Automatic Asymptotics

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[summary by Bruno Salvy]

Joris van der Hoeven sets up an ambitious research program of automating the derivation of asymptotic expansions in various contexts.

The problem has already been solved in several cases. For exp-log functions—functions obtained from a variable x and the set of rational numbers $\mathbb{Q}$ by closure under field operations and the application of exp and log—John Shackell gave a procedure in [5] which he extended in [6] to Liouvillian functions. In [7], he showed how to handle composition and in [4] B. Salvy and J. Shackell showed how to compute an expansion of $y(x)$ subject to $F(y) = x$ for F an exp-log function. Asymptotic expansions for differential algebraic equations were also made effective by J. Shackell in [8], however the algorithm in this case pays for its generality by an exponential complexity with respect to the order [9]. Another approach to the exp-log function problem was used by D. Gruntz to implement the new `limit` function in Maple [2].

While J. Shackell and several others have based their work on the theory of Hardy fields [3], the approach followed by J. van der Hoeven is inspired by Écalle's theory of transseries [1]. Informally, there are two main ingredients in this work. The first one consists in computing with asymptotic scales suitable for exponentiation and logarithm (so-called *normal bases*, see below). The second one consists in working simultaneously with expansions in these scales and a handle on the exact full information related to them. This handle (named *algorithmic multiseries*) makes it possible to compute more terms of an expansion whenever necessary and to invoke an oracle for zero-equivalence of functions in order to prevent indefinite cancellation. More precise definitions are as follows.

Definition 1. An asymptotic scale is a finite ordered set $\{g_1, \dots, g_n\}$ of positive unbounded exp-log functions such that $\log g_i = o(\log g_{i+1})$, for $i = 1, \dots, n-1$.

Definition 2. Let g be a positive unbounded exp-log function. A *multiseries* with respect to g is a formal sum

$$f = \sum_{\alpha \in S} f_\alpha g^\alpha,$$

the coefficients f_α being exp-log functions and the support $S \subset \mathbb{R}$ having finitely generated support:

$$S = \alpha_1 \mathbb{N} + \alpha_2 \mathbb{N} + \cdots + \alpha_k \mathbb{N} + \beta,$$

where the α_i's are strictly positive real numbers and $\beta \in \mathbb{R}$.

Definition 3. A multiseries expansion with respect to $\{g_1, \dots, g_n\}$, is a multiseries with respect to g_n, where each coefficient can recursively be expressed as a multiseries with respect to $\{g_1, \dots, g_{n-1}\}$, each multiseries with respect to g_1 having constant coefficients.

DEFINITION 4. A *normal basis* is an asymptotic scale $\{g_1, \ldots, g_n\}$ satisfying the following conditions

(1) $g_1 = \log_k x$ ($k \in \mathbb{N}$), where $\log_k$ denotes the logarithm iterated k times ($\log_0 x = x$);
(2) $\log g_i \in \mathbb{R}[[g_1; \ldots; g_{i-1}]]$, for $2 \leq i \leq n$.

DEFINITION 5. A multiseries with respect to a normal basis $\{g_1, \ldots, g_n\}$ is said to be *algorithmic* when it converges to a function, and for any $k \in \mathbb{N}^\star$ its first k coefficients with respect to g_n can be computed and are themselves algorithmic (constants being algorithmic).

The theorem J. van der Hoeven aims at proving in various contexts of asymptotic expansions is that there always exists a normal basis and (under suitable restrictions) algorithmic multiseries. He has proved this theorem for exp-log functions [10], where the inversion problem and algebraic equations have been at least partially treated too. Here is a theorem from [10].

THEOREM 1. *Schanuel's conjecture implies that the field of real algebraic exp-log functions is an automatic expansion field.*

An automatic expansion field is a field where normal bases and multiseries with respect to them can be computed for any element. Schanuel's conjecture is related to the zero-equivalence problem for constants. Algebraic differential equations and the zero-equivalence problem in a general setting should be treated in [11].

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