Short and Easy Computer Proofs of Partition and q-Identities

Peter Paule

RISC Linz, Austria

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[summary by Bruno Salvy]

1. Binomial and hypergeometric identities ($q = 1$)

Binomial identities are identities which involve binomial coefficients like the famous *Saalschütz identity:*

$$\sum_{k=0}^{n} \frac{\binom{n}{k}\binom{x}{k}\binom{y}{k}}{\binom{x+y+z+n}{k}\binom{z+k}{k}} = \frac{\binom{x+z+n}{n}\binom{y+z+n}{n}}{\binom{z+n}{n}\binom{x+y+z+n}{n}}. \tag{1}$$

Tables like [3] list several hundred such identities. Since binomial coefficients satisfy many relations, the expression on the left-hand side may appear under numerous disguises, which makes it difficult to locate it in such tables (or to implement table lookup in a computer algebra system). However, a sort of *normal form* follows from the observation that in many identities with left-hand side $\sum_k f(k)$, the function $f(k)$ satisfies

$$\frac{f(k+1)}{f(k)} \in F(k), \tag{2}$$

for some suitable field of coefficients F. Thus f is completely determined by $f(0)$ and a rational function, for which a normal form is available. A function f satisfying this property is called a *hypergeometric term*. In a suitable algebraic extension, $f(k)$ can be made explicit:

$$f(k) = \frac{(a_1)_k \cdots (a_m)_k}{(b_1)_k \cdots (b_n)_k} \frac{z^k}{k!} f(0),$$

where $(a)_k = a(a+1)\cdots(a+k-1)$ denotes the rising factorial. The sum of f (when $f(0) = 1$) is usually called the *hypergeometric series* with the following notation

$${}_mF_n\left(\left.\begin{matrix} a_1, \ldots, a_m \\ b_1, \ldots, b_n \end{matrix}\right| z\right) = \sum_{k=0}^{\infty} f(k).$$

According to G. E. Andrews, "By using hypergeometric series one can reduce 450 of the 577 entries in Gould's table to 32 entries." Thus for instance, the Saalschütz identity is obtained as

$${}_3F_2\left(\left.\begin{matrix} -x, -y, -n \\ z+1, -x-y-z-n \end{matrix}\right| 1\right) = \frac{(x+z+1)_n (y+z+1)_n}{(z+1)_n (x+y+z+1)_n}.$$

Given a function $F(n,k)$ hypergeometric in both parameters, plus a technical condition (holonomy), D. Zeilberger gave an algorithm to compute a linear recurrence satisfied by the definite sum with respect to one of the parameters. The technique is based on *creative telescoping* [11] which applies to a larger context of holonomic identities. To compute $\sum_k F_{n,k}$ from a first-order recurrence like (2) in n and a second one in k, the idea is to determine a recurrence satisfied by $F_{n,k}$ where k does not appear *in the coefficients*. In the case of the Saalschütz identity, this gives

$$
\begin{aligned}
&(n+3+z)(n+1+x+y+z)_3 F_{n+3,k+1}\\
&-(n+1+x+y+z)_2\{[(x+y+z+2n+5)(2n+z+5)-2(n+2)(n+3)]F_{n+2,k+1}\\
&\qquad +(n+2-y)(n+2-x)F_{n+2,k}\}\\
&+(n+1+x+y+z)[(n+2)(n+2+x+y+2z)(n+2+x+y+z)F_{n+1,k+1}\\
&\qquad +(n+2)(2n^2+6n+2nz-x^2-xz-yz+3z-y^2+5)F_{n+1,k}]\\
&-(n+1)(n+2)(n+y+z+1)(n+x+z+1)F_{n,k}=0.
\end{aligned}
\tag{3}
$$

In the holonomic universe, such an elimination is always possible. The above identity is then rewritten

$$
\begin{aligned}
&(x+y+z+n+1)_3(n+z+3)F_{n+3,k}\\
&-(x+y+z+n+1)_2\times\\
&\qquad (z^2+xz+10z+yz+4nz+xy+14n+3y+ny+17+nx+3x+3n^2)F_{n+2,k}\\
&+(x+y+z+n+1)(n+2)\times\\
&\qquad (2xz+2xy+2nx+4x+2z^2+9z+5nz+3n^2+9+10n+4y+2yz+2ny)F_{n+1,k}\\
&-(n+2)(n+1)(n+1+x+z)(n+1+y+z)F_{n,k}=G_{n,k+1}-G_{n,k},
\end{aligned}
\tag{4}
$$

with

$$
\begin{aligned}
G_{n,k}=&(x+y+z+n+2)(x+y+z+n+1)[(x+y+z+n+3)(n+z+3)F_{n+3,k}\\
&-(z^2+4nz+10z+xz+2n^2+10n+5x+13+2nx+2ny+yz+5y)F_{n+2,k}\\
&+(n+2)(2z+2+n+x+y)F_{n+1,k}].
\end{aligned}
\tag{5}
$$

Now summing with respect to k shows that the left-hand side of (4) is the desired recurrence for the sum. Using M. Petkovšek's algorithm [7], it is then possible to find the right-hand side of (1).

H. Wilf and D. Zeilberger have designed a *fast* algorithm (as opposed to general non-commutative elimination) to compute recurrences of the type (3) for terminating hypergeometric summation and multi-summation [9]. The analogous of $G_{n,k}$ in (5) is called a *certificate* of the computation, since it makes (4) easy to check by mere rational function manipulations. This algorithm has been at least partially implemented in Maple by D. Zeilberger [10] and T. Koornwinder [4] and in Mathematica by P. Paule and M. Schorn [5]. It has the extra advantage that the recurrence it returns instead of (3) is of order 1.

2. q-identities

A natural generalization of hypergeometric identities is provided by q-hypergeometric identities. In this context, rising factorials are replaced by $(a)_n=(1-a)(1-aq)\cdots(1-aq^{n-1})$, $n!$ becomes $(q)_n/(1-q)^n$ and the binomial coefficients become the q-binomial coefficients or Gaussian polynomials

$$\begin{bmatrix} n\\ k\end{bmatrix}=\frac{(q)_n}{(q)_k(q)_{n-k}},\qquad 0\le k\le n.$$

The classical counterpart of these numbers or identities involving them is obtained by letting q tend to 1. Like binomial coefficients, the Gaussian polynomials have nice combinatorial interpretations and properties (see [2]).

A q-hypergeometric term is a function $f(k)$ such that $f(k+1)/f(k)$ is a rational function of q and q^k. The techniques of Wilf and Zeilberger extend to q-hypergeometric identities [9] and a Mathematica implementation is available [8]. For instance, the following identity is equivalent to the q-binomial theorem:

$$\sum_{k=-n}^{n} q^{\binom{k}{2}} \begin{bmatrix} 2n \\ n+k \end{bmatrix} x^k = (-x)_n(-q/x)_n. \tag{6}$$

Let the summand be $f_{n,k}$, which satisfies

$$\frac{f_{n,k+1}}{f_{n,k}} = x\frac{q^k - q^n}{1-q^{n+k+1}}, \qquad \frac{f_{n+1,k}}{f_{n,k}} = \frac{(1-q^{2n+1})(1-q^{2n+2})q^k}{(q^k-q^{n+1})(1-q^{n+k+1})}.$$

Then it can be found that $f_{n,k}$ also satisfies the following recurrence where k does not appear in the coefficients:

$$q^{n+1}f_{n,k+2} + x(1+q^{2n+1})f_{n,k+1} + x^2q^nf_{n,k} - xf_{n+1,k+1} = 0.$$

This is then rewritten as

$$xf_{n+1,k} - (x^2q^n + xq^{2n+1} + x + q^{n+1})f_{n,k} = g_{n,k+1} - g_{n,k}, \tag{7}$$

with certificate

$$g_{n,k} = xf_{n+1,k} - q^{n+1}f_{n,k+1} - xq(q^{2n}+q^n+1)f_{n,k}.$$

From this follows that the left-hand side of (7) gives a recurrence satisfied by the sum. Since this recurrence is of order 1, solving it is easy and this yields the right-hand side of (6).

It is not difficult to see that $\lim_{n\to\infty} \left[{2n \atop n+k}\right] = 1/(q)_\infty$. Now, taking the limit in (6), changing q into q^2 and x into qz yields the famous *Jacobi triple-product identity:*

$$\sum_{k=-\infty}^{\infty} q^{k^2}z^k = \prod_{n=0}^{\infty}(1-q^{2n+2})(1+zq^{2n+1})(1+\frac{q^{2n+1}}{z}). \tag{8}$$

3. Partition identities

There is a strong connection between identities about partitions and q-calculus. For instance the q-binomial coefficient $\left[{N+M \atop M}\right]$ is the generating function (in the variable q) of the number of partitions of n into at most M parts, each $\leq N$. Probably the most famous partition identities in this category are the *Rogers-Ramanujan identities*, an example of which is

$$1 + \sum_{k=1}^{\infty}\frac{q^{k^2}}{(1-q)(1-q^2)\cdots(1-q^k)} = \prod_{m=0}^{\infty}\frac{1}{(1-q^{5m+1})(1-q^{5m+4})}. \tag{9}$$

It is easy to see that the right-hand side is the generating function of partitions into parts equal to 1 or 4 mod 5. It turns out that the left-hand side can be read as the generating function of partitions into parts with minimal difference 2 (see [2]). This identity states that these numbers are identical. For instance, the coefficient of q^{10} is 6 on both sides, corresponding to (10), (1,9), (2,8), (3,7) (4,6), (1,3,6) on the left and (1^{10}), $(1^6,4)$, $(1^2,4^2)$, $(1^4,6)$, (1,9), (4,6) on the right (exponent denoting repetition).

D. Zeilberger has used his algorithm to prove (9) by proving the following finite version due to G. Andrews:

$$\sum_k \frac{q^{k^2}}{(q)_k(q)_{n-k}} = \sum_k \frac{(-1)^kq^{(5k^2-k)/2}}{(q)_{n-k}(q)_{n+k}}. \tag{10}$$

Letting n tend to infinity and making use of Jacobi's identity (8) yields (9). However, as in many other instances, the WZ-technique (for Wilf & Zeilberger) yields a recurrence whose order is not minimal (here 5 instead of 2). Not only is it not mathematically aesthetic, but it generally leads to computations that require much more memory and computer time than necessary.

P. Paule's key observation [6] is that *by taking advantage of the symmetry in the summands of (10) (and of many other similar identities) the order of the equation obtained by the WZ-algorithm becomes minimal!* Thus for sums with even summand $f(k)$, the idea is to try summing $(f(k)+f(-k))/2$ instead. This leads to the following three-line proof of (9).

THEOREM 1. *The Rogers-Ramanujan identity*

$$\sum_{k=0}^{\infty} \frac{q^{k^2}}{(q)_k} = \frac{1}{(q)_\infty} \sum_k (-1)^k q^{(5k^2-k)/2}$$

is the limit when $n \to \infty$ of

$$\sum_k \frac{2q^{k^2}}{(q)_k (q)_{n-k}} = \sum_k \frac{(-1)^k q^{(5k^2-k)/2}(1+q^k)}{(q)_{n-k}(q)_{n+k}}. \tag{11}$$

In addition, both sides of (11) satisfy the recurrence

$$(1-q^n)u_n = (1+q-q^n+q^{2n-1})u_{n-1} + qu_{n-2}.$$

PROOF. The initial conditions $u_0 = 2$ and $u_1 = 2(1+q)/(1-q)$ are easily seen to hold for both sides. The proof that both sides satisfy the recurrence relation is easy once given their certificates: $-q^{2n-1}(1-q^{n-k})$ and $q^{2n+3k}(1-q^{n-k})/(1+q^k)$. □

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