Asymptotics of Mahler Recurrences: Binary Partitions Weighted by the Number of Summands

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[summary by Hsien-Kuei Hwang]

Abstract

The asymptotic behaviour of the number of binary partitions (summands being powers of two) weighted by the number of summands is investigated. The methods of proof relies on the Mellin-Perron formula.

1. Introduction

Consider the generating function

$$f(z) = \sum_{n\geq 0} a_n z^n = \prod_{k\geq 0} \frac{1}{1-\rho z^{2^k}}, \qquad (\rho > 0), \tag{1}$$

which satisfies the functional equation of Mahler type:

$$f(z)(1-\rho z) = f(z^2). \tag{2}$$

The general problem of interest here is the asymptotic behaviour of the sequence a_n, as $n \to +\infty$. Different values of ρ give rise to different behaviours of a_n. The simplest case is when $\rho > 1$. One easily deduces from (2)

$$a_n = f\left(\rho^{-2}\right)\rho^n + O\left(\rho^{n/2}\right), \qquad (n \to +\infty).$$

When $\rho = 1$, a_n represents the number of partitions of n into summands which are powers of two. Asymptotics of a_n were originally studied by C. L. Siegel and by K. Mahler, and later by N. G. de Bruijn [2]. The principal methods used by De Bruijn are Mellin transform (without explicit mention) and the saddle-point method. His result is

$$\begin{aligned}\log a_{2n} = \log a_{2n+1} = \frac{1}{2\log 2}\left(\log\frac{n}{\log n}\right)^2 + \left(\frac{1}{2} + \frac{1}{\log 2} + \frac{\log\log 2}{\log 2}\right)\log n \,-\\ -\left(1 + \frac{\log\log 2}{\log 2}\right)\log\log n + \varpi\left(\frac{\log n - \log\log n}{\log 2}\right) + O\left(\frac{(\log\log n)^2}{\log n}\right),\end{aligned}$$

as $n \to +\infty$, where $\varpi(u)$ is a periodic function of period 1 whose Fourier expansion is explicited. Another approach (with weaker error term) to similar problems by Pennington [7] proceeds along Ingham's Tauberian theorem.

The problem becomes very complex when $\rho = e^{it}$, $0 < |t| \leq \pi$, t real. A study of the various behaviours of a_n using elementary methods is contained in van der Hoeven's DEA memoir.

This talk is concerned with the case when $0 < \rho < 1$. As individual term are highly irregular, one considers the summatory function of a_n.

THEOREM 1. *Let a_n be defined in (1) with $0 < \rho < 1$ and set $F_n = \sum_{0 \le k \le n} a_k$. Then F_n satisfies*

$$F_n = P(\log_2 n) n^{\alpha} + O\left(n^{\alpha+\varepsilon-1/2}\right), \qquad (\varepsilon > 0), \tag{3}$$

as $n \to +\infty$, where $\alpha = -\log(1-\rho)/(\log 2) > 0$ and $P(u)$ is a periodic function of u whose mean value is approximately given by

$$\frac{e^{-\lambda/\log 2}}{\Gamma(\alpha+1)}(1-\rho)^{\gamma/(\log 2)-1/2}, \qquad \textit{where} \qquad \lambda = \sum_{k \ge 1} \frac{\log k}{k} \rho^k.$$

2. Proof

To derive (3), one starts from the Mellin-Perron formula:

$$\sum_{1 \le j \le n} a_j = \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} \frac{n^{s+1}}{s} \varphi(s) \mathrm{d}s, \tag{4}$$

where c is taken to be larger than the abscissa of absolute convergence of the Dirichlet series

$$\varphi(s) := \sum_{j \ge 1} a_j j^{-s}.$$

To apply (4), one requires the analytic continuation of φ to a larger half-plane (than its original domain of analyticity) and the magnitude of growth of the continued function at $\sigma \pm i\infty$.

The abscissa of convergence of φ is determined by the growth order of F_n, cf. [8, §9.14]. From the defining equation (1), one readily obtains the recurrence

$$\begin{cases} a_0 &= 1; \\ a_n &= \rho a_{n-1} + a_{n/2}, \end{cases} \tag{5}$$

with the convention that $a_x = 0$ when $x \notin \mathbb{Z}$. From this, one deduces the following relations for F_n,

$$\begin{cases} F_0 &= 1; \\ F_n &= \rho F_{n-1} + F_{n/2} + F_{(n-1)/2}, \end{cases}$$

again with the convention that $F_x = 0$ when $x \notin \mathbb{Z}$. From this last recurrence, one can verify, by induction, that

$$F(n) = O\left(n^{\alpha}\right).$$

Thus, by [8, §9.15], the abscissa of absolute convergence of the Dirichlet series $\varphi(s)$ is not greater than α. A probabilistic argument concerning the distribution of the number of summands in a binary partition permits to show that the abscissa of absolute convergence of $\varphi(s)$ is in fact less than α.

The analytic continuation of φ can be computed by (5) and the technique used in [1]. One thus obtains

$$(1 - \rho - 2^{-s})\varphi(s) = \rho + \rho g(s),$$

where

$$g(s) = \sum_{j \ge 1} a_j \left(j^{-s} - (j+1)^{-s}\right) = \sum_{k \ge 1} \binom{s+k-1}{k} (-1)^{k+1} \varphi(k+s).$$

The second expression provides the required analytic continuation of φ to the whole s-plane.

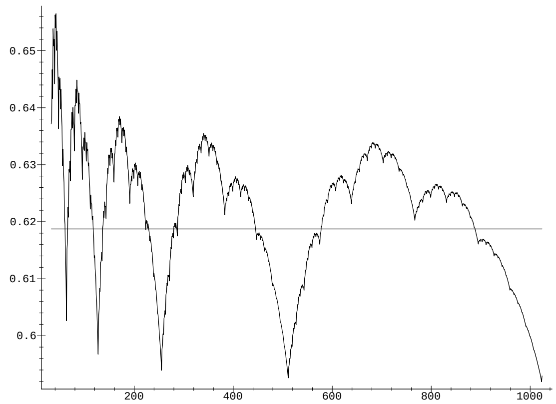

FIGURE 1. The reduced sequence f_n/n^α associated to $\rho = 1/2$ contrasted with the mean value of the periodic function $P(v)$.

The remaining analysis follows closely the lines for the number of odd numbers in Pascal triangle in [4]. The expression for the approximate mean-value of $P(u)$ is obtained by Mellin transform starting from

$$\log f(e^{-t}) = \sum_{j\geq 0} \log \frac{1}{1-\rho e^{-2^j t}} \sim -\alpha \log t + p(\log_2 t) + q(t),$$

as $t \to 0^+$, where p is a 1-periodic function and q is an entire function. The key point is the strong correspondence (see [5]) between the asymptotic behaviour of a function in the vicinity of 0 and the singularities of its Mellin transform. This provides a precise estimation for the residues of the Mellin transform of $f(e^{-t})$, which is $\Gamma(s)\varphi(s)$. The theorem of residues is then applied, hence the theorem stated. See Figure 1 for an illustration.

3. Concluding remarks

The number of unrestricted partitions (whose summands are positive integers) weighted by the number of summands has generating function

$$\sum_{n\geq 0} p(n) z^n = \prod_{k\geq 1} \frac{1}{1-\rho z^k}.$$

The corresponding asymptotics have been thoroughly studied in the literature. The case $\rho = 1$ leads to the famous Hardy-Ramanujan-Rademacher formula, and other cases were completed by Wright [9].

The methods presented in this talk, adapted from [4], become more or less standard and are powerful enough to be applicable to other problems, like q-multiplicative functions [6], the Goldberg problem of determining the asymptotic behaviour of the coefficients

$$[z^n] \exp\left(\sum_{j\geq 0} z^{2^j}\right),$$

divide-and conquer recurrences, etc.

Bibliography

[1] Allouche (Jean-Paul) and Cohen (Henri). – Dirichlet series and curious infinite products. *Bulletin of the London Mathematical Society*, vol. 17, 1985, pp. 531–538.

[2] de Bruijn (N. G.). – On Mahler's partition problem. *Indagationes Mathematicae*, vol. 10, 1948, pp. 210–220. – reprinted from *Koninklijke Nederlandse Academie van Wetenschappen, Proceedings.*

[3] Dumas (Ph.). – Asymptotics of Mahlerian recurrences: binary partitions weighted by the number of summands, 1996. To appear.

[4] Flajolet (P.), Grabner (P.), Kirschenhofer (P.), Prodinger (H.), and Tichy (R. F.). – Mellin transforms and asymptotics: digital sums. *Theoretical Computer Science*, vol. 123, n° 2, 1994, pp. 291–314.

[5] Flajolet (Ph.), Gourdon (X.), and Dumas (Ph.). – Mellin transforms and asymptotics : Harmonic sums. *Theoretical Computer Science*, vol. 144, n° 1–2, June 1995, pp. 3–58.

[6] Grabner (P.). – Completely q-multiplicative functions: the Mellin transform approach. *Acta Arithmetica*, vol. 65, 1993, pp. 85–96.

[7] Pennington (W. B.). – On Mahler's partition problem. *Annals of Mathematics*, vol. 57, 1953, pp. 531–546.

[8] Titchmarsh (E. C.). – *The theory of functions.* – Oxford University Press, Oxford, 1939, second edition.

[9] Wright (E. M.). – Asymptotic partition formulae: (II) weighted partitions. *Proceedings of the London Mathematical Society*, vol. 36, 1933, pp. 117–141.