The Height of a Random Tree

Tomasz Luczak
Adam Mickiewicz University, Poznan, Poland
March 29, 1993

[summary by Wojtek Szpankowski]

1. Introduction

Let $T_n$ be a random labelled rooted tree on the vertex set $[n] = \{1, 2, \ldots, n\}$ with the root $v_0 \in [n]$ (here and below we assume that a root is always the vertex number 1). The limit distribution of the height of $\bar{H} = \bar{H}(n)$ of $T_n$, was found by Rényi and Szekeres [3] who proved the following result.

**Theorem 1.** For every constant $\beta > 0$

$$
\lim_{n \to \infty} \left( \bar{H} = \left\lfloor \frac{\sqrt{2\pi}}{\beta} \right\rfloor \right) = 2 \sqrt{\frac{2\pi}{n}} \beta \sum_{i=1}^{\infty} \left( \frac{2i^4 \pi^4}{\beta} - 3i^2 \pi^2 \right) \exp \left( -\frac{i^2 \beta}{\sqrt{n}} \right),
$$

where the convergence is uniform for $\beta \in (c, C)$ for every constants $0 < c < C < \infty$.

Furthermore, they proved that the $s$-th moment of random variable $h(T_n)/\sqrt{2\pi}$ tends to $2\Gamma(s/2 + 1)(s - 1)\zeta(s)$. In particular, for the expectation and the variance of $h(T_n)$, one obtains

$$
\lim_{n \to \infty} \frac{\mathbb{E} h(T_n)}{\sqrt{n}} = \sqrt{2\pi} = 2.50663\ldots
$$

$$
\lim_{n \to \infty} \frac{\text{Var} h(T_n)}{n} = \frac{2\pi(\pi - 3)}{3} = 0.29655\ldots
$$

(See also [1] for a generalization of this result to other simply generated families of trees.)

Consider now the following greedy algorithm. For a tree $T$ with the root $v_0$ let $\mathcal{F}(T)$ be the forest of rooted trees obtained from $T$ by removing the root, where as the root of a tree $T' \in \mathcal{F}(T)$ we choose the vertex adjacent to $v_0$ in $T$. The height of a tree can be estimated by using the following simple greedy algorithm, which finds in a tree a path starting from the root. The algorithm starts with a tree $T^{(0)} = T$ on $n$ vertices, removes its root $v_0$, chooses the largest tree $T^{(1)}$ from $\mathcal{F}(T^{(0)})$ (if there are more than one of them it picks the one with the lexicographically first root), appends its root to a path, and repeats this procedure until for some $h$ tree $T^{(h)}$ consists only of one vertex.

This talk concerns the study of the height $H = H(n)$ found in a random tree by the above greedy algorithm. The limiting distribution of $H$ is found and it is shown that the expected value of $H/\sqrt{n}$ tends to an absolute constant $C$, where

$$
C = \frac{\sqrt{2\pi}}{2\sqrt{2} - \ln(3 + 2\sqrt{2})} = 2.353139\ldots
$$

Thus, on average, the algorithm finds a path whose length is roughly 93% of the expected height of the tree.
2. Main Results

We need some definitions. Let us define recursively two sequences of random variables \( \{H_i\} \) and \( \{W_i\} \) by setting \( \tilde{H}_0 = \min_j \{|T_n^{(j)}| \leq n/2\} \), \( W_0 = |T_n^{(\tilde{H}_0)}| \) whereas for \( i \geq 1 \) let

\[
\tilde{H}_i = \min_j \{|T_n^{(j)}| \leq W_{i-1}/2\}
\]

and \( W_i = |T_n^{(\tilde{H}_i)}| \). Furthermore, set \( H_0 = \tilde{H}_0 \) and \( H_i = \tilde{H}_i - \tilde{H}_{i-1} \) for \( 1 \leq r \leq n - 1 \). Thus, \( W_i \) denotes the size of the tree \( T_n^{(r)} \) when it first drops under \( W_{i-1}/2 \) and \( H_i \) is the number of steps of the algorithm between two such moments. Note that for every \( i \geq 0 \) we have \( W_i \leq 2^{i-1} n \).

Clearly, the length of the path found by the algorithm can now be written as a sum of \( H_i \)'s, so

\[
\Pr(H > k) = \Pr\left(\sum_{i \geq 0} H_i > k\right)
\]

\[
= \Pr(H_0 > k) + \sum_{j \geq 1} \Pr\left(\sum_{i=0}^{j-1} H_i > k \land \sum_{i=0}^{j-1} H_i \leq k\right)
\]

In order to characterize the behaviour of the probabilities \( \Pr(\sum_{i=0}^{j-1} H_i > k \land \sum_{i=0}^{j-1} H_i \leq k) \) let us define an integral operator \( A \) by setting

\[
(Ag)(x) = \int_0^x \int_{1/4}^{1/2} f(z, y) g((x - z)/\sqrt{y}) dy dz,
\]

where \( f \) is a function defined as

\[
f(x, y) = \frac{1}{2\pi} \int_{1/2-y}^{\sqrt{y}} \frac{x}{t^3} \exp\left(-\frac{x^2}{2(1 - y + t)}\right) dt.
\]

Furthermore, let

\[
g_0(x) = \int_x^\infty \int_{1/4}^{1/2} f(z, y) dy dz,
\]

and for \( j \geq 1 \)

\[
g_j = Ag_{j-1} = A^j g_0.
\]

The next result shows that functions \( g_j \) are closely related to our problem.

**Lemma 1.** For every \( x > 0 \) we have

\[
\Pr(H_0 > x) = (1 + o(1))g_0,
\]

and for \( j \geq 1 \)

\[
\Pr\left(\sum_{i \geq 0} H_i > x \land \sum_{i \geq 0} H_i \leq x\right) = (1 + o(1))g_1(x),
\]

where, for given constants \( c, C \), the quantity \( o(1) \) tends to \( \theta \) uniformly for every \( x \in (c, C) \).

As a consequence of Lemma 1 one proves the limiting distribution of \( H \).

**Theorem 2.** For every constant \( x > 0 \)

\[
\lim_{n \to \infty} \Pr(H > x) = h(x),
\]

where

\[
h(x) = \sum_{j=0}^{\infty} g_j(x) = \sum_{j=0}^{\infty} (A^j g_0)(x).
\]

122
In particular, the function $h$ is the only continuous solution of the integral equation

$$h(x) = g_0(x) + (Ah)(x) = \int_x^{\infty} \int_{1/4}^{1/2} f(z, y) dy \, dz + \int_0^x \int_{1/4}^{1/2} f(z, y) h((x - z)/\sqrt{y}) dy \, dz,$$

Having computed the distribution of $H$, it is not hard to guess the value of its mean. Clearly, $E H / \sqrt{n}$ should converge to the expected value of the random variable $Z$, where $P(Z > x) = h(x)$ and $h(x)$ is given by Theorem 2. But $x h(x) \to 0$ as $x \to \infty$ (as a matter of fact Theorem 1 says that the probability that the actual height of a random tree is larger than $x$ decreases exponentially with $x$), so

$$\mu = E Z = \int_0^{\infty} h(x) dx.$$  

Integrating both sides of the formula on $h(x)$ in Theorem 2, after elementary calculations, one obtains

$$\mu = \int_0^{\infty} \int_{1/4}^{1/2} x f(x, y) dy \, dx + \mu \int_0^{\infty} \int_{1/4}^{1/2} \sqrt{y} f(x, y) dy \, dx.$$  

Consequently,

$$\mu = \frac{\int_0^{\infty} \int_{1/4}^{1/2} x f(x, y) dy \, dx}{1 - \int_0^{\infty} \int_{1/4}^{1/2} \sqrt{y} f(x, y) dy \, dx}.$$  

Finally, one proves the following result.

**Theorem 3.** The average height obtained by the greedy algorithm is

$$\lim_{n \to \infty} \frac{E H}{\sqrt{n}} = \mu,$$

where

$$\mu = \frac{\int_0^{\infty} \int_{1/4}^{1/2} x f(x, y) dy \, dx}{1 - \int_0^{\infty} \int_{1/4}^{1/2} \sqrt{y} f(x, y) dy \, dx} = \frac{\sqrt{2\pi}}{2\sqrt{2} - \ln(3 + 2\sqrt{2})} = 2.353139\ldots.$$  

This completes our presentation of main results of the talk. More details can be found in [2].

**Bibliography**

