Asymptotic Analysis of Finite Differences and Rice Integrals

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[summary by Philippe Dumas and Danièle Gardy]

Rice's method is designed to estimate sums

$$Df_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} f_k, \tag{1}$$

where the sequence f_n can be extended as an analytic function $\phi(n)$. The asymptotics of the sequence f_n are assumed to be known, and the problem is to obtain the asymptotic behaviour of the sequence Df_n. The obvious bound

$$|Df_n| \leq 2^n \max_k |f_k|$$

is often disappointing, due to cancellation phenomena and an accurate evaluation of the f_k's cannot provide a direct estimate of the Df_n's. Hence more sophisticated techniques, presented in this talk, are needed. The complete paper is presented in [3].

Many problems in the analysis of algorithms lead to a sequence Df_n. In the sixties, Knuth [4, p. 131] encountered the sum

$$U_n = \sum_{k=2}^{n} \binom{n}{k} \frac{(-1)^k}{2^{k-1} - 1}$$

in the study of radix exchange sorting. On can find numerous others examples in the analysis of digital structures [4, p. 501], [1, 6] or conflict resolution in broadcast communications [7].

There are two classical approaches to estimate such alternating sums :

- one can arrange the sum to obtain harmonic sums, which can be tackled by Mellin transforms. This is the standpoint of De Bruijn[1] ;
- Rice proposed a direct approach, which relies on the formula

$$Df_n = \frac{(-1)^n}{2i\pi} \int_{\mathcal{C}} \phi(s) \frac{n!}{s(s-1)\cdots(s-n)}\, ds. \tag{2}$$

  The path $\mathcal{C}$ is a contour which encloses the points $0, 1, \ldots, n$, but no singularity of $\phi(s)$. It is assumed that $\phi(s)$ is an analytic function which extends the sequence f_n, and has a polynomial growth at infinity.

[1] based on an original of De Bruijn to Knuth ca. 1965 to be found in the middle pages of Knuth's personal copy of the book *Asymptotic Methods in Analysis*

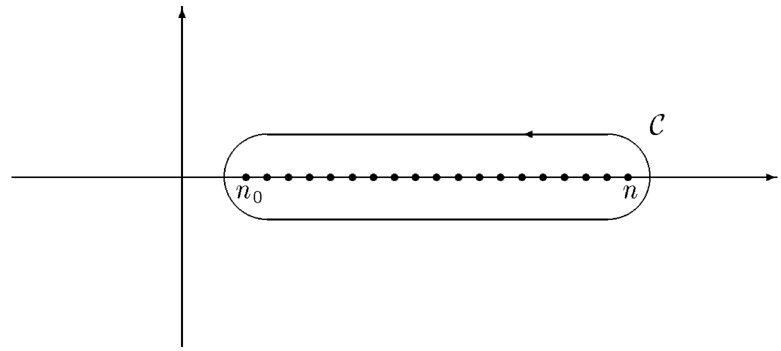

FIGURE 1. A Rice contour.

Here, we develop the second approach, which we call Rice's method. Following Prodinger *et alii*, we write the so-called Rice kernel as follows,

(3) $$[n,s] := \frac{(-1)^n n!}{s(s-1)\cdots(s-n)} = -\frac{\Gamma(-s)\Gamma(n+1)}{\Gamma(n+1-s)}.$$

1. Finite differences

The transformation of sequences (Δ is the forward difference operator, $\Delta f_r = f_{r+1} - f_r$)

$$D : f_n \longmapsto g_n = (-1)^n \Delta^n f_0 = \sum_{k=1}^{n} \binom{n}{k} (-1)^k f_k$$

can be translated into the language of ordinary generating, $F(z) = \sum_n f_n z^n$, or into the language of exponential generating series $f(z) = \sum_n f_n \frac{z^n}{n!}$. In the first case, we obtain essentially Euler's transformation of series

$$G(z) = \frac{1}{1-z} F\left(\frac{-z}{1-z}\right).$$

The second case leads to the formula

$$g(z) = e^z f(-z),$$

which shows the involutive character of the transformation. It is also possible to introduce the Poisson generating series

$$\hat{f}(t) = e^{-t} f(t) = \sum_n f_n e^{-t} \frac{t^n}{n!},$$

which is a simple variant of the exponential generating series. From this point of view, the transformation becomes very simple,

$$g(z) = \hat{f}(-z).$$

Euler transforms and Poisson transforms occur in the analysis of quadtrees [2] and digital structures.

2. Integral representation

The next lemma has been known since the 19th century [5, chap. 8].

LEMMA 1 (RICE'S LEMMA). *Let $\phi(s)$ be an analytic function defined in a neighbourhood Ω of the positive real axis $[0,+\infty)$. Let $\mathcal{C}$ be a contour enclosing the integers $n_0, \ldots, n$, but no singularity of $\phi(s)$. Then*

$$\sum_{k=n_0}^{n} \binom{n}{k} (-1)^k \phi(k) = \frac{1}{2i\pi} \int_{\mathcal{C}} \phi(s) \frac{(-1)^n n!}{s(s-1)\cdots(s-n)}\, ds. \tag{4}$$

The proof is a mere application of Cauchy's residue formula.

The principle of the method is to use hypotheses about the growth of $\phi(s)$ in order to deform the integration contour and obtain an estimate of the sum. More precisely, the function $\phi(s)$ is of polynomial order k if

$$\phi(s) = O(|s|^k) \qquad \text{as } s \to \infty,\ s \in \Omega.$$

With this assumption, the integrand $\phi(s)[n,s]$ tends to 0 when s goes to infinity, if n is large. This permits to modify the path of integration.

3. Rational case

In this section, we consider the basic case of a rational function.

THEOREM 1. *Let $\phi(s)$ be a rational function which is analytic in a neighbourhood of $[n_0,+\infty)$. Then, when n is large enough,*

$$\sum_{k=n_0}^{n} (-1)^k \binom{n}{k} \phi(k) = -\sum_{\omega} \operatorname{Res}\left[\phi(s) \frac{(-1)^n n!}{s(s-1)\cdots(s-n)}, s = \omega\right]. \tag{5}$$

The ω's which appear in the sum are the poles of the integrand not in $[n_0,+\infty)$.

The proof relies on Cauchy's formula used with a contour which is the union of a Rice contour $\mathcal{C}$ and a circle $\mathcal{C}_R$ (cf. Fig. 2). When R tends to infinity, the integral over $\mathcal{C}_R$ tends to 0 provided n is greater than the degree of $\phi(s)$.

Let $s_0 = \sigma_0 + it_0$ be a pole of order r of $\phi(s)$, which we assume not to be a non-negative integer for the sake of simplicity. Then,

$$\operatorname{Res}\left[\phi(s) \frac{(-1)^n n!}{s(s-1)\cdots(s-n)}, s = s_0\right] \underset{n\to\infty}{\asymp} n^{s_0} \log^{r-1} n = n^{\sigma_0} e^{it_0 \log n} \log^{r-1} n.$$

It is possible to be more precise and we refer to [3] for details. The authors obtain an asymptotic equivalent of the type above. Hence Rice's method epitomises a standard asymptotic behaviour mixed with some fluctuations. Moreover it must be pointed out that the rightmost poles of $\phi(s)$ give the most significant part of the asymptotic estimate of Df_n.

EXAMPLE. Let us consider the sum

$$S_n(m) = \sum_{k=1}^{n} \binom{n}{k} \frac{(-1)^k}{k^m},$$

with m a positive integer. A direct application of the preceding result gives

$$S_n(m) = -\frac{1}{m!} \sum_{k=0}^{m} (-1)^k \binom{m}{k} \Gamma^{(k)}(1) (\log n)^{m-k} + O\left(\frac{\log^m n}{n}\right).$$

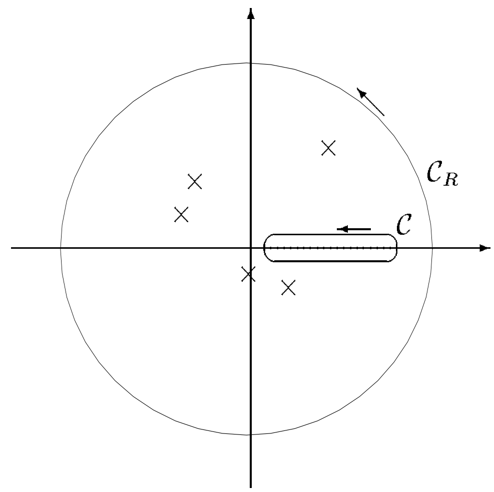

FIGURE 2. For R large enough, all poles (tagged by a cross $\times$) of $\phi(s)$ are inside the circle $\mathcal{C}_R$ but outside the Rice contour $\mathcal{C}$.

EXAMPLE. For the sum

$$T_n = \sum_{k=1}^{n} \binom{n}{k} \frac{(-1)^k}{k^2+1},$$

the function involved is $\phi(s) = 1/(s^2+1)$ and we expect some term $n^{\pm i}$ to occur. Actually we have

$$T_n = \frac{\pi}{\sqrt{\sinh \pi}} \cos(\log n + \vartheta_0) + o(1).$$

It is to be noticed that T_n remains bounded whereas the central term of the sum is of order $2^n/n^2$.

4. Meromorphic case

The meromorphic case is a mere extension of the rational case.

EXAMPLE. The sum

$$U_n = \sum_{k=2}^{\infty} (-1)^k \binom{n}{k} \frac{1}{2^{k-1}-1}$$

is associated with the meromorphic function

$$\phi(s) = \frac{1}{2^{s-1}-1}, \qquad \text{whose poles are the} \qquad \chi_k = 1 + \frac{2ik\pi}{\log 2}.$$

One integrates over the circles $\mathcal{C}_R$ with center 1 and radius $R = (2k+1)\pi/\log 2$. The circles go between the poles and the function $\phi(s)$ is only of polynomial order on these circles. In this way, one obtains

$$\begin{aligned} U_n &= \frac{n}{\log 2}(H_{n-1} - 1) - \frac{n}{2} + 2 + \frac{1}{2} \sum_{\substack{k \neq 0 \\ k \in \mathbb{Z}}} \frac{\Gamma(n+1)\Gamma(-1+\chi_k)}{\Gamma(n+\chi_k)} \\ &= n \log_2 n + Cn + \frac{n}{\log 2} \sum_{k \neq 0} \Gamma(-\chi_k) e^{2i\pi \log_2 n} + O(\sqrt{n}). \end{aligned}$$

There are many others examples for which we refer to [3] in order to keep this summary short. Let us say simply that Rice's method provides a correspondence between the singularity of $\phi(s)$ and the asymptotic behaviour of Df_n, as summarised below.

Type of singularity		Asymptotic behaviour
Simple pole	$1/(s-s_0)$	$-\Gamma(-s_0)n^{s_0}$
Multiple pole	$1/(s-s_0)^r$	$-\Gamma(-s_0)n^{s_0}\dfrac{(\log n)^{r-1}}{(r-1)!}$
Algebraic singularity	$(s-s_0)^\lambda$	$-\Gamma(-s_0)n^{s_0}\dfrac{(\log n)^{-\lambda-1}}{\Gamma(-\lambda)}$
Logarithmic singularity	$(s-s_0)^\lambda(\log(s-s_0))^r$	$-\Gamma(-s_0)n^{s_0}\dfrac{(\log n)^{-\lambda-1}}{\Gamma(-\lambda)}(\log\log n)^r$

5. Poisson–Mellin–Newton cycle

A Rice integral may often be written

$$\frac{1}{2i\pi}\int_{c-i\infty}^{c+i\infty}\phi(s)\frac{(-1)^n n!}{s(s-1)\cdots(s-n)}\,ds.$$

For n large, this is approximately

$$\frac{1}{2i\pi}\int_{-c-i\infty}^{-c+i\infty}\phi(-s)\Gamma(-s)n^s\,ds.$$

Hence an idea that we write informally

$$\text{Rice} \approx \text{Mellin}^{-1}.$$

The argument is only heuristic and the right standpoint is the so-called Poisson–Mellin–Newton cycle.

Let us start from a sequence f_n. We associate with it the Poisson series

$$\hat{f}(t)=\sum_{n=0}^{\infty}f_n\frac{e^{-t}t^n}{n!}.$$

The Mellin transform of $\hat{f}(t)$ is

$$\begin{aligned}\hat{f}^*(s)&=\sum_{n=0}^{\infty}f_n\int_0^\infty e^{-t}\frac{t^{s+n-1}}{n!}\,dt\\&=\sum_{n=0}^{\infty}f_n\frac{\Gamma(s+n)}{n!}\\&=\Gamma(s)\sum_{n=0}^{\infty}f_n\frac{s(s+1)\cdots(s+n-1)}{n!}.\end{aligned}$$

Hence we have $\hat{f}^*(-s)=\Gamma(-s)\nu(s)$, where $\nu(s)$ is the Newton series

$$\nu(s)=\sum_{n=0}^{\infty}(-1)^n f_n\frac{s(s-1)\cdots(s-n+1)}{n!}.$$

But we can find the coefficient f_n again by the difference operator,

$$f_n = (-1)^n \Delta^n \nu(0).$$

We recognise the expression $D\nu(n)$ and by Rice's lemma we have

$$f_n = \frac{1}{2i\pi} \int_{\mathcal{C}} [n, s] \nu(s)\, ds.$$

Eventually, Rice's transform appears to be the inverse of Mellin transform, composed with Poisson transform.

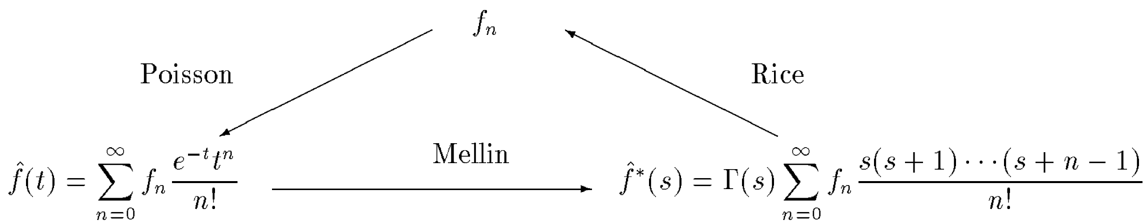

FIGURE 3. The Poisson–Mellin–Newton cycle.

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