Symbolic and Numerical Manipulations of Divergent Power Series

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[summary by Bruno Salvy]

Abstract
Divergent series arise naturally in many different contexts. This talk describes mixed symbolic-numerical algorithms to deal with these series when they arise from linear differential equations.

Introduction
A simplified example of a divergent power series is obtained when summing a Taylor series outside its circle of convergence. More generally, when solving a linear differential equation like the Euler equation

$$x^2y'' + y = x$$

by an undetermined coefficient method. The power series obtained is the Euler series

$$
\sum_{n \geq 0} (-1)^n n! x^{n+1},
$$

which has a radius of convergence equal to 0. This problem also occurs in nonlinear differential equations, singular perturbations, differential equations, or asymptotic analysis (e.g., by the Laplace method).

The Borlitz-Ritt theorem states that any power series in any sector of finite opening in the complex plane is the asymptotic expansion of a function which is analytic in the sector. However, this analytic function is far from being uniquely determined, which makes numerical valuation hopeless. In the context of differential equations, the situation is much better because of the following result.

Theorem 1. Let $G(x, y_0, \ldots, y_n)$ be an analytic function of $n+2$ variables and $\hat{f} \in \mathbb{C}[[x]]$ a formal power series solution of $G(x, y, \ldots, y^{(n)}) = 0$. Then there exists a real number $k > 0$ such that for all open sectors $V$ with vertex at the origin, opening $< \pi/k$ and small enough radius, there exists a function $f$ which is a solution of the differential equation $G(x, y, \ldots, y^{(n)}) = 0$ asymptotic to $\hat{f}$ on $V$.

Thus the main numerical problem is to develop techniques that will sum the divergent series not to values of any analytic function asymptotic to it, but to values of the actual solution of the differential equation corresponding to it.
1. Elementary methods

To compute the sum of a convergent power series \( \sum_{n=0}^{\infty} a_n x^n \) outside its circle of convergence, one first has to define a path connecting the origin to the point where the sum is defined. A basic subproblem is that of summing along a ray originating at the origin and avoiding singularities of the function. Lindelöf gave a simple way of doing this by computing this value as the limit of

\[
a_0 + \lim_{t \to 0} \sum_{n \geq 1} a_n x^n e^{-t \ln n}.\]

When the series is convergent at \( x \), this result is the sum of all the series of the same type as the Euler series. By computing

\[
a_0 + a_1 x + a_2 x^2 + \lim_{t \to 0} \sum_{n \geq 3} a_n x^n e^{-t \log n \log \log n},\]

unfortunately, this technique does not hold for power series numerically.

The simplest technique to deal with divergent series is to sum the Euler series for the real part of its equation, namely 0.09156333394. Using a convergent integral representations for the Euler series, it is not difficult to show (as [5]) that the error made by truncating this series at its last term is exponentially small (with respect to 1/x). This property is actually much more general (as below). The drawback of this good precision is the impossibility of obtaining an arbitrary precision by this method. This is to be contrasted with the direct summation of convergent power series, where the terms grow rapidly first as \( x \) becomes large, but numerous terms are necessary to obtain a good precision. As a consequence, many techniques to converge from various representations of a function to a divergent power series have been developed [3].

2. Gevrey asymptotics, Borel transform, \( k \)-summability

A general framework to account for the nicest behavior of common divergent power series is provided by Gevrey asymptotics. A Gevrey series is a power series whose coefficients' growth is bounded by \( C(n)!^{1/k} A^n \), for some fixed constants \( C, A, k > 0 \). Gevrey asymptotic expansions are for the main point of bounded Morera's criterion, which is the following.

**Definition 1.** Let \( k \) be a positive real number and \( l, r, V, b \) an open sector with \( V \) at 0. Let \( f \) be an analytic function on \( V \). The formal power series \( \sum_{n \geq 0} a_n x^n \) is Gevrey asymptotic to \( f \) of order \( s = 1/k \) on \( V \) if for all compact subsectors \( W \) of \( V \) and for all \( n \in \mathbb{N} \), there exist \( C_W > 0 \) and \( A_W > 0 \) such that

\[
|x|^{-n} \left| f(x) - \sum_{p=0}^{n-1} a_p x^p \right| < C_W (n)!^{1/k} A_W^n, \quad \forall x \in W, \quad x \neq 0
\]
By Stirling’s formula, truncating a G. v. r. y asymptotic expansion of order 1/k to the last term gives an exponentially small error (in 1/x^k). This is on of the facts of the integral of G. v. r. y asymptotics. Another crucial property, due to Watson, is that the r.h.s. is at most an analytic function of G. v. y asymptotic of order 1/k to a s r i s f on a sector of o. n. g. r large r than π/k. This provides the unique necessary for numerical computations bas d on th s r i s alon.

With the same hypothesis as in Th or m 1, an old th or m of Maill t stat s that th r. exists k > 0 such that f is a G. v. r. y s r i s of order 1/k. This r. sult is us ful in conjunction with a count part of Th o r m 1 du to Ramis and Sibuya which states that if f is G. v. y of order 1/k th n th r. exists k' ≥ k such that for any sector V with v. r. x 0, op. n. g. < π/k' and sufficiently small radius, th r. exists a function f solution to the di f f r ntial equation which is G. v. y asymptotic of order r k to f. Combing th s two r. sults explains why summing to th l ast term is a good m. thod for formal s r i s solutions of di f f r ntial equations.

If f = ∑ a_n x^n is a G. v. r. y s r i s of order 1/k, th n its Borel transform of order k is d f n. d a

\[(\hat{B}_k f)(\xi) = \sum_{n \geq 0} \frac{a_n}{\Gamma(1 + n/k)} \xi^n.\]

For k = 1, this cor. sponds to dividing the co effi cients by n!. Estimat s on th co effi cients show that this transform is an analytic function φ(ξ). Th n if th Laplac. transform of ind x k

\[f(x) = \int_d^\infty k \phi(\xi) e^{-\xi x / x^k} \xi^{k-1} d\xi\]

cor. s the s. of f in th di r. on d, wh r. d is a straight line from 0 to infinity. Th s r i s f is th n said to be k-summable in the direction d. Th conv r. nc. of th int. ral is r. lat. d to th growth of φ at infinity. It is asy to s th. that Taylor s r i s of f is pr. cis. ly f so th th th proc ss yi. ds a conv r. nc. r pr s. n. ation for f. Th sum ho. r d p nds on th path of int. ration d, in th s. way an analytic continuation d p nds on a path. This d p nd. nc. is r. lat. d to th Stokes pheno. menon.

Num rically, in th case of conv r. nc., th. probl m is r. duc. d to finding k and computing th. analytic continuation of φ. In th case of solutions of linear di f f r ntial equations, this computation is simplifi d by noticing that k can be duc. d from th slop s of a N. wton polygon associat. d with th. lin. ar di f f r ntial equation and that φ sati fi s a lin ar di f f r ntial equation d r. v. d formally from th sati fi d by f. Th r for its Taylor co effi cients satisfy a computa. lin ar r. curr. nc. which can be us. d to obtain many co effi cients f. s t. y. B. sid s, th. poss. bl. singulariti s of φ are locat. d at th z ro s of th lading co effi. nt of th. lin. ar di f f r ntial. equ. ation it sati fi s, so th. it is poss. bl. to comp. th. continuation along a path which avoids singulariti s, with a knowl. ng of th. exact radius of conv r. nc. of th. pow. r s r i s on. is computing. This proc ss can also be appli d to th. div r. nc. s r i s th. occur as part of th. asymptotic. xpansion of solutions of lin ar di f f r ntial equations at an ir. gular singular point, by first computing a lin ar di f f r ntial. equ. ation sati fi d by th s. r i s.

3. Multisummability

Not all solutions of lin ar di f f r ntial. equations are k summab. for som. k. On r. ason for this is that th ord r of growth of an analytic function at infinity is r. lat. d to th. growth of its Taylor co effi cients at th. origin. Thus by adding a 1 summab. and a 2 summab. div r. nc. s r i s, one obtains a s r i s which is G. v. r. y of order r 1, but th. growth at infinity of its Borel transform of 1 v. 1 l is expon. ntial of order r 2. This l. ads to th. consi. ration of a mor. g. n. ral class of div r. nc. s r i s.
Definition 2. Let \( k_1, \ldots, k_r \) be real numbers such that \( k_1 > \cdots > k_r > 0 \) and \( l \leq d \) be a line from 0 to infinity. A formal power series \( f(x) \) is \((k_1, \ldots, k_r)\)-summable in the direction \( d \) if there exists a positive integer \( g \) such that \( \hat{f}(x^{1/m}) \) is a sum of \( r \) series \( \hat{f}_1, \ldots, \hat{f}_r \), each \( \hat{f}_i \) being \( k_i/m \) summable in the direction \( d \).

A result of Jurkat is that Hardy's summation technique (1) will sum any multisummable series without having to know \( k_1, \ldots, k_r \).

The following result due to Braaksma demonstrates the vanishing of multisummability.

Theorem 2. Let \( G(x, y_0, \ldots, y_n) \) be an analytic function of \( n + 2 \) variables and \( \hat{f} \in \mathbb{C}[[x]] \) a formal power series solution of \( G(x, y, \ldots, y^{(n)}) = 0 \). Let \( k_1 > \cdots > k_r > 0 \) be the positive slopes of the associated Newton polygon. Then \( \hat{f} \) is \((k_1, \ldots, k_r)\) summable in every direction \( d \), except possibly a finite number of them.

Braaksma's proof uses Écalle's theory of accretion summability.

In the linear case, a technique due to Balsam makes it possible to compute the sum by computing successive transforms of indicial \( k_j \)'s by \( 1/k_j = 1/k_1 + \cdots + 1/k_j \) and transforms of order \( r \) \( k_j \) in the corresponding Laplace transforms of order \( r \). At each step, exact linear differential equations can be computed for each Taylor series and exact linear currents for the coefficients.

Conclusion

Numerically, the difficulty is that each line of integration is time-consuming and induces a precision loss. At the moment, this process is still largely inactive, notably the choice of paths of integration at each step.

Bibliography


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