Mellin Transforms and Asymptotics: Harmonic Sums

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[summary by Hsien-Kuei Hwang]

Abstract

This talk gives a general introduction to the asymptotic study of harmonic sums arising in many concrete applications, especially the analysis of algorithms.

1. Introduction

Mellin transform is a precious tool in analytic number theory and in algorithmic analysis as the growth order of the quantity involved is usually polynomial. Given a locally integrable function $f(t)$ on $(0, \infty)$, the Mellin transform $M[f(t); s]$ of f is defined by [11, Ch. III]

$$M[f(x); s] := \int_0^\infty x^{s-1} f(x)\, dx,$$

whenever the integral converges. An essential feature of the function $M[f(x); s]$ is that its domain of analyticity is usually an infinite strip $-\alpha < \Re s < \beta$, the two boundaries $-\alpha, \beta$ being determined, respectively, by the asymptotic behaviours of $f(x)$ when the parameter $x \to 0^+$ and $x \to \infty$. More precisely,

$$\alpha = \sup\{a : f(x) = O(x^a), x \to 0^+\}, \quad \text{and} \quad \beta = \sup\{b : f(x) = O(x^{-b}), x \to \infty\}.$$

Thus the inversion formula

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{-s} M[f(t); s]\, ds \qquad (-\alpha < c < \beta) \tag{1}$$

offers the flexibility of capturing the asymptotic behaviours of $f(x)$ as x gets small or large by merely shifting the line of integration (here $\Re s = c$) to the left or to the right, respectively, and by collecting the contributions of the singularities encountered (usually poles).

Globally, the important step of "transforming" the given source (function, sequence, etc.) by considering either the associated weighted sums (ordinary/exponential generating function, Dirichlet series, factorial series, etc) or weighted integrals (Laplace, Fourier, Mellin, Hilbert, etc) has the effect of smoothing our "raw data" which becomes more manageable, at least from an analytic point of view.

In view of illustrating the application of Mellin transform to harmonic sums, this talk is mainly example-oriented.

2. Basic Properties

Let $f, M[f;s], \alpha, \beta$ be as in the previous section. Then, cf. [11, Ch. III],

(1) [analyticity] $M[f;s]$ is analytic in the *fundamental strip* $-\alpha < \Re s < \beta$;

(2) [harmonic sums] For any sequences $\{\lambda_k\}$ and $\{\mu_k\}$,

$$M\left[\sum_k \lambda_k f(\mu_k t); s\right] = \left(\sum_k \lambda_k \mu_k^{-s}\right) M[f(x);s];$$

(3) [Riemann-Lebesgue lemma] If $s = \sigma + it$, where $-\alpha < \sigma < \beta$, then

$$\lim_{|t|\to\infty} M[f(x); \sigma + it] = 0;$$

(4) [inversion formula] Under certain regularity conditions,

$$f(x) \sim \pm \sum_{\varpi \in S} \operatorname{Res}\left[M[f(x);s]x^{-s}; s = \varpi\right],$$

as $x \to 0^+$ (the $+$ sign being taken) or $x \to \infty$ (the $-$ sign), where S denotes the set of all singularities (poles) of $M[f(x);s]$ to the left ($x \to 0^+$) or to the right ($x \to \infty$) of the fundamental strip.

3. Two Examples

Example. Find the asymptotic behaviour of

$$F(x) = \sum_{k\ge 1} d(k) e^{-kx} \qquad \text{as} \quad x \to 0, \Re x > 0,$$

where $d(k) = \sum_{d|k} 1$ denotes the number of divisors of k. By Property (2),

$$M[F(x);s] = \left(\sum_{k\ge 1} d(k) k^{-s}\right) M[e^{-x}; s] = \zeta^2(s)\Gamma(s) \qquad (\Re s > 1),$$

where ζ denotes Riemann's zeta function and Γ Euler's Gamma function. Standard facts about these two functions [10] and Cauchy's residue theorem lead to the following expansion due to Wigert, cf. [9, p. 163],

$$F(x) \sim \frac{1}{x}\log\frac{1}{x} + \frac{\gamma}{x} + \frac{1}{4} - \sum_{k\ge 1} \frac{B_{2k}^2}{2k\,(2k)!} x^{2k-1} \qquad (x \to 0, \Re x > 0), \tag{2}$$

the B_k being the Bernoulli numbers.

Remark by Hwang. From (2), we obtain

$$\sum_{k\ge 1} d(k) z^k \sim \frac{1}{1-z}\log\frac{1}{1-z} + \frac{\gamma}{1-z} + \cdots \qquad (z \sim 1, |z| < 1),$$

and, in a purely formal way (transferring to coefficient), cf. [4],

$$\sum_{1\le k\le n} d(k) = n\log n + (2\gamma - 1)n + \cdots,$$

the determination of the error term constitutes the well-known Dirichlet divisor problem [9, Ch. XII].

EXAMPLE. Find the asymptotic behaviour of

$$F(x) = \sum_{k\geq 0}(1 - e^{-x/2^k}) \qquad \text{as} \quad x \to \infty, |\arg x| < \pi/2,$$

Again, by Property (2), we obtain

$$M[F(x); s] = \left(\sum_{k\geq 0} 2^{ks}\right) M[1 - e^{-x}; s] = -\frac{\Gamma(s)}{1 - 2^s} \qquad (-1 < \Re s < 0).$$

Thus, by the inversion formula (1) and Cauchy's theorem,

$$F(x) = \log_2 x + \frac{1}{2} + \frac{\gamma}{\log 2} + Q(\log_2 x) + R(x) \qquad (x \to \infty, |\arg x| < \pi/2),$$

where $Q(u)$ is a continuous periodic function whose Fourier series is given by

$$Q(u) = -\frac{1}{\log 2} \sum_{k\in\mathbb{Z}\setminus\{0\}} \Gamma\left(\frac{2k\pi i}{\log 2}\right) e^{-2k\pi i u},$$

and $R(x) = O(x^{-M})$ for any $M > 0$.
Remark by Hwang. The error term $R(x)$ can be easily replaced by an asymptotic expansion as follows.

$$R(x) = \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{\Gamma(s)}{1-2^s} x^{-s}\, ds = -\sum_{k\geq 1} \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \Gamma(s)(2^k x)^{-s}\, ds = -\sum_{k\geq 1} e^{-2^k x},$$

the interchange of the sum and the integral being justified by absolute convergence. This expression for $R(x)$ gives not only an asymptotic expansion but also an exact formula for $F(x)$ as long as $\Re x > 0$.

4. Three Applications to the Analysis of Algorithms

EXAMPLE (AVERAGE HEIGHT OF BINARY (OR PLANAR) TREES). The problem in question [1] (after some reductions) is the asymptotic behaviour of

$$\mu_n := \frac{S_n}{A_n} \qquad \text{as} \quad n \to \infty,$$

where $A_n = \frac{1}{n}\binom{2n-2}{n-1}$ are the Catalan numbers and

$$S_{n+1} = \sum_{k\geq 1} d(k)\left[\binom{2n}{n+1-k} - 2\binom{2n}{n-k} + \binom{2n}{n-1-k}\right].$$

By elementary approximations (using Stirling's formula), we obtain

$$\mu_{n+1} = -2g_0(n) + \frac{4}{n} g_2(n) + O\left(\frac{\log n}{\sqrt{n}}\right),$$

where $g_b(n) = G_b(1/n)$, with $G_b(x) = \sum_{k\geq 1} k^b d(k) e^{-k^2 x}$. Applying Mellin transform to $G_b(x)$ and after some simplification, we obtain [1]

$$\mu_n = \sqrt{\pi n} - \frac{1}{2} + O\left(\frac{\log n}{\sqrt{n}}\right) \qquad (n \to \infty).$$

Example (Average external path length in a trie). Let ℓ_n denote the expected path length of a random trie with n keys. Then [3, 6, 7] ℓ_n satisfies $\ell_0 = \ell_1 = 0$ and

$$\ell_n = n + 2^{-n} \sum_{0\le k\le n} \binom{n}{k}(\ell_k + \ell_{n-k}) \qquad (n = 2, 3, 4, \ldots).$$

The associated exponential generating function satisfies

$$\ell(z) = \sum_{n\ge 1} \frac{\ell_n}{n!} z^n = z(e^z - 1) + 2\ell(z/2)e^{z/2}.$$

Thus

$$\ell_n = \sum_{2\le k\le n} \binom{n}{k} \frac{(-1)^k k}{1-2^{1-k}} = n \sum_{k\ge 0} \left[1 - (1-2^{-k})^{n-1}\right] = n \sum_{k\ge 0} (1 - e^{-n/2^k}) + O(1),$$

and the result of Example 2 gives [6]

$$\ell_n = n \log_2 n + \left(\frac{1}{2} + \frac{\gamma}{\log 2}\right) n + Q(\log_2 n) n + O(1) \qquad (n \to \infty).$$

Two different approaches leading to full asymptotic expansions for ℓ_n are outlined in [2].

Example (Average number of Carry Propagations). Let $t(x, y)$ denote the number of carry propagations when adding two numbers x and y and

$$P_{n,k} = 4^{-n} \#\{(x, y) : 0 \le x, y < 2^n, t(x, y) > k\} \qquad (n, k \ge 0).$$

The quantity of interest is $t_n = \sum_{k\ge 0} P_{n,k}$, as $n \to \infty$, which reduces to [5]

$$t_n = \sum_{k\ge 0} [z^n] \left(\frac{1}{1-z} - \frac{1}{1 - z + z^k/2^{k+1}}\right) = \sum_{k\ge 1} \left(1 - e^{-n/2^k}\right) + O\left(\frac{\log^4 n}{n}\right),$$

where in the last step the localization of the smallest (in modulus) zero of the polynomials $1 - z + z^k/2^{k+1}$ is needed. As in Example 2, we have [5]

$$t_n = \log_2 n + \frac{\gamma}{\log 2} - \frac{1}{2} + Q(\log_2 n) + O\left(\frac{\log^4 n}{n}\right) \qquad (n \to \infty).$$

Remark by Hwang. More calculations show that the O-term can be replaced by an asymptotic expansion of the form

$$\sum_{k\ge 1} n^{-k} \sum_{0\le j\le k} \pi_{k,j}(\log_2 n) \log^j n,$$

the $\pi_{k,j}(u)$ being periodic functions in u.

5. Some Harmonic Sums

Let $\{\lambda_k\}$ and $\{\mu_k\}$ be two given sequences. Let $f(x)$ be exponentially small at infinity and

$$f(x) \sim \sum_{k\geq 0} f_k x^{\alpha_k} \qquad (x \to 0^+, 0 = \alpha_0 < \alpha_1 < \alpha_2 < \cdots, \alpha_k \to \infty).$$

This assumption implies [11, p. 153] that the function $M[f(x); s]$ admits meromorphic continuation into the whole s-plane with simple poles at $-\alpha_k$, the residues being f_k, for $k = 0, 1, 2, \ldots$. From this fact, we can deduce the following asymptotics as $x \to 0^+$, cf. [2].

(1) [Euler-Maclaurin-Barnes formula]

$$\sum_{n\geq 1} f(nx) \sim \frac{1}{x}\int_0^\infty f(t)\,dt + \frac{f(0)}{2} + \sum_{k\geq 1} f_k \zeta(-\alpha_k) x^{\alpha_k} \qquad (x \to 0^+);$$

(2)

$$\sum_{n\geq 1} (-1)^{n-1} f(nx) \sim \sum_{k\geq 0} (1 - 2^{1+\alpha_k}) f_k \zeta(-\alpha_k) x^{\alpha_k} \qquad (x \to 0^+);$$

(3)

$$\sum_{n\geq 1} f(2^n x) \sim f(0)\log_2 \frac{1}{x} + \frac{f(0)}{2} + \frac{\gamma_f}{\log 2} + \delta(\log_2 x) + \sum_{k\geq 1} \frac{f_k}{1 - 2^{\alpha_k}} x^{\alpha_k} \qquad (x \to 0^+),$$

where

$$\begin{aligned} \gamma_f &= \int_0^1 \frac{f(t) - f(0)}{t}\,dt + \int_1^\infty \frac{f(t)}{t}\,dt \\ \delta(u) &= \frac{1}{\log 2} \sum_{k\in\mathbb{Z}\setminus\{0\}} M[f; 2k\pi i/\log 2] e^{-2k\pi i u}, \end{aligned}$$

the latter being a continuous periodic function of period 1.

Many other types of harmonic power sums can be found in [8, Ch. III].

6. Some Amusing Sums

Let $F(x) = \sum_{m,n\geq 1} f((m^2+n^2)x)$. To derive an asymptotic expansion for $F(x)$, as $x \to 0^+$, we observe that

$$M[F(x); s] = \left(\sum_{m,n\geq 1} (m^2+n^2)^{-s}\right) M[f(x); s] = \frac{M[\Theta^2(x); s]}{\Gamma(s)} M[f(x); s].$$

where $\Theta(x) = \sum_{n\geq 2} e^{-n^2 x}$. The singularities of $M[\Theta^2(x); s]$ to the right of the vertical line $\Re s = 1$ (included) are determined by the asymptotic behaviour of $\Theta(x)$ as $x \to 0^+$, for which we apply once again Mellin transform. We thus obtain the functional equation, cf. [9, §2.6],

(3)
$$\Theta(x) = \frac{1}{2}\sqrt{\frac{\pi}{x}} - \frac{1}{2} + \sqrt{\frac{\pi}{x}}\Theta\left(\frac{1}{x}\right) \qquad (x > 0),$$

the last term being exponentially small as $x \to 0^+$. Once the singularities of the function $M[\Theta^2(x); s]$ are explicited, the asymptotic behaviour of $F(x)$ can easily be derived.

In the same manner, we can consider harmonic sums of the types

$$\sum_{n\geq 1} \lfloor \sqrt{n} \rfloor f(nx), \quad \sum_{n\geq 1} \lfloor \log_2 n \rfloor f(nx), \ldots$$

7. Conclusion

Besides harmonic sums, Mellin transform finds applications to the asymptotics of integrals (especially of convolution type), to the asymptotic behaviour of generating functions, Laplace transform, etc, and to many interesting identities and functional equations like (3).

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