Probabilistic Recurrence Relations for Divide-and-Conquer Algorithms

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[summary by Paul Zimmermann]

Abstract

Probabilistic recurrence relations occur frequently in the analysis of randomized algorithms. After an introduction to the work of Karp on probabilistic recurrence relations of the form

$$T(n) = a(n) + T(h(n)) \tag{1}$$

where $h(n)$ is a random variable, we discuss probabilistic recurrence relations for divide-and-conquer algorithms. In (sequential and parallel) divide-and-conquer algorithms a problem of size n is usually divided in subproblems of size $h_1(n), \ldots, h_k(n)$ where the $h_i(n)$ are (not necessarily independent) random variables. The analysis of the time complexity of parallel divide-and-conquer algorithms or the space complexity of sequential divide-and-conquer algorithms leads to a recurrence of the form

$$T(n) = a(n) + \max(T(h_1(n)), \ldots, T(h_k(n))). \tag{2}$$

On the other hand, the analysis of the time complexity of sequential divide-and-conquer algorithms and the space complexity of parallel divide-and-conquer algorithms leads to a recurrence of the form

$$T(n) = a(n) + T(h_1(n)) + \cdots + T(h_k(n)).$$

For both types of probabilistic recurrences, we give an upper bound for the probability distribution on $a(n)$, if the distribution of the $h_i(n)$ is unknown. The only informations needed are upper bounds on the expected values of the $h_i(n)$. (Joint work with Marek Karpinski.)

The problem here is to find an upper bound for $T(n)$, with little information about the distribution of $h(n)$, $h_1(n), \ldots, h_k(n)$.

Karp studied Equation (1) with the following assumptions [1]:

(i) $a(n) \geq 0$,
(ii) $h(n)$ is a random variable over $[0, n]$,
(iii) $E[h(n)] \leq m(n)$ where $0 \leq m(n) < n$, and $m(n)$ and $m(n)/n$ are non-decreasing.

Under these conditions, if one defines $u(n)$ to be the least non-negative solution of

$$\tau(n) = a(n) + \tau(m(n)),$$

we have two results according to the function $a(n)$.

THEOREM 1. *If* $a(n) = 0$ *for* $n < d$ *and* $a(n) = 1$ *for* $n \geq d$, *let* $c_t = \min\{x \mid u(x) \geq t\}$, *then*

$$\Pr[T(n) \geq u(n) + n] \leq \left(\frac{m(n)}{n}\right)^{n-1} \frac{m(n)}{c_{u(n)}}.$$

THEOREM 2. *If* $a(n)$ *is continuous and strictly increasing, then* $m(n)$ *is continuous and*

$$\Pr[T(n) \geq u(n) + na(n)] \leq \left(\frac{m(n)}{n}\right)^{n}.$$

In the general case of equation 2, Wolf Zimmermann and Marek Karpinski obtained the following result [2].

THEOREM 3. *Let* $a(n)$ *be continuous, non-decreasing, and strictly increasing on* $\{n \mid a(n) > 0\}$. *Also let the* $m_i(n)$ *be strictly increasing. Then, for every instance* z *of size* n *and every positive integer* j, *we have*

$$\Pr[T(z) \geq u(n) + ja(n)] \leq \left(\frac{m_1(n) + \cdots + m_k(n)}{n}\right)^{j}.$$

Bibliography

[1] Karp (Richard M.). – Probabilistic recurrence relations. In *Proceedings of the Twenty Third Annual ACM Symposium on Theory of Computing.* pp. 190–197. – ACM Press, 1991.

[2] Karpinski (M.) and Zimmermann (W.). – *Probabilistic Recurrence Relations for Parallel Divide-and-Conquer Algorithms.* – Technical Report n° TR-91-067, Berkeley, ICSI, 1991.