Average Case Analysis of Tree Rewriting Systems

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Abstract
A general technique is presented to easily compute the order of the average complexity of a tree rewriting system from its matrix representation. It can be used for example to prove that the average cost of the k-th differentiation is of order \( n^{1+k/2} \).

1. Introduction

We aim at studying the order of the average complexity of regular tree rewriting systems. We deal with simple families of trees \( T \), in the sense of Meir and Moon [4]. The corresponding generating function (GF) is defined by \( T(z) = z\phi(T(z)) \) where \( \phi(y) \) is a polynomial whose \( n \)-th coefficient is the number of constructors of arity \( n \). For example, the GF of binary trees is defined by \( T(z) = z(1 + T^2(z)) = z\phi(T(z)) \) with \( \phi(y) = 1 + y^2 \).

Asymptotics of \( T(z) \). We define \( \tau > 0 \) as the solution of \( \tau\phi'(\tau) - \phi(\tau) = 0 \) and we denote \( \rho = \tau/\phi(\tau) = 1/\phi'(\tau) \). Then \( \rho \) is the dominant singularity of \( T(z) \) with the Puiseux expansion

\[
T(z) = \tau - \sqrt{\frac{2\phi(\tau)}{\phi'(\tau)}} \left(1 - \frac{z}{\rho}\right)^{1/2} + \sum_{n \geq 2} d_n \left(1 - \frac{z}{\rho}\right)^{n/2}.
\]

From singularity analysis, we deduce the estimate of the \( n \)-th coefficient of \( T(z) \)

\[
[z^n]T(z) = \sqrt{\frac{\phi(\tau)}{2\pi\phi'(\tau)}} \rho^{-n} n^{-3/2} \left(1 + O\left(\frac{1}{n}\right)\right).
\]

An example: differentiation of trees. A typical example of a tree rewriting system is formal differentiation. We describe the action of the differentiation and copy operators on trees constructed with a binary constructor \( \ast \) and a variable \( a \)

\[
d(a) \to a \\
d(u \ast v) \to d(u) \ast cp(v) + d(v) \ast cp(u) \\
\text{cp}(a) \to a \\
\text{cp}(u \ast v) \to \text{cp}(u) \ast \text{cp}(v)
\]

If \( B(z) \) denotes the GF of binary trees, this translates in terms of cost generating functions in the form

\[
C_d(z) = B(z) + 2zB(z)C_d(z) + 2zB(z)C_{cp}(z),
\]

\[
C_{cp}(z) = B(z) + 2zB(z)C_{cp}(z).
\]
Equivalently, we have the matrix representation

\[
\begin{pmatrix}
C_d(z) \\
C_{cp}(z)
\end{pmatrix}
= \begin{pmatrix}
2zB(z) & 2zB(z) \\
0 & 2zB(z)
\end{pmatrix}
\begin{pmatrix}
C_d(z) \\
C_{cp}(z)
\end{pmatrix}
+ \begin{pmatrix}
B(z) \\
0
\end{pmatrix}.
\]

The solution is

\[
C_d(z) = \frac{B(z)}{(1 - 2zB(z))^2} = \frac{B(z)(1 + B^2(z))}{(1 - B(z))^2(1 + B(z))^2}, \quad C_{cp}(z) = \frac{B(z)(1 + B^2(z))}{(1 - B(z))(1 + B(z))}.
\]

Since \( B(z) = z\phi(B(z)) \) with \( \phi(w) = 1 + w^2 \), \( \tau \) and \( \rho \) are easily determined: \( \tau = 1, \rho = 1/2 \).

**Average complexity.** The cost GF \( C_{cp}(z) \) writes as \( F(B(z)) \), where \( F(w) \) is a rational function. The dominant pole of \( F(w) \) is \( w = \tau \) and it is simple. An application of transfer lemma of singularity analysis [2] then leads to the estimate \( \tilde{C}_{cp}^n \sim c_1 n \) with \( c_1 > 0 \) for the cost of the copy operator over trees of size \( n \). As for the cost GF \( C_d(z) \), it writes as a rational functional in \( B(z) \) with the double dominant pole \( \tau \), and we deduce an average asymptotic value of the form \( \tilde{C}_d^2 \sim c_2 n^{3/2}, c_2 > 0 \).

**2. Regular rewriting systems**

The matrix representation (1) for the cost GF’s can be generalised for all regular rewriting systems [1].

**Theorem 1 (Matrix representation for regular rewriting systems).** The cost GF’s of operators \( f_1, \ldots, f_n \) of a regular system satisfy a system of the form

\[
\begin{pmatrix}
C_{f_1}(z) \\
\vdots \\
C_{f_n}(z)
\end{pmatrix}
= \begin{pmatrix}
M(z,T(z)) \\
\vdots \\
M(z,T(z))
\end{pmatrix}
\begin{pmatrix}
C_{f_1}(z) \\
\vdots \\
C_{f_n}(z)
\end{pmatrix}
+ \begin{pmatrix}
T^{\tau_1}(z) \\
\vdots \\
T^{\tau_n}(z)
\end{pmatrix},
\]

where the \( \tau_i \) are the arities of the \( f_i \)’s, and where the coefficient of the square matrix \( M(z,T(z)) \) are polynomials in \( z \) and \( T(z) \) with non negative coefficients.

Thus, the expression of each cost GF is

\[
C_{f_i}(z) = \frac{\det^{[i]}(\Id - M(z,T(z)))}{\det(\Id - M(z,T(z)))},
\]

where \([i]A\) denotes the matrix in which the \( i \)-th column of \( A \) has been substituted by the rightmost vector of equation (2). We deduce, since \( z = T(z)/\phi(T(z)) \), that \( C_{f_i}(z) \) writes as

\[
C_{f_i}(z) = \frac{P_i(T(z))}{Q_i(T(z))},
\]

where \( P_i(w) \) and \( Q_i(w) \) are polynomials. The average complexity of the operator \( f_i \), defined by

\[
\overline{C}_n^{f_i} = \frac{[z^n]C_{f_i}(z)}{[z^n]T^{\tau_i}(z)},
\]

is determined by the relative position of \( \rho \) with respect to the smallest positive solution \( \rho_{0,i} \) of \( Q_i(T(\rho_{0,i})) = 0 \) (see [1]).

**Theorem 2 (Average cost estimate).** The average cost satisfies

(i) If \( Q_i(T(z)) \) does not vanish on \((0, \rho]\), then \( \overline{C}_n^{f_i} = c_1(1 + O(1/n)) \);

(ii) if \( \rho = \rho_{0,i} \), then \( \overline{C}_n^{f_i} = c_2 n^{k/2}(1 + O(1/\sqrt{n})) \);
(iii) if $\rho > \rho_{0,i}$, then $C_{II} = c_3(\rho/\rho_{0,i})^n n^{r+1/2}(1 + O(1/\sqrt{n}))$, with $c_j > 0$ and $k$, $q$ positive integers.

In the case $\rho = \rho_{0,i}$, we have $k = s + 1$ where $s$ is the multiplicity of the factor $(T(z) - \tau)$ in $Q_i(T(z))$.

3. Computation of the order of the average cost

It is possible to derive directly from the matrix $M(z, T(z))$ the order of the average cost of the operators of a regular system. The substance relies on Frobenius theory of matrices with nonnegative coefficients (see for instance [5]). The general technique proceeds as follows. First, decompose $M(z, T(z))$ into diagonal blocks of irreducible matrices (Definition 1), then work on each irreducible block.

3.1. Irreducible matrix case.

**Definition 1.** A square matrix $M$ is irreducible if there does not exist any permutation matrix $P$ such that $P^{-1}MP = (A \quad 0 \quad 0)$ with $A$, $B$ and $C$ square matrices.

In other terms, an irreducible matrix is associated to a strongly connected graph. If the matrix $M(z, T(z))$ is irreducible, the order of the average complexity of the operators are easily found.

**Theorem 3.** Let $\{f_i\}$ be a set of operators of a regular rewriting system represented by an irreducible matrix $M(z, T(z))$. Then all the $\rho_{0,i}$ are equal to the smallest positive root $\rho_0$ of the equation $\det(\text{Id} - M(z, T(z))) = 0$ (take $\rho_0 = +\infty$ if there is no positive solution). The relative position of $\rho_0$ with respect to $\rho$ is determined from the dominant eigenvalue $r(\rho, \tau)$ of $M(\rho, \tau)$. We have

$$r(\rho, \tau) < 1 \text{ iff } \rho_0 > \rho, \quad r(\rho, \tau) = 1 \text{ iff } \rho_0 = \rho, \quad r(\rho, \tau) > 1 \text{ iff } \rho_0 < \rho.$$  

When $r(\rho, \tau) = 1$, or equivalently $\rho_0 = \rho$, it is possible to get the exponent of $n$ in the estimate (ii) of Theorem 1. This is the polynomial case.

**Theorem 4.** Let $\{f_i\}$ be a set of operators of a regular rewriting system represented by an irreducible matrix $M(z, T(z))$. If the dominant eigenvalue of $M(\rho, \tau)$ is 1, then the $f_i$’s have an average complexity which is linear or of order $n^{3/2}$.

The case $n^{3/2}$ occurs only in the degenerate case where $M(z, T(z))$ does not depend on $T(z)$.

3.2. General case. In the general case, we start by finding a permutation matrix $P$ such that $P^{-1}MP$ writes as a block diagonal matrix, each block being of the form

$$B = \begin{pmatrix}
A_1 & 0 \\
\vdots & \ddots \\
0 & \cdots & A_k
\end{pmatrix},$$

where each $A_i = A_i(z, T(z))$ is an irreducible square block. We also need the constraint that for all $i < j$, the submatrix of $B$ whose lines are those of $A_j$ and columns are those of $A_i$ is not zero. Considering the graph represented by the matrix $M$, this task can be achieved thanks to Tarjan algorithm on strongly connected components (see [3, pp. 441–448] for example).

Now, each block of the form $B$ can be considered independently. Let $C_{f_i}(z)$ be a cost GF associated to an irreducible square block $A_i$. Expression (3) together with Theorem 3 show that the position of $\rho_{0,i}$ is the smallest positive root of $\prod_{i=1}^k \det(\text{Id} - A_i)$. Thus, if $\rho_{k}$ denotes the smallest positive root of $\det(\text{Id} - A_i)$, for each $\ell$, we need to compare $\rho_{k}$ with $\min_{1 \leq i \leq \ell-1} \rho_{i}$ in order to get
the order of the average complexity of the operators \( f_i \). In fact, Theorem 3 asserts that this task can be achieved by comparing only the dominant eigenvalues \( r_i(\rho, \tau) \) of the \( A_i(\rho, \tau)'s.\)

In the polynomial case, the multiplicity of the factor \( (T(z) - \tau) \) in the denominator of the cost GF is obtained by adding the multiplicities of this factor in the determinants \( \det(\Id - A_i) \), yielding the exponent of \( n \) in equation (ii) of theorem 2.

3.3. Examples.

Tree shuffle. We consider binary trees \( B = a + o(B, B) \) and operators \( f \) and \( g \) defined on \( B^2 \) by

\[
\begin{align*}
    f(a, a) &\rightarrow a & f(o(u, v), a) &\rightarrow o(u, v) & f(a, o(u, v)) &\rightarrow g(u, v) \\
    g(a, a) &\rightarrow a & g(o(u, v), a) &\rightarrow o(u, v) & g(a, o(u, v)) &\rightarrow g(u, v)
\end{align*}
\]

\[
\begin{align*}
    f(o(u_1, v_1), o(u_2, v_2)) &\rightarrow o(f(u_1, u_2), f(v_1, v_2)) \\
    g(o(u_1, v_1), o(u_2, v_2)) &\rightarrow o(f(u_1, u_2), g(v_1, v_2))
\end{align*}
\]

The matrix representation of the shuffle is

\[
M(z, T(z)) = \begin{pmatrix}
2z^2T^2(z) & 2z^2 \\
2z^2T^2(z) & z^2T^2(z) + 2z^2
\end{pmatrix}.
\]

This matrix is irreducible. The eigenvalues of \( M(\rho, \tau) \) are 1 and 1/4, thus we are in the polynomial case with a linear average complexity of the operators \( f \) and \( g \).

Formal differentiation. The classical formal double differentiation on unary-binary trees \( T \) with constructors \(*\), \( \exp \) and a variable has the matrix representation

\[
M(z, T(z)) = \begin{pmatrix}
z + 2zT(z) & 0 & 0 \\
2zT(z) + z & z + 2zT(z) & 0 \\
2zT(z) + 2z & 4zT(z) + 2z & z + 2zT(z)
\end{pmatrix}.
\]

The diagonal coefficients of \( M(\rho, \tau) \) are only 1’s, thus we are in the polynomial case with three blocks of irreducible matrices on the diagonal, all giving a contribution to the order of the complexity. Thus, the average complexity of the double differentiation operator is \( cn^2 \). By induction on \( k \), it can be proved that the average cost of the \( k \)-th differentiation is of order \( n^{k/2+1} \).

Bibliography


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