Oscillating Rivers

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[summary by Jacques Carette]

Abstract

An oscillating river is an oscillating asymptotic solution of an ordinary differential equation where there is, at infinity, an exponential concentration of solutions of the differential equation. The aim of this talk is to present a few cases where such can be proved to occur.

1. Introduction

F. and M. Diener have recently studied some cases of solutions of ordinary differential equations which are exponentially close to each other at infinity [1, 2, 3]. This new type of attractor was baptised "fleuve", or "river" a name suggested by the corresponding phase portraits (see Figure 1). If one considers a scalar equation

$$\frac{dY}{dX} = \sum_{j=0}^{n} P_j(X)Y^j$$

where the $P_j(X)$ are finite sums of rational powers of X with complex coefficients, then there exist effective methods to ascertain the presence of rivers. The associated solutions are either attractive, in which case there is an infinity of solutions which share the same asymptotic behaviour, else they are repulsive, in which case there is a unique asymptotically unstable solution. These rivers generally possess divergent asymptotic expansions in fractional powers of X, but they are always Gevrey.

2. The Periodic Model

The results established for the $P_j(X)$ in the class defined above, are still valid if these functions possess a pole at plus infinity, but not if one of them has an essential singularity. As the figures show, this is not an obstacle for such rivers to occur. However, F. Michel believes that it is the periodic structure of the functions considered which makes the phenomenon possible. This leads to the study of the following model:

$$\frac{dY}{dX} = \sum_{i\in I} a_i(X)X^{m_i}Y^{n_i} \tag{1}$$

where $m_i \in \mathbb{Q}, n_i \in \mathbb{N}, I$ is a finite set, $a_i(X) \in \mathcal{C}$, where $\mathcal{C}$ is the algebra of $\mathcal{C}^\infty$ periodic functions of fixed period τ.

The asymptotic behaviour of these rivers does not follow from those of the previous case (that we shall refer to as the polynomial case). In fact, there will be three cases, depending on a parameter c which depends on the m_i and the n_i. The method used to prove the theorems will be to give qualitative results about solutions that bound (above and below) the trajectories of interest. This is to be contrasted to the methods used for the polynomial case, which were singular perturbations and non-standard techniques.

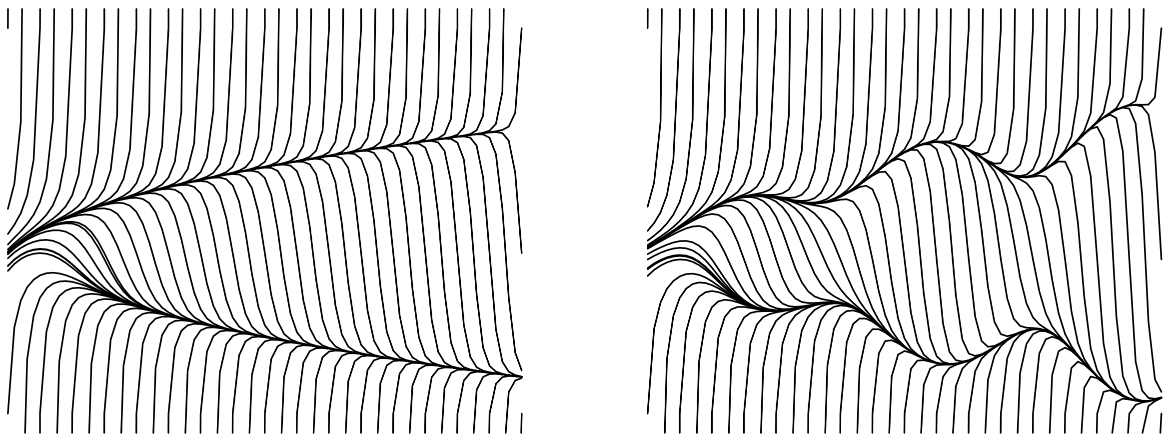

FIGURE 1. Rivers: $Y' = Y^2 - X$ and $Y' = Y^2 - (1 + \sin X/2)X$.

3. Notation

Even though the results are straightforward to state in vague terms, the exact formulation needs a number of preliminary concepts to be defined.

DEFINITION 1. If $k(X) \in \mathcal{C}$ is a non-zero function and $Y(X)$ is a real function defined on a neighbourhood of $+\infty$ and $r \in \mathbb{Q}$ then we say that $Y(X)$ is asymptotic to $k(X)X^r$ (which we denote $Y(X) \sim k(X)X^r$) if

$$Y(X) = k(X)X^r + o(X^r), \qquad X \to \infty.$$

Note that if a function is asymptotic to $k(X)X^r$ then the term $k(X)$ is necessarily unique which makes $\mathcal{C}$ an appropriate space for studying the asymptotic behaviour of solutions of our model.

As is usual in dealing with differential equations, crucial information is contained in the Newton polygon $\mathcal{P}$, the convex hull of the set of horizontal half-lines going left from the points (m_i, n_i). A rational number r will be called a co-slope of $\mathcal{P}$ if r is the slope of one of the normals to the segments of $\mathcal{P}$.

Note $Q(S, X, Y) = \sum_{i \in I} a_i(S) X^{m_i} Y^{n_i}$ where $a_i(X) \in \mathcal{C}$ and are not identically zero. The number $\bar{a}_i$ is the average of a_i over one period, and $\tilde{a}_i(X) = a_i(X) - \bar{a}_i$. We also generalise this notation to any function of X, by which we mean that a bar indicates an average and a tilde the zero-average translation.

Let $r \in \mathbb{Q}$, then we define $\mu_0 = \max(m_i + rn_i)$ and $c = 1 - r - \mu_0$. For the slope r, c measures the attraction (or repulsion) of the associated solution, if it exists. Let q be the smallest positive integer such that $r - n/q$ for $n \in \mathbb{N}$ takes on all values of $m_i + rn_i$ for $i \in I$ and the value $r - 1$. From these values we can define $p = cq$ and $\mu_n = \mu_0 - n/q$. Furthermore, we set

$$Q_r^n(S, X, Y) = \sum_{m_i + rn_i = \mu_n} a_i(S) x^{m_i} Y^{n_i}, \qquad \bar{Q}_r^n(X, Y) = \sum_{m_i + rn_i = \mu_n} \bar{a}_i x^{m_i} Y^{n_i}.$$

We will only discuss the various Q_r^0 functions here, but the other functions can be used to determine the successive terms in the asymptotic expansion. We denote by a $'$ the derivative of the above functions with respect to Y. The differential equation

$$\frac{dY}{dX} = Q_r^0(X,1,Y) \tag{2}$$

will be important. Finally, note by $(*)$ an expansion of the form

$$\sum_{i\geq 0} \alpha_i(X) X^{r-i/q}$$

for $\alpha_i(X) \in \mathcal{C}$. This is the model used for an asymptotic expansion.

4. Results

Each of the following three theorems has two sub-cases, sub-case (a) corresponds to the attractive case, and sub-case (b) to the repulsive case.

THEOREM 1. *Let $r \in \mathbb{Q}$, $k(X) \in \mathcal{C}$, $k \neq 0$ be such that*

(1) *r is a co-slope of $\mathcal{P}$;*
(2) *$c > 1$;*
(3) *$\forall X$, $Q_r^0(X,1,k(X)) = 0$;*
(4) *(a) $\forall X$, $(Q_r^0)'(X,1,k(X)) < 0$, or (b) $\forall X$, $(Q_r^0)'(X,1,k(X)) > 0$.*

Then there exists a series of the type $(*)$ which is a formal solution of (1) with $\alpha_0(X) = k(X)$. Furthermore, there exists an infinite number of solutions asymptotic to $k(X)X^r$ in the attractive case, and a unique one in the repulsive case. The series $(*)$ is the asymptotic expansion of those solutions.

Conditions (3) and (4) express that $k(X)X^r$ is the first term in the asymptotic expansion for a trajectory with constant 0 derivative along that trajectory. The first condition is to insure that the function Q_r^0 is not reduced to one term (where only the function $k(X) \equiv 0$ would satisfy the third condition). The cases in Condition (4) indicate whether the branch of the solution we are considering is attractive or repulsive. The fact that $c > 1$ indicates that geometrically the nearby solutions oscillate along with the trajectory considered.

THEOREM 2. *Let $r \in \mathbb{Q}$, $k \in \mathbb{R}$, $k \neq 0$ be such that*

(1) *r is a co-slope of $\mathcal{P}$;*
(2) *$0 < c < 1$;*
(3) *$\bar{Q}_r^0(1,k) = 0$;*
(4) *(a) $(\bar{Q}_r^0)'(1,k) < 0$, or (b) $(\bar{Q}_r^0)'(1,k) > 0$.*

Then there exists a series of the type $(*)$ which is a formal solution of (1) with $\alpha_0(X) = k(X)$. Furthermore, there exists an infinite number of solutions asymptotic to kX^r in the attractive case, and a unique one in the repulsive case. The series $(*)$ is the asymptotic expansion of those solutions.

Since $0 < c < 1$, the rivers do not oscillate starting at the first approximation, and thus it is necessary to look at the averaged function $\bar{Q}_r^0$ instead, but the meaning of the third and fourth conditions are essentially the same as in the previous theorem.

THEOREM 3. *Let $r \in \mathbb{Q}$, $k(X) \in \mathcal{C}$, $k(X) \neq 0$ be such that*

(1) *r is a co-slope of $\mathcal{P}$;*
(2) *$c = 1$;*

(3) *$k(X)$ is a periodic solution of (2);*

(4) *(a) $\overline{(Q_r^0)'(X,1,k(X))} < 0$, or (b) $\overline{(Q_r^0)'(X,1,k(X))} > 0$.*

Then there exists a series of the type $(*)$ which is a formal solution of (1) with $\alpha_0(X) = k(X)$. Furthermore, there exists an infinite number of solutions asymptotic to $k(X)X^r$ in the attractive case, and a unique one in the repulsive case. The series $(*)$ is the asymptotic expansion of those solutions.

The case $c = 1$ is an intermediate situation: there are oscillations of the type $k(X)X^r$, but $k(X)$ does not correspond exactly to the oscillations of the trajectory. When it exists, we observe that it is generally out of phase and of smaller amplitude.

We will then call *oscillating river* the solutions described by each of the preceding theorems.

5. Example

Let us consider the equation $Y' = (Y^2 - (2 + \sin(X)))X^\alpha$. The first condition in all theorems implies that necessarily $r = 0$. From this, we can calculate $c = 1 + \alpha$. Thus, if $\alpha > 0$, the first theorem gives that we have a river asymptotic to $\pm(2 + \sin(X))^{1/2}$, if $1 < \alpha < 0$, the second one gives rivers asymptotic to $\pm\sqrt{2}$ and if $\alpha = 0$, the last theorem leads us to search for periodic solutions of a periodic equation, where one can consult the large literature on this subject.

6. Proof Ideas

To prove that there exists solutions asymptotic to $k(X)X^r$, the notion of *tunnels* is used. If there exists $X_0, \nu_-, \nu_+ \in \mathbb{R}$ with $\nu_- < \nu_+$ such the right hand side of (1) is positive for all $X \geq X_0$ for $Y = \nu_-$ and negative for $Y = \nu_+$, then the set $\{(X,Y) \mid X \geq X_0, \nu_- < Y < \nu_+\}$ is called a tunnel. In the appropriate coordinates, the hypotheses imply easily that such tunnels exist, which then force the existence of the asymptotic solutions. Technical computations show the existence of a formal series solution. And finally, a few more arguments with tunnels and leading term comparisons allow us to conclude that this series is actually an asymptotic series expansions for the solutions shown previously to exist.

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