is a holonomic ideal. This means that all ∂-finite functions with respect to differentiation are also holonomic. Finally, creative telescoping always works in holonomic ideals.

3. Conclusions

The algorithms we have given work in a very general context of Ore algebras. However, creative telescoping is never guaranteed *a priori* to give an answer in the general case, unless the existence of the result is ensured, for instance by holonomy. An advantage of our approach is that it may well return results in non-holonomic cases.

An important difficulty will be the subject of future work. Even in the Weyl algebra case, the ideals $\mathcal{I}$ we are dealing with have a natural description in *rational* Ore algebras $\mathbb{K}(\mathbf{x})\langle \boldsymbol{\partial} \rangle$. However, for creative telescoping what we need is a basis of $\mathcal{I} \cap \mathbb{K}[\mathbf{x}]\langle \boldsymbol{\partial} \rangle$. At the moment, we do not have any algorithm to produce this basis. However, algorithms exist to deal with the same problem in the commutative case, and they might extend to this framework.

This problem is illustrated by the computation of the diagonal of $1/(1 - x - y)$. This can be obtained via a residue computation as the definite integral of $f = (s^2 - s + x)^{-1}$ which is holonomic. Thus creative telescoping applies and there exists an operator free of s in the ideal. The annihilating ideal $\operatorname{Ann} f$ of f in $\mathbb{K}(s, x)\langle \partial_s, \partial_x \rangle$ is generated by $\mathcal{S} = \{(s^2 - s + x)\partial_s + 2s - 1, (s^2 - s + x)\partial_x + 1\}$. However, the ideal generated by $\mathcal{S}$ in the Weyl algebra $\mathcal{A} = \mathbb{K}[s, x]\langle \partial_s, \partial_x \rangle$ is smaller than $\operatorname{Ann} f \cap \mathcal{A}$ and does not contain any polynomial free of s. To get such a polynomial, it is necessary to augment $\mathcal{S}$, for instance with $(s^2 - s + x)\partial_s\partial_x + 2\partial_s$.

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