Riordan Arrays and their Applications

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[summary by Danièle Gardy]

Abstract

A Riordan array is a doubly indexed sequence of coefficients of a bivariate generating function. This talk presents some of their properties, then shows how they can be useful in combinatorial problems.

1. Riordan arrays

The term *Riordan array* was introduced recently to denote a concept familiar in combinatorics; it is a doubly indexed sequence $\{d_{n,k}; n, k \in \mathbb{N}\}$, defined for two formal series $d(t)$ and $h(t)$ by

$$d_{n,k} = [t^n]\{d(t)(th(t))^k\}, \qquad \text{or} \qquad \sum_{n,k} d_{n,k} t^n u^k = \frac{d(t)}{1 - uth(t)}. \tag{1}$$

We use the notation $(d, h) := \{d_{n,k}\}$. A Riordan array is *proper* when $d_{n,n} \neq 0$ for all n, i.e. when $h(0) \neq 0$.

Property 1. *The $d_{n,k}$ satisfy a recurrence relation*

$$d_{n+1,k+1} = a_0 d_{n,k} + a_1 d_{n,k+1} + \cdots . \tag{2}$$

The a_i's define a series $A(z) = \sum_i a_i z^i$ and the series h satisfies the equation $h(t) = A(th(t))$.

If the recurrence relation (2) holds for some sequence $d_{n,k}$, then this sequence is a proper Riordan array, with $d(t)$ the generating function of the sequence $\{d_{n,0}\}$: $d(t) = \sum_n d_{n,0} t^n$, and $h(t)$ the (unique) solution of the equation $Y = A(tY)$.

Theorem 1. *Let $f(z) = \sum_k f_k z^k$; then*

$$\sum_k d_{n,k} f_k = [t^n]\{d(t) f(th(t))\}.$$

Example. The binomial numbers $\binom{n}{k}$ are defined by $d(t) = h(t) = 1/(1-t)$:

$$\binom{n}{k} = [t^{n-k}]\left\{\frac{1}{(1-t)^{k+1}}\right\} = [t^n]\left\{\frac{t^k}{(1-t)^{k+1}}\right\}.$$

The generating function of the associated sequence $\{a_i\}$ is simply $A(t) = 1 + t$, and the so-called Euler transform is derived from Theorem 1:

$$\sum_k \binom{n}{k} f_k = [t^n]\left\{\frac{1}{1-t} f\left(\frac{1}{1-t}\right)\right\}.$$

2. Combinatorial sums

THEOREM 2. *The Euler transform generalizes as*

$$\sum_k \binom{n+ak}{m+bk} f_k = [t^n]\left\{\frac{t^m}{(1-t)^{m+1}} f\left(\frac{t^{b-a}}{(1-t)^b}\right)\right\} \qquad (b > a),$$
$$= [t^m]\left\{(1+t)^n f\left((1+t)^a t^{-b}\right)\right\} \qquad (b < 0).$$

Theorem 2 applies to sums involving the Catalan numbers $C_k = \binom{2k}{k}/(k+1)$; for example

$$\sum_k \binom{n+k}{m+2k}(-1)^k C_k = \binom{n-1}{m-1}.$$

It does not apply directly to Stirling numbers of the first and second kind $\left[{n \atop k}\right]$ and $\left\{{n \atop k}\right\}$; however the simple identities

$$\sum_n \frac{k!}{n!}\left[{n \atop k}\right] t^n = \log^k \frac{1}{1-t}; \qquad \sum_n \frac{k!}{n!}\left\{{n \atop k}\right\} t^n = (e^t - 1)^k$$

allow us to use a modified form of it. For example, Theorem 2 after some algebra gives

$$\sum_k \left\{{m \atop k}\right\}\left[{k+1 \atop p}\right]\frac{(-1)^{k-p+1}}{k+1} = \frac{1}{m+1}\binom{m+1}{p} B_{m-p+1},$$

where B_n is a Bernoulli number.

3. Inversion formulæ

The *product* of two Riordan arrays $D = (d(t), h(t)) = \{d_{n,k}\}$ and $F = (f(t), g(t)) = \{f_{n,k}\}$ is the double sequence $\{g_{n,k} = \sum_j d_{n,j} f_{j,k}\}$.

PROPERTY 2. *The product of two Riordan arrays is a Riordan array:*

$$D \cdot F = (d(t)f(th(t)), h(t)g(th(t))).$$

The identity element is $I = (1, 1)$.

PROPERTY 3. *A proper Riordan array $D = (d(t), h(t))$ has an inverse $d^{-1} = \{\bar{d}_{n,k}\} = (\bar{d}(t), \bar{h}(t))$.*

As a consequence, we obtain an inversion formula for sums:

$$\sum_k d_{n,k} f_k = g_n \iff \sum_k \bar{d}_{n,k} g_k = f_n.$$

Such formulæ date back a long time; see for example Riordan's book [9]. The inversion formulæ in this book were recently revisited by Sprugnoli [11], who proved them again using the theory of Riordan arrays; see also [10].

4. Coloured walks

We consider three types of steps on a square lattice: North (N), East (E) and North-East (NE). An underdiagonal walk is entirely below or on the main diagonal $x = y$. A *weakly underdiagonal walk* is a walk such that its final point is on or below the diagonal. Let $p_{n,k}$ and $q_{n,k}$ be respectively the number of underdiagonal walks and the number of weakly underdiagonal walks, with n steps and ending at a distance k from the diagonal; the distance k can be either along the x-axis or along the y-axis.

If there is only one kind of step of each type, then we have a Motzkin walk, corresponding to Motzkin words. A generalization allows for different kinds of horizontal (E), vertical (N) or diagonal (NE) steps [7]. Let a, b and c be the number of different steps in the East, North-East and North directions; then we have the following result.

THEOREM 3. *The $\{p_{n,k}\}$ and $\{q_{n,k}\}$ are Riordan arrays such that the associated A function is $A(t) = a + bt + ct^2$, and, with $\Delta = 1 - 2bt + (b^2 - 4ac)t^2$,*

$$\{p_{n,k}\} = \left(\frac{1 - bt - \sqrt{\Delta}}{2act^2}, \frac{1 - bt - \sqrt{\Delta}}{2ct^2}\right);$$

$$\{q_{n,k}\} = \left(\frac{1}{\sqrt{\Delta}}, \frac{1 - bt - \sqrt{\Delta}}{2ct^2}\right).$$

Symmetric walks. When there is the same number of colours for East and North steps ($a = c$), then it is possible to derive some interesting identities. For example, let $\{f_k\}$ be the periodic sequence $\{1, 0, -1, 0, 1, 0, -1, \ldots\}$; then $\sum_k p_{n,k} f_k = b^n$. The algebraic proof of this equality is easy; there also exists a combinatorial interpretation: There is a bijection between the underdiagonal walks ending at distance k, whose last non-NE step is N, and the walks ending at a distance $k + 2$ whose last non-NE step is E.

5. Asymptotics for convolution matrices

The reference for this section is [8]. Let F be an analytic function s.t. $F(0) = 1$, and define $F_n(x) = [z^n]\{F(z)^x\}$. This is a polynomial function of degree n, satisfying a convolution property:

$$F_n(x + y) = \sum_{k=0}^{n} F_{n-k}(x) F_k(y).$$

The $f_{n,k} = [x^k]\{n! F_n(x)\} = [x^k z^n]\{n! F(z)^x\}$ define an infinite *convolution matrix* [1, 6]. Let $d_{n,k} := \frac{k!}{n!} f_{n,k}$; then

$$d_{n,k} = \sum \frac{k!}{k_1! k_2! k_3! \cdots} \left(\frac{f_1}{1!}\right)^{k_1} \left(\frac{f_2}{2!}\right)^{k_2} \left(\frac{f_3}{3!}\right)^{k_3} \cdots,$$

which shows that $\{d_{n,k}\}$ is a Riordan array: $\{d_{n,k}\} = (1, (\ln F(z))/z)$. In other terms, $f_{n,k} = (n!/k!)[z^n] \{(\ln F(z))^k\}$.

For a fixed ratio $p = n/k$, we have an asymptotic equivalent of $d_{n,k}$, or equivalently of $f_{n,k}$: Define $m = n - k$ and

$$\Phi_p(z) = \left(\frac{\ln F(z)}{z}\right)^{1/(p-1)} = \left(\frac{\ln F(z)}{z}\right)^{\frac{k}{m}};$$

then $d_{n,k} = [z^m]\{\Phi_p(z)^m\}$. As long as p is fixed, we can define $C(u) = \sum_m [z^m]\{\Phi_p(z)^m\} u^m$. A form of the function $C(u)$ can be computed as follows. There exists a unique analytical function

$w(z)$ s.t. $w(0) = 0$ and $w(z) = z\Phi_p(w(z))$. Define a function G by $G'(u) = 1/\Phi_p(u)$; then, by the Lagrange Inversion Formula,

$$[z^n]\{G(w(z))\} = \frac{1}{n}[u^{n-1}]\left\{G'(z)\Phi_p(u)^n\right\} = \frac{1}{n}[u^{n-1}]\left\{\Phi_p(u)^{n-1}\right\}.$$

Applying this formula backwards to $[z^m]\{\Phi_p(z)^m\}$ gives

$$d_{n,k} = [u^m]\{C(u)\} \qquad \text{with} \qquad C(u) = \frac{uw'(u)}{w(u)} = \frac{1}{1 - u\Phi_p'(w(u))}.$$

When the function Φ_p is well-behaved, an asymptotic equivalent of $d_{n,k}$ can be obtained by singularity analysis. For example, assume that the radius of convergence of $w(z)$ is finite, and that w has a single singularity r on its circle of convergence, defined as the solution of smallest modulus of the equation $z\Phi_p'(w(z)) = 1$. Let $s = w(r)$; as $s = r\Phi_p(s)$, we get that $s\Phi_p'(s) = \Phi_p(s)$. This leads finally to the asymptotic formula

(3)
$$d_{n,k} \sim \frac{\Phi_p'(s)}{\sqrt{2\Phi_p(s)\Phi_p''(s)}} \frac{\Phi_p'(s)^m}{4^m}\binom{2m}{m}.$$

If desired, this equivalent can be expanded into an asymptotic expansion to any order.

Example. Asymptotic estimates for Stirling numbers of the second kind have been obtained by several authors (see for example [12] for a survey of results and for uniform expansions); some estimates can also be derived from (3): For $f_{n,k} = \left\{ {n \atop k} \right\}$, we have $d_{n,k} = (k!/n!)\left\{ {n \atop k} \right\}$ and $\Phi_p(z) = h(z)^{1/(p-1)}$ with $h(z) = (e^z - 1)/z$. The equation defining s simplifies into $e^s = p/(p-s)$. Then $\Phi_p(s) = 1/(p-s)^{k/m}$, $\Phi_p'(s) = \Phi_p(s)/s$ and $\Phi_p''(s) = \Phi_p(s)(p(s-p+1))/(s^2(p-1))$. The asymptotic equivalent (3) gives after some computation

$$\left\{ {n \atop k} \right\} \sim \frac{n!}{k!}\sqrt{\frac{\Phi_p(s)}{2s^2\Phi_p''(s)}}\binom{2n-2k}{n-k}\left(\frac{\Phi_p(s)}{4s}\right)^m$$
$$\sim \frac{n!}{k!}\sqrt{\frac{k(n-k)}{2n(ks-n+k)}}\binom{2n-2k}{n-k}\frac{k^k}{(4s)^{n-k}(n-sk)^k}.$$

Numerical results for Stirling numbers of the second kind are then presented for given n and k, and hence p; the conditions of application of the formula (3) are not satisfied (the formula holds for *constant* p and $n \to +\infty$) but the approximations computed are very close to the actual values, which suggests that the range of application of (3) is much wider than indicated, and that some kind of uniformity result should hold.

Indeed, for a fixed ratio $p = n/k$, the asymptotic expansion given by (3) can also be obtained by a saddle-point approximation. We give here the computation for the first term of the expansion; the full asymptotic expansion can probably be obtained in a similar way. For $n - k = m \to +\infty$, we have to compute $[z^m]\Phi_p^m(z)$. The saddle-point ρ_0 is defined by the equation $z\Phi_p'(z)/\Phi_p(z) = 1$; the uniqueness of the solution on $]0, +\infty[$ shows that $\rho_0 = s$. Then

(4)
$$[z^m]\{\Phi_p^m(z)\} \sim \frac{\Phi_p(s)^m}{s^m\sqrt{2\pi m\sigma^2}} \qquad \text{with} \qquad \sigma^2 = s^2\left(\frac{\Phi_p''}{\Phi_p}(s) - \frac{\Phi_p'^2}{\Phi_p^2}(s) + \frac{\Phi_p'(s)}{s\Phi_p(s)}\right).$$

As $\Phi_p'(s)/\Phi_p(s) = 1/s$ here, σ^2 is simply $s^2\Phi_p''(s)/\Phi_p(s)$. Injecting this into Equation (4), and with $\Phi_p(s) = s\Phi_p'(s)$, we get

$$[z^m]\{\Phi_p^m(z)\} \sim \frac{\Phi_p'(s)^{m+1}}{\sqrt{2\pi m \Phi_p(s)\Phi_p''(s)}},$$

which is exactly (3) if we use Stirling's approximation for the factorial: $\binom{2m}{m}4^{-m} \sim 1/\sqrt{\pi m}$.

Thus, the approach presented in this talk can be seen as an alternative to the saddle-point approach; instead of solving the equation $s\Phi_p'(s) = \Phi_p(s)$, it leads to solving the equation $w(r) = r\Phi_p(w(r))$, which may be simpler in some cases.

To show how we can obtain asymptotic expansions for a large range of n and k, we write

$$[z^m]\{\Phi_p^m(z)\} = [z^m]\left\{f(z)^k\right\} \qquad \text{with} \qquad f(z) = \frac{\log F(z)}{z}.$$

Now $f^k(z) = \Phi_p(z)^m$ and the saddle-point approximation gives

$$[z^m]\{f^k(z)\} \sim \frac{f(\rho_1)^k}{\sigma \rho_1^m \sqrt{2\pi k}}, \tag{5}$$

for ρ_1 the (unique) real positive solution of the equation $zf'(z)/f(z) = m/k$ and

$$\sigma^2 = \rho_1^2((f''/f)(\rho_1) - (f'^2/f^2)(\rho_1) + (f'(\rho_1)/\rho_1 f(\rho_1)).$$

Now $f'/f = (m/k)(\Phi_p'/\Phi_p)$ and the saddle-point is still $\rho_1 = s$; also $\sigma^2 = (m/k)\rho_1^2(\Phi_p''/\Phi_p)(\rho_1)$; hence the equation (5) is simply another way of writing (3) or (4). However, in this last form, it is easy to understand why the approximation (3) holds for n/k no longer fixed: The equivalent approximation (5) has been proved for $m = \Theta(k)$ [2, 5] or for $m = o(k)$ [3, 4]. This indicates that the asymptotic expansion (3) is valid without restriction on $p = n/k$, as long as $n = k + O(k)$, and $m = n - k \to +\infty$.

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