Introduction to $q$-calculus

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Abstract

Many mathematical formulæ can be generalised by adding a new parameter $q$, leading to what is called a $q$-analogue, because the original formula can be obtained as the limit when $q$ tends to 1. We present here a combinatorial introduction to the $q$-calculus.

1. Partitions and words

1.1. Partitions. A partition $\lambda$ is a decreasing sequence $(\lambda_1, \ldots, \lambda_k)$ of positive integers: $\lambda_i \in \mathbb{N}^*$ and $\lambda_i \geq \lambda_{i+1}$ for all $i$. The length of $\lambda$ is $\ell(\lambda) = k$, its height is $|\lambda| = \sum_{i=1}^{k} \lambda_i$. If $|\lambda| = n$, we say that $\lambda$ is a partition of $n$. For all $\ell, m \in \mathbb{N}$, let

$$P(\ell, m) = \{ \lambda : \ell(\lambda) = \ell \quad \text{and} \quad \lambda_i \leq m \}.$$ 

It is possible to determine a partition $\lambda$ from the numbers $m_i = \text{Card}\{j : \lambda_j = i\}$ denoting the multiplicity of $i$ in $\lambda$. In this way, the partition $\lambda$ can be written as $\lambda = (1^{m_1}, 2^{m_2}, \ldots)$.

A nice way to represent a partition $\lambda$ is to use its Ferrers diagram

$$D_\lambda = \{(i,j) \in \mathbb{Z}^2 : 1 \leq i \leq \lambda_j \quad \text{and} \quad 1 \leq j \leq \ell(\lambda)\}.$$ 

The number $P(\ell, m)$ can be viewed as $P(\ell, m) = \{ \lambda : D_\lambda \subset (m^\ell) \}$. The conjugate partition of $\lambda$ is the partition $\lambda'$ whose Ferrers diagram is symmetric from $D_\lambda$ with respect to the first bisecting line. We have $|\lambda'| = |\lambda|$ and $(\lambda')' = \lambda$.

1.2. Gaussian polynomials. Ferrers diagrams enable to establish a correspondence between partitions of $P(\ell, m)$ and paths joining $(0, \ell)$ to $(m, 0)$ with the steps $(0, -1)$ and $(1, 0)$. Such paths have $m + \ell$ steps ($m$ horizontal and $\ell$ vertical) so $\text{Card} P(\ell, m) = \binom{m + \ell}{m}$. To take into account the height in this statistic, we introduce its generating function with respect to a new variable $q$. We have

$$\sum_{\lambda \in P(\ell, m)} q^{|\lambda|} = \frac{(q)_m q^\ell}{(q)_\ell} \quad \text{where} \quad \begin{cases} (q)_k = \prod_{i=1}^{k}(1 - q^i) & k \geq 1, \\
(q)_0 = 1. \end{cases}$$  

Letting $q \to 1$ in this identity, we find again $\text{Card} P(\ell, m) = \binom{m + \ell}{m}$. This motivates the definition of a $q$-analogue of the binomial coefficients, denoted by

$$\binom{m + \ell}{\ell} = \frac{(q)_m q^\ell}{(q)_\ell}.$$  

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and called Gaussian polynomials. They satisfy several $q$-properties like Pascal recurrences or symmetry.

By letting $\ell \to \infty$ in identity (1), we get

\[
\sum_{\lambda: \lambda_1 \leq m} q^{\lambda_1} = \frac{1}{(q)_m} = \sum_{\lambda: a(\lambda) \leq m} q^{\lambda_1}.
\]

This last equality is obtained from the conjugate partitions. Then letting $m \to \infty$, we find

\[
\sum_{\lambda} q^{\lambda_1} = \sum_{n \geq 0} p(n) q^n = \frac{1}{(q)_{\infty}},
\]

where $p(n)$ is the total number of partitions of $n$.

1.3. Infinite products. Formula (2) can be refined by introducing a new variable $x$. More precisely, denoting

\[
(x)_\infty = \prod_{i=0}^{\infty} (1 - x q^i),
\]

we have the identity

\[
\frac{1}{(x)_\infty} = \sum_{m_0, m_1, \ldots} \left( \prod_{i=0}^{\infty} x^{m_i} q^{m_i} \right) = \sum_{\ell \geq 0} \sum_{\lambda: a(\lambda) \leq \ell} q^{\lambda_1} = \sum_{\ell \geq 0} \frac{x^\ell}{(q)_{\ell}}.
\]

In the same vein, by expanding $(-x)_\infty$ we have

\[
(-x)_\infty = \sum_{\ell \geq 0} \frac{q^{\ell}}{(q)_{\ell}} x^\ell.
\]

These two identities are sometimes called Euler identities.

Jacobi identity. The triple product identity of Jacobi is

\[
(q)_\infty (x)_\infty (qx^{-1})_\infty = \sum_{n \in \mathbb{Z}} (-1)^n x^n q^{\binom{n}{2}}.
\]

As a corollary, we have the formulæ

\[
(q)_\infty = \sum_{n \in \mathbb{Z}} (-1)^n q^{n(3n+1)/2}
\]

(4)

(5)

\[
(q)^3 = \sum_{n \in \mathbb{R}} (-1)^n (2n + 1) q^{n(n+1)/2}.
\]

The first one is Euler’s pentagonal number theorem, and can be used with (3) to establish several congruences relations satisfied by the partition numbers $p(n)$.
2. Words

2.1. Correspondence between partitions and binary words. Consider \( \lambda \in P(\ell, m) \). Its Ferrers diagram can be considered as a path joining points \((0, \ell)\) and \((m, 0)\) by \(m\) horizontal steps and \(\ell\) vertical steps. We encode this path with a word on \(\{1, 2\}^*\), associating a 1 for each vertical step, a 2 for each horizontal step (see figure 1). This construction defines a correspondence between \(P(\ell, m)\) and \(M(\ell, m)\), the words of \(\{1, 2\}^*\) with \(\ell\) “1” and \(m\) “2”.

We define the number of inversions of a word \(w\) in \(\{1, 2\}^*\) by

\[
\text{Inv } w = \text{Card}\{(i, j): 1 \leq i < j \leq \ell + m \text{ and } 2 = w_i > w_j = 1\}.
\]

We have \(\text{Inv } w = |\lambda|\), where \(w\) is the word obtained from \(\lambda\) by the correspondence, thus

\[
\sum_{w \in M(\ell, m)} q^{\text{Inv}(w)} = \begin{bmatrix} m + \ell \\ \ell \end{bmatrix}.
\]

Another interesting parameter is the major index defined by

\[
\text{Maj } w = \sum_{w_i > w_{i+1}} i,
\]

and surprisingly, its generating function is the same as the one of \(\text{Inv}\).

2.2. Statistics on words over \(n\) letters. The previous discussion finds a natural generalisation by considering \(M(a_1, \ldots, a_n)\), the set of words with \(n\) letters where the \(i\)-th letter appears exactly \(a_i\) times. The length of such a word \(w\) is \(a_1 + \cdots + a_n\). The parameters \(\text{Inv } w\) and \(\text{Maj } w\) are defined as in (6) and (7). The \(Z\)-statistic (called like this because of Zeilberger work [2]) of a word \(w\) is defined as

\[
z(w) = \sum_{1 \leq i < j \leq n} \text{Maj } w_{i,j}
\]

where \(w_{i,j}\) is the word obtained from \(w\) by keeping only the \(i\)-th and the \(j\)-th letter. These parameters satisfy

\[
\sum_{w \in M(a_1, \ldots, a_n)} q^{\text{Inv}(w)} = \sum_{w \in M(a_1, \ldots, a_n)} q^{\text{Maj}(w)} = \sum_{w \in M(a_1, \ldots, a_n)} q^z(w) = \begin{bmatrix} a_1 + \cdots + a_n \\ a_1, \ldots, a_n \end{bmatrix} = \frac{(q)_{a_1 + \cdots + a_n}}{(q)_{a_1} \cdots (q)_{a_n}},
\]

providing a \(q\)-analogue of multinomial coefficients.
3. Basic hypergeometric functions

We use the notations

\[(a)_\infty = (a; q)_\infty = \prod_{i=0}^{\infty} (1 - aq^i), \quad (a)_n = (a; q)_n = \frac{(a)_\infty}{(aq^n)_\infty}\]

and we define the basic hypergeometric series as

\[\phi_r\left( \begin{array}{c} \alpha_1, \ldots, \alpha_r \\ \beta_1, \ldots, \beta_s \end{array} ; x \right) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_r)_n}{(\beta_1)_n \cdots (\beta_s)_n} x^n.\]

The \(q \to 1\) limit in this expression leads to a classical hypergeometric series, thus we have defined a \(q\)-analogue of hypergeometric series. A good survey of basic hypergeometric series can be found in [3].

3.1. The \(q\)-binomial theorem. The relation \((1 - x)_1 \phi_0 (a; x) = (1 - ax)_1 \phi_0 (a; qx)\) together with \(1 \phi_0 (a; 0) = 1\) leads to the \(q\)-binomial theorem:

\[(8) \quad 1 \phi_0 (a; x) = \frac{(ax)_\infty}{(x)_\infty}.\]

By setting \(a = q^n\) then \(x = -q^{-n}\), we deduce

\[\prod_{i=0}^{n-1} (1 + q^ix) = \sum_{k=0}^{n} \binom{n}{k} q^{k^2} x^k.\]

When \(q \to 1\), this leads to the classical binomial theorem.

3.2. Heine transforms. Like classical hypergeometric functions, the basic hypergeometric functions satisfy several identities. A first family is the Heine transforms:

\[2 \phi_1\left( \begin{array}{c} \alpha, \beta \\ \gamma \end{array} ; x \right) = \frac{(\beta)_\infty (ax)_\infty}{(\gamma)_\infty (x)_\infty} 2 \phi_1\left( \begin{array}{c} \gamma/\beta, x \\ \alpha x \end{array} ; \beta \right)\]

\[2 \phi_1\left( \begin{array}{c} \alpha, \beta \\ \gamma \end{array} ; x \right) = \frac{(\alpha \beta x)_\infty}{(x)_\infty} 2 \phi_1\left( \begin{array}{c} \gamma/\alpha, \gamma/\beta \\ \gamma \end{array} ; \alpha \beta x \right).\]

3.3. Pfaff-Saalschütz \(q\)-theorem. This result applies to functions of the type \(3 \phi_2\). For all non-negative integer \(n\), we have

\[(9) \quad \phi_2\left( \begin{array}{c} a, b, q^{-n} \\ c^{1-n}, c^{1-n} \end{array} ; q \right) = \frac{(c/a)_n (c/b)_n}{(c)_n (c/cab)_n}.\]

There exists several equivalent forms of this theorem. By letting \(n \to +\infty\) in (9), we obtain the Gauss \(q\)-theorem

\[2 \phi_1\left( \begin{array}{c} a, b \\ c \end{array} ; x \right) = \frac{(c/a)_\infty (c/b)_\infty}{(c)_\infty (c/cab)_\infty}.\]

Another corollary of the Pfaff-Saalschütz \(q\)-theorem is the \(q\)-formula of Chu-Vandermonde, obtained by setting \(a = q^{n+1}, b = q^{-k}\) and \(c = q^{n+1}\) in (9)

\[\sum_{i=0}^{k} (-1)^i q^{i(1-1/2+n-n)i} \binom{n+i}{i} \binom{m+k}{k-i} = \binom{k+m-n-1}{k}.\]
There exists several generalizations of the Pfaff-Saalschütz $q$-theorem. One is called the Dougall $q$-theorem, it applies to functions of the type $s\phi_7$.

4. $q$-analogues of usual tools

4.1. $q$-derivative. The $q$-derivative of a function $f$ is defined as
\[ \delta_q f(t) = \frac{f(t) - f(qt)}{(1 - q)t}. \]
The formulae of the classical derivative have their $q$-analogues with respect to the $q$-derivative.

4.2. $q$-integration. The function $g(t) = \int_{0}^{t} f(x) \, dqx$ must satisfy $\delta_q g = f$, so
\begin{align*}
g(t) - g(qt) &= t(1 - q)f(t) \\
g(qt) - g(q^2t) &= qt(1 - q)f(qt)
\end{align*}
\[ \ldots = \ldots \]
thus $g(t) = g(t) - g(0) = \sum_{n \geq 0} q^n t (1 - q) f(q^n t)$, and we define
\[ \int_{0}^{t} f(x) \, dqx = t(1 - q) \sum_{n \geq 0} q^n f(q^n t). \]
Like the classical integral, there exists a $q$-formula of integration by parts. There are several ways of defining an improper integral, for example
\[ \int_{0}^{+\infty} f(t) \, dq t = (1 - q) \sum_{n \in \mathbb{N}} q^n f(q^n) \quad \text{and} \quad \int_{0}^{+\infty} f(t) \, dq t = \int_{0}^{1/(1-q)} f(t) \, dt \]
are $q$-analogues of $\int_{0}^{+\infty} f(t) \, dt$.

4.3. $q$-differential equations. The $q$-differential equation $\delta_q f(t) = f(t)$ admits the solution
\[ f(t) = \frac{f(qt)}{1 - t(1 - q)} = \cdots = \frac{f(t)}{(1 - t(1 - q))_\infty}, \]
thus the solution $f$ with $f(0) = 1$ is
\[ e_q(t) = \frac{1}{(1 - t(1 - q))_\infty} = \sum_{n \geq 0} (1 - q)^n (q)_n t^n, \]
(the last identity is obtained from the $q$-binomial theorem (8) with $a = 0$ and $x = t(1 - q)$) providing a $q$-analogue of the expansion of $\exp(t)$.

As for the $q$-differential equation $\delta_q f(t) = f(qt)$, the solution which takes the value 1 at 0 is
\[ E_q(t) = -t(1 - q)_\infty = \sum_{n \geq 0} q^n (1 - q)^n (q)_n t^n, \]
the last identity being a consequence of the $q$-binomial theorem applied with $a = -t(1 - q)/x$ and $x \to 0$. This second $q$-analogue of the expansion of $\exp(t)$ satisfy the obvious relation $e_q(t) E_q(-t) = 1$. Nevertheless, there does not exist any simple relation between $e_q(x) e_q(y), E_q(x) E_q(y)$ and
$e_q(x + y), E_q(x + y)$. A $q$-analogue of the relation $\exp(x + y) = \exp(x)\exp(y)$ is given by the formula

$$e_q(x)E_q(y) = \sum_{n=0}^{\infty} \frac{1}{\prod_{k=0}^{n-1} (1 - q^{-k})} \frac{(q)_{n-1}}{(1 - q)^n} (x + q^k y),$$

obtained from the $q$-binomial theorem with $a = -y/x$ and $x = x(1 - q)$.

4.4. The $q$-gamma function. The $q$-gamma function is defined as

$$\Gamma_q(s) = \frac{(q)_{\infty}}{(q^s)^{\infty}} (1 - q)^{1-s}, \quad s \in \mathbb{C} \setminus \{0, -1, -2, \ldots\},$$

which tends to $\Gamma(s)$ as $q \to 1$. The functional equation of $\Gamma_q$ is

$$\Gamma_q(s + 1) = \frac{1 - q}{1 - q} \Gamma_q(s),$$

and since $\Gamma_q(1) = 1$, we have for all positive integer $n$

$$\Gamma_q(n + 1) = \prod_{k=1}^{n} \frac{1 - q^k}{1 - q} = \frac{(q)_n}{(1 - q)^n},$$

which is a $q$-analogue of $\Gamma(n + 1) = n!$. The function $\log \Gamma_q(x)$ is convex for $x > 0$. An integral representation of $\Gamma_q$ is

$$\Gamma_q(s) = \int_0^{1/(1-q)} t^{s-1} E_q(-qt) \, dq.$$

There also exists a $q$-analogue of the Gauss duplication formula.

4.5. The $q$-beta function. An equivalent form of the $q$-binomial theorem is

$$\int_0^1 t^{x-1} \frac{(qt)_{\infty}}{(q^x t)_{\infty}} \, dq = \frac{\Gamma_q(x)\Gamma_q(y)}{\Gamma_q(x + y)}, \quad (\Re(x) > 0).$$

This expression can be used to define the most well known and the most useful $q$-analogue of the beta function.

Bibliography

