Enumerations related to automorphisms of rooted tree structures

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[summary by Dominique Gouyou-Beauchamps]

Abstract

The goal of this paper is to present a panorama of the fundamental properties of cycle index series and asymmetry index series within enumerative combinatorics, as well as a few concrete applications. A given structure is said to be asymmetric if its automorphism group reduces to the identity. We introduce an asymmetry indicator series $\Gamma_F(x_1, x_2, x_3, \ldots)$ by means of which we study the correspondence $F \to \overline{F}$ in connection with the various operations existing in the theory of species of structures. It is shown that all these operations are automatically computable but this aspect is not developed in the summary.

1. Species and Asymmetry Index Series

Given any finite set $U$, let us denote by $\mathfrak{N}[U]$ the set of all rooted trees having $U$ as underlying set of vertices. Clearly, every bijection $\beta : U \to V$ between finite sets induces another bijection which we denote by $\mathfrak{N}[\beta] : \mathfrak{N}[U] \to \mathfrak{N}[V]$ and call the transportation of rooted trees along $\beta$ (we replace each vertex $u$ in $a$ by the corresponding vertex $\beta(u)$). Of course, transportation commutes with composition in the following way: given any two successive bijections $\beta : U \to V, \beta' : V \to W$, we have $\mathfrak{N}[\beta' \circ \beta] = \mathfrak{N}[\beta'] \circ \mathfrak{N}[\beta]$ and $\mathfrak{N}[1_U] = 1_{\mathfrak{N}[U]}$ (where $1_U$ denotes, as usual, the identity bijection of a finite set $U$ into itself).

A combinatorial species [5] is a functor from the category of finite sets and bijections into itself. In other words, a combinatorial species is a rule $F$ that associates a finite set $F[U]$ to any finite set $U$ and a bijection $F[\beta] : F[U] \to F[V]$ to any bijection $\beta : U \to V$. An element $s \in F[U]$ is called an $F$-structure on the underlying set $U$. The bijection $F[\beta]$ is called the transportation of $F$-structures along $\beta$.

In the case of weighted species $F$, each $F$-structure $s$ is given a weight $w_F(s)$ in a certain commutative ring $\mathcal{R}$ and the transportation $F[\beta]$ must preserve these weights.

Given a species $F$ and two $F$-structures $s \in F[U]$ and $s' \in F[V]$, an isomorphism $\beta$ from $s$ to $s'$ is a bijection $\beta : U \to V$ such that $F[\beta](s) = s'$. Two isomorphic $F$-structures are said to be of the same type. An automorphism of $s$ is an isomorphism from $s$ to $s$. The automorphisms of any given $F$-structure $s$ form a group called the automorphism group of $s$. When this group is trivial, the structure $s$ is said to be asymmetric.

For each integer $n$, consider now the set $\mathfrak{N} = \{1, 2, \ldots, n\}$. It is easy to see that any species $F$ induces, by transportation, a countable family of actions of the symmetric group $S_n$:

$$S_n \times F[\mathfrak{N}] \to F[\mathfrak{N}], \quad n = 0, 1, 2, \ldots.$$
Given a weighted species $F$, the formal power series

$$F(x) = \sum_{n \geq 0} f_n \frac{x^n}{n!}, \quad \tilde{F}(x) = \sum_{n \geq 0} \tilde{f}_n x^n, \quad \bar{F}(x) = \sum_{n \geq 0} \bar{f}_n x^n,$$

whose coefficients are defined by

- $f_n$ is the sum of the weights of the $F$-structures on any $n$-element set
  
  = the sum of the weights of the elements of $F[n]$,

- $\tilde{f}_n$ is the sum of the weights of the types of $F$-structures on any $n$-element set
  
  = the sum of the weights of the orbits of the action $S_n \times F[n] \to F[n]$,

- $\bar{f}_n$ is the sum of the weights of the types of asymmetric $F$-structures on any $n$-element set
  
  = the sum of the weights of the $n!$-point orbits of the action $S_n \times F[n] \to F[n]$.

are respectively called the (exponential) generating series of $F$, the types generating series of $F$, and the asymmetry types generating series of $F$.

**Example.** For every $n \geq 0$

- $a_n$ is the number of rooted trees on $n$ given vertices
  
  = the number of elements of $\mathfrak{X}[n]$,

- $\tilde{a}_n$ is the number of types of rooted trees on $n$ vertices
  
  = the number of orbits of the action $S_n \times \mathfrak{X}[n] \to \mathfrak{X}[n]$,

- $\bar{a}_n$ is the number of types of asymmetric rooted trees on $n$ vertices
  
  = the number of $n!$-point orbits of the action $S_n \times \mathfrak{X}[n] \to \mathfrak{X}[n]$.

These sequences of numbers can be “encoded” into the series

$$\mathfrak{X}(x) = \sum_{n \geq 0} a_n \frac{x^n}{n!} = x + 2 \frac{x^2}{2!} + 9 \frac{x^3}{3!} + 64 \frac{x^4}{4!} + 625 \frac{x^5}{5!} + 7776 \frac{x^6}{6!} + 117649 \frac{x^7}{7!} + \cdots,$$

$$\tilde{\mathfrak{X}}(x) = \sum_{n \geq 0} \tilde{a}_n x^n = x + x^2 + 2x^3 + 4x^4 + 9x^5 + 20x^6 + 48x^7 + 115x^8 + \cdots,$$

$$\bar{\mathfrak{X}}(x) = \sum_{n \geq 0} \bar{a}_n x^n = x + x^2 + x^3 + 2x^4 + 3x^5 + 6x^6 + 12x^7 + 25x^8 + \cdots.$$

Given two species $F$ and $G$, other species can be constructed: the sum $F + G$, the product $F \cdot G$, the substitution $F(G)$ (also denoted $F \circ G$), and the derivative $F'$ (also denoted $dF/dX$). The generic structures belonging to each of these species are described as follows:

1. $s$ is an $(F + G)$-structure on $U$ iff $s$ is an $F$-structure on $U$ or a $G$-structure on $U$ (the “or” is an “exclusive or”),
2. $s$ is an $(F \cdot G)$-structure on $U$ iff $s = (f, g)$ where $f$ is an $F$-structure on $U_1$, $g$ is a $G$-structure on $U_2$, and $U_1 \cup U_2 = U$, $U_1 \cap U_2 = \emptyset$,
3. $s$ is an $(F(G))$-structure on $U$ iff $s = (f, \gamma)$ where $\gamma$ is a set of $G$-structures having disjoint underlying sets whose union is $U$, and $f$ is an $F$-structure on the set $\gamma$ (the assumption $G[\emptyset] = \emptyset$ is made in order to have a finite number of $F(G)$-structures on each $U$),
4. $s$ is an $F'$-structure on $U$ iff $s$ is an $F$-structure on the augmented set $U \cup \{\star\}$, where $\star$ denotes a point outside $U$. 

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The passage from species to series satisfies the following properties:

- The transformation \( F \to F(x) \) commutes with combinatorial sums, products, substitutions, and derivations:

\[
(F + G)(x) = F(x) + G(x), \quad (F \cdot G)(x) = F(x) \cdot G(x), \\
(F \circ G)(x) = F(G(x)), \quad F'(x) = dF(x)/dx.
\]

- The transformations \( F \to \tilde{F}(x) \) and \( F \to \tilde{G}(x) \) commute with combinatorial sums and products but do not commute, in general, with substitutions and derivations:

\[
(\tilde{F} + \tilde{G})(x) = \tilde{F}(x) + \tilde{G}(x), \quad (\tilde{F} + \tilde{G})(x) = \tilde{F}(x) + \tilde{G}(x), \\
(\tilde{F} \cdot \tilde{G})(x) = \tilde{F}(x) \cdot \tilde{G}(x), \quad (\tilde{F} \cdot \tilde{G})(x) = \tilde{F}(x) \cdot \tilde{G}(x), \\
(\tilde{F} \circ \tilde{G})(x) \neq \tilde{F}(\tilde{G}(x)), \quad (\tilde{F} \circ \tilde{G})(x) \neq \tilde{F}(\tilde{G}(x)), \\
\tilde{F}'(x) \neq d\tilde{F}(x)/dx, \quad \tilde{F}'(x) \neq d\tilde{F}(x)/dx.
\]

Consider an infinite sequence \( t = (t_1, t_2, t_3, \ldots) \) of distinct formal “weights” and, given a finite set \( U \), define an \( F_t \)-structure on \( U \) as being a couple \( s = (f, v) \) where \( f \) is an \( F \)-structure on \( U \) and \( v : U \to \{1, 2, 3, \ldots\} \) is a function that assigns an arbitrary positive integer to each element of \( U \). Define the \( t \)-weight of the structure \( s \) by \( w(s) = \prod_{v \in U} t_v(v) \).

Given a bijection \( \beta : U \to V \), define the transportation \( F_t[\beta] : F_t[U] \to F_t[V] \) by

\[
F_t[\beta](s) = (F_t[\beta](f), v \circ \beta^{-1}).
\]

Of course, the two series \( \tilde{F}_t(x) \) and \( \tilde{F}_t(x) \) can be associated to the weighted species \( F_t \) and each series is easily seen to be a symmetric function of the \( t_i \)'s [11].

Let \( F \) be any species and \( t = (t_1, t_2, t_3, \ldots) \) be a countable sequence of formal variables related to the variables \( x_1, x_2, x_3, \ldots \) by the equations

\[
x_k = t_1^k + t_2^k + t_3^k + \cdots, \quad (k\text{-th power sum}), \quad k = 1, 2, 3, \ldots.
\]

The cycle index series \( Z_F \) and the asymmetry index series \( \Gamma_F \) are defined by

\[
Z_F(x_1, x_2, x_3, \ldots) = \text{the expression of the symmetric function } \tilde{F}_t(x) \mid_{x=1}
\]

of \( t_1, t_2, t_3, \ldots \) in terms of the variable \( x_1, x_2, x_3, \ldots \),

\[
Z_t(x_1, x_2, x_3, \ldots) = \text{the expression of the symmetric function } \tilde{F}_t(x) \mid_{x=1}
\]

of \( t_1, t_2, t_3, \ldots \) in terms of the variable \( x_1, x_2, x_3, \ldots \).

It turns out that the cycle index series \( Z_F \) is the sum, over \( n \), of the classical Polya’s cycle index polynomials of the family of actions \( S_n \times F[n] \to F[n] \), of the symmetric group \( S_n \), \( n \geq 0 \). Examples show that \( \Gamma_F \) contains informations independent of \( Z_F \) (and vice versa). Using the theory of symmetric functions and collecting monomials in \( x_1, x_2, x_3, \ldots \), both series can be written in the “standard form”

\[
f(x_1, x_2, x_3, \ldots) = \sum_{n \geq 0} \sum_{\sigma \vdash n} f_\sigma \frac{x_1^{\sigma_1} x_2^{\sigma_2} \cdots x_n^{\sigma_n}}{\sigma_1! \sigma_2! \cdots \sigma_n!},
\]

where the coefficients \( f_\sigma \) satisfy

\[
f_\sigma \in \mathbb{N} \text{ if } f = Z_F, \quad \text{while} \quad f_\sigma \in \mathbb{Z} \text{ if } f = \Gamma_F.
\]
The notation $\sigma \vdash n$ means that $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ runs through the partitions of $n$, and $\sigma_i$ is the number of parts of size $i$ in $\sigma$.

The transformations $F \rightarrow Z_F$ and $F \rightarrow \Gamma_F$ both commute with combinatorial sums, products, substitutions and derivations:

\begin{align*}
(1) \quad & Z_{F+G} = Z_F + Z_G, \quad Z_{FG} = Z_F \cdot Z_G, \quad Z_{F \circ G} = Z_F \circ Z_G, \quad Z_{F'} = \frac{\partial Z_F}{\partial x_1}, \\
(2) \quad & \Gamma_{F+G} = \Gamma_F + \Gamma_G, \quad \Gamma_{FG} = \Gamma_F \cdot \Gamma_G, \quad \Gamma_{F \circ G} = \Gamma_F \circ \Gamma_G, \quad \Gamma_{F'} = \frac{\partial \Gamma_F}{\partial x_1},
\end{align*}

where $Z_F \circ Z_G$ (resp. $\Gamma_F \circ \Gamma_G$) denotes the plethystic substitution of the series $Z_F$ and $Z_G$ (resp. $\Gamma_F$ and $\Gamma_G$). The plethysms $Z_F \circ Z_G$ of two series $Z_F = f(x_1, x_2, x_3, \ldots)$ and $Z_G = g(x_1, x_2, x_3, \ldots)$ is the series $f(g_1, g_2, g_3, \ldots)$, where $g_k(x_1, x_2, x_3, \ldots) = g(x_k, x_{2k}, x_{3k}, \ldots)$ [2].

The series $F(x), \tilde{F}(x)$, and $\bar{F}(x)$ can be computed from $Z_F$ and $\Gamma_F$ by making use of the following remarkable formulas:

\begin{align*}
(3) \quad & F(x) = Z_F(x, 0, 0, \ldots) = \Gamma_F(x, 0, 0, \ldots), \\
(4) \quad & \tilde{F}(x) = Z_F(x, x^2, x^3, \ldots), \quad \bar{F}(x) = \Gamma_F(x, x^2, x^3, \ldots).
\end{align*}

The following explicit formulas are direct consequences of (1)-(4):

\begin{align*}
(\hat{F} \circ G)(x) &= Z_F(\tilde{G}(x), \tilde{G}(x^2), \ldots), \quad \hat{F}(x) = \frac{\partial Z_F}{\partial x_1}(x, x^2, x^3, \ldots), \\
(\bar{F} \circ G)(x) &= Z_F(\bar{G}(x), \bar{G}(x^2), \ldots), \quad \bar{F}(x) = \frac{\partial \Gamma_F}{\partial x_1}(x, x^2, x^3, \ldots).
\end{align*}

The series $F$, $\tilde{F}$, $\bar{F}$, $Z_F$, and $\Gamma_F$ have been computed for many elementary species. Table 1 gives a short table.

Given any species $F$ and any integer $n \in \mathbb{N}$ we can extract a subspecies $F_n \subseteq F$ by collecting all those $F$-structures having an underlying cardinality $n$. If $F = F_n$ we say that $F$ is concentrated,

<table>
<thead>
<tr>
<th>Species</th>
<th>$F$</th>
<th>$F$</th>
<th>$F$</th>
<th>$F$</th>
<th>$Z_F$</th>
<th>$\Gamma_F$</th>
</tr>
</thead>
<tbody>
<tr>
<td>singleton</td>
<td>$X$</td>
<td>$x$</td>
<td>$x$</td>
<td>$x_1$</td>
<td>$\frac{1}{2}(x_1^2 + x^2)$</td>
<td>$\frac{1}{2}(x_1^2 - x^2)$</td>
</tr>
<tr>
<td>pair</td>
<td>$E_2$</td>
<td>$\frac{x^2}{2!}$</td>
<td>$x^2$</td>
<td>0</td>
<td>$\exp\left(\sum_{n \geq 1} \frac{x^n}{n}\right)$</td>
<td>$\exp\left(\sum_{n \geq 1} (-1)^n \frac{x^n}{n}\right)$</td>
</tr>
<tr>
<td>set</td>
<td>$E$</td>
<td>$\exp(x)$</td>
<td>$\frac{1}{1-x}$</td>
<td>$1+x$</td>
<td>$\exp\left(\sum_{n \geq 1} \frac{x^n}{n}\right)$</td>
<td>$\exp\left(\sum_{n \geq 1} (-1)^n \frac{x^n}{n}\right)$</td>
</tr>
<tr>
<td>subset</td>
<td>$P$</td>
<td>$\exp(2x)$</td>
<td>$\frac{1}{(1-x)^2}$</td>
<td>$(1+x)^2$</td>
<td>$\exp\left(\sum_{n \geq 1} (-1)^n \frac{x^n}{n}\right)$</td>
<td>$\exp\left(\sum_{n \geq 1} (-1)^n \frac{x^n}{n}\right)$</td>
</tr>
<tr>
<td>list</td>
<td>$L$</td>
<td>$\ln\left(\frac{1}{1-x}\right)$</td>
<td>$\frac{1}{1-x}$</td>
<td>$1-x$</td>
<td>$\phi(n) \ln\left(\frac{1}{1-x}\right)$</td>
<td>$\sum_{n \geq 1} \phi(n) \ln\left(\frac{1}{1-x}\right)$</td>
</tr>
<tr>
<td>cycle</td>
<td>$C$</td>
<td>$\frac{1}{1-x}$</td>
<td>$\prod_{n \geq 1} \frac{1}{1-x^n}$</td>
<td>$1+x$</td>
<td>$\prod_{n \geq 1} \frac{1}{1-x^n}$</td>
<td>$\prod_{n \geq 1} \frac{1}{1-x^n}$</td>
</tr>
</tbody>
</table>

Table 1. Basic species and their generating series. Here $\phi(n)$ and $\mu(n)$ respectively denote the classical Euler and Möbius functions of $n$. 
2. General Explicit and Recursive Formulas

Consider the combinatorial equation $\mathfrak{A} = X \cdot E(\mathfrak{A})$ which characterizes the species $\mathfrak{A}$ of rooted trees. We get in a purely mechanical way the following classical result [3]

\[
\mathfrak{A}(x) = x e^{A(x)}, \quad \mathfrak{A}(x) = x \exp \left( \sum_{n \geq 1} \frac{\mathfrak{A}(x^n)}{n} \right), \quad \mathfrak{A}(x) = x \exp \left( \sum_{n \geq 1} \frac{(-1)^{n-1} \mathfrak{A}(x^n)}{n} \right),
\]

\[
Z_A = x_1 \exp \left( \sum_{n \geq 1} \frac{(Z_A)_n}{n} \right), \quad \Gamma_A = x_1 \exp \left( \sum_{n \geq 1} \frac{(-1)^{n-1} (\Gamma_A)_n}{n} \right).
\]

The fundamental Otter-Robinson-Leroux [12, 14, 10] equation

\[
\mathcal{A} + \mathfrak{A}^2 = \mathfrak{A} + E_2(\mathfrak{A}),
\]

between the species $\mathfrak{A}$ of rooted trees and the species $\mathcal{A}$ of ordinary trees, gives the following results

\[
\mathcal{A}(x) = \mathfrak{A}(x) - \frac{1}{2} (\mathfrak{A}(x))^2, \quad \mathfrak{A}(x) = \mathfrak{A}(x) - \frac{1}{2} (\mathfrak{A}(x))^2 + \frac{1}{2} \mathfrak{A}(x^2) \text{ (Otter [12])},
\]

\[
\mathfrak{A}(x) = \mathfrak{A}(x) - \frac{1}{2} (\mathfrak{A}(x))^2 - \frac{1}{2} \mathfrak{A}(x^2) \text{ (Harary-Prins [3])},
\]

\[
Z_A = Z_A - \frac{1}{2} (Z_A)^2 + \frac{1}{2} (Z_A)_2 \text{ (Robinson [14])}, \quad \Gamma_A = \Gamma_A - \frac{1}{2} (\Gamma_A)^2 - \frac{1}{2} (\Gamma_A)_2.
\]

3. R-enriched rooted trees and R-enriched trees

The species $\mathfrak{A}_R$ of $R$-enriched rooted trees (Labelle 1981) is recursively characterized by the following combinatorial equation (i.e. natural isomorphism between species):

\[
(6) \quad \mathfrak{A}_R = X \cdot R(\mathfrak{A}_R).
\]

Depending on the choice of “enriching species” $R$, this definition includes: ordinary rooted trees ($R = E$), cyclic rooted trees ($R = 1 + C$), binary rooted trees ($R = 1 + E_2$), plane rooted trees ($R = L$), oriented rooted trees ($R = E^2$), and permutation rooted trees ($R = S$).

A variant to the notion of $R$-enriched rooted tree is that of $R'$-enriched tree. It is a tree in which the set of “immediate neighbours” of each node is equipped with an $R'$-structure. The species of $R'$-enriched rooted trees is denoted by $\mathcal{A}_R$.

**Lemma 1 (Labelle 1981).** The species $\mathcal{A}_R^* = X \frac{d\mathfrak{A}_R}{dx}$ of pointed $R$-enriched trees satisfies

\[
\mathcal{A}_R^* = XR(\mathfrak{A}_R'),
\]

where $R' = \frac{dR}{dx}$ and $\mathfrak{A}_R' = XR'(\mathfrak{A}_R')$ is the species of $R'$-enriched rooted trees.

**Lemma 2.** The species $\mathcal{A}_R$ of $R$-enriched trees and the species $\mathfrak{A}_R'$ of $R'$-enriched rooted trees are related by the combinatorial equation

\[
(7) \quad \mathcal{A}_R + \mathfrak{A}_R^2 = XR(\mathfrak{A}_R') + E_2(\mathfrak{A}_R').
\]
Theorem 1. From equations (6) and (7) we obtain the following ten formulas:

\[ \widetilde{A}_R(x) = xR(\widetilde{A}_R(x)), \]
\[ \overline{A}_R(x) = xZ_R(\overline{A}_R(x)), \]
\[ \overline{A}_R(x) = x\Gamma_R(\overline{A}_R(x)), \]
\[ \overline{A}_R(x) = xR(\overline{A}_R(x)) - \frac{1}{2}(\overline{A}_R(x))^2, \]
\[ \overline{A}_R(x) = xZ_R(\overline{A}_R(x)), \]
\[ \overline{A}_R(x) = x\Gamma_R(\overline{A}_R(x)), \]
\[ \overline{A}_R(x) = xR(\overline{A}_R(x)) - \frac{1}{2}(\overline{A}_R(x))^2 - \frac{1}{2}(\overline{A}_R(x))^2, \]
\[ \overline{A}_R(x) = xZ_R(\overline{A}_R(x)), \]
\[ \overline{A}_R(x) = x\Gamma_R(\overline{A}_R(x)), \]
\[ \overline{A}_R(x) = xR(\overline{A}_R(x)) - \frac{1}{2}(\overline{A}_R(x))^2 - \frac{1}{2}(\overline{A}_R(x))^2. \]

Theorem 2. Let \( \mathfrak{A}_R \) be the species of \( R \)-enriched rooted trees. Then, for every partition \( \sigma = (\sigma_1, \sigma_2, \ldots) \) and every species \( F \),

\[ \text{coeff}_\sigma Z_{A_R} = \text{coeff}_\sigma x_1 \prod_{k \geq 1} \left( 1 - \frac{x_1 \partial Z_R / \partial x_1}{Z_R} \right)_k \left( Z_R \right)^{\sigma_k}_k, \]
\[ \text{coeff}_\sigma Z_{F(A_R)} = \text{coeff}_\sigma Z_F \prod_{k \geq 1} \left( 1 - \frac{x_1 \partial Z_R / \partial x_1}{Z_R} \right)_k \left( Z_R \right)^{\sigma_k}_k, \]
\[ \text{coeff}_\sigma \Gamma_{A_R} = \text{coeff}_\sigma x_1 \prod_{k \geq 1} \left( 1 - \frac{x_1 \partial \Gamma_R / \partial x_1}{\Gamma_R} \right)_k \left( \Gamma_R \right)^{\sigma_k}_k, \]
\[ \text{coeff}_\sigma \Gamma_{F(A_R)} = \text{coeff}_\sigma \Gamma_F \prod_{k \geq 1} \left( 1 - \frac{x_1 \partial \Gamma_R / \partial x_1}{\Gamma_R} \right)_k \left( \Gamma_R \right)^{\sigma_k}_k. \]

Theorem 3. Let \( A_R \) be the species of \( R \)-enriched trees. Then, for every partition \( \sigma = (\sigma_1, \ldots) \),

\[ \text{coeff}_\sigma Z_{A_R} = \begin{cases} \omega_{\sigma_1, \ldots, \sigma_2, \ldots} & \text{if } \sigma_1 \neq 0, \\ 2^\sum_{k=1}^{n} \sigma_k - 1 \cdot b_{\sigma_2, \ldots} & \text{if } 0 = \sigma_1 = \sigma_3 = \cdots, \\ 0 & \text{otherwise}, \end{cases} \]
\[ \text{coeff}_\sigma \Gamma_{A_R} = \begin{cases} \omega_{\sigma_1, \ldots, \sigma_2, \ldots} & \text{if } \sigma_1 \neq 0, \\ 2^\sum_{k=1}^{n} \sigma_k - 1 \cdot b_{\sigma_2, \ldots} & \text{if } 0 = \sigma_1 = \sigma_3 = \cdots, \\ 0 & \text{otherwise}, \end{cases} \]
where

\[ \omega_\sigma = \text{coeff}_\sigma Z_R \prod_{k \geq 1} \left( 1 - \frac{x_1 \partial^2 Z_R / \partial x_1^2}{\partial Z_R / \partial x_1} \right)_k \left( \partial Z_R / \partial x_1 \right)^{\sigma_k}_k, \]

\[ b_\sigma = \text{coeff}_\sigma x_1 \prod_{k \geq 1} \left( 1 - \frac{x_1 \partial^2 Z_R / \partial x_1^2}{\partial Z_R / \partial x_1} \right)_k \left( \partial Z_R / \partial x_1 \right)^{\sigma_k}_k, \]

\[ \omega^*_\sigma = \text{coeff}_\sigma \Gamma_R \prod_{k \geq 1} \left( 1 - \frac{x_1 \partial^2 \Gamma_R / \partial x_1^2}{\partial \Gamma_R / \partial x_1} \right)_k \left( \partial \Gamma_R / \partial x_1 \right)^{\sigma_k}_k, \]

\[ b^*_\sigma = \text{coeff}_\sigma x_1 \prod_{k \geq 1} \left( 1 - \frac{x_1 \partial^2 \Gamma_R / \partial x_1^2}{\partial \Gamma_R / \partial x_1} \right)_k \left( \partial \Gamma_R / \partial x_1 \right)^{\sigma_k}_k. \]

4. Examples

Every formula developed above can be implemented on symbolic computation systems such as MAPLE, MATHEMATICA, MACSYMA, or DARWIN (Bergeron 1988). Examples of computation are given in [7]. In the sequence, this paragraph contains concrete applications of some of our results for particular choices of the enriching species \( R \).

\(( R = 1 + C )\).

\# of types of planes trees on \( n \) vertices =

\[ \frac{1}{2(n-1)} \sum_{d(n-1)} \phi \left( \frac{n-1}{d} \right) \left( \frac{2d}{d} \right) - \frac{1}{2} c_{n-1} + \frac{1}{2} \chi_{\text{even}}(n)c_{(n/2)-1}, \]

\# of types of asymmetric planes trees on \( n \) vertices =

\[ \frac{1}{2(n-1)} \sum_{d(n-1)} \mu \left( \frac{n-1}{d} \right) \left( \frac{2d}{d} \right) - \frac{1}{2} c_{n-1} - \frac{1}{2} \chi_{\text{even}}(n)c_{(n/2)-1}, \]

where \( \chi_{\text{even}} \) is the characteristic function of the set of even numbers, \( \phi(n) \) and \( \mu(n) \) respectively denote the classical Euler and Möbius functions of \( n \), and \( c_n = \frac{1}{n+1} \binom{2n}{n} \) are the usual Catalan numbers.

\(( R = E )\). In the case \( \mathcal{A}_R = \mathcal{A}_E = \mathcal{A} \) (the species of rooted trees), (15) and (17) can be rewritten as

\[ \text{coeff}_\sigma Z_A = \begin{cases} 0 & \text{if } \sigma_1 = 0, \\ \sigma_1^{-1} \prod_{k \geq 2} (\phi_k^\sigma - k \sigma_k \phi_k^{\sigma_k-1}) & \text{otherwise,} \end{cases} \]

\[ \text{coeff}_\sigma \Gamma_A = \begin{cases} 0 & \text{if } \sigma_1 = 0, \\ \sigma_1^{-1} \prod_{k \geq 2} (\theta_k^\sigma - k \sigma_k \theta_k^{\sigma_k-1}) & \text{otherwise,} \end{cases} \]

where \( \phi_k = \sum d(k) d \sigma d \) and \( \theta_k = \sum d(k)(-1)^{(k/d)-1} d \sigma d \).
In the case \( \mathcal{A}_R = \mathcal{A}_E = \mathcal{A} \) (the species of trees), (19) and (20) can be rewritten as

\[
\text{coeff}_\sigma Z_{\mathcal{A}_R} = \begin{cases} 
    a_\sigma / \sigma_1 & \text{if } \sigma_1 \neq 0, \\
    2^{(\sum \sigma_k) - 1} a_{\sigma_2, \sigma_3, \ldots} & \text{if } 0 = \sigma_1 = \sigma_3 = \cdots, \\
    0 & \text{otherwise,}
\end{cases}
\]

\[
\text{coeff}_\sigma \Gamma_{\mathcal{A}_R} = \begin{cases} 
    a_\sigma^* / \sigma_1 & \text{if } \sigma_1 \neq 0, \\
    2^{(\sum \sigma_k) - 1} a_{\sigma_2, \sigma_3, \ldots}^* & \text{if } 0 = \sigma_1 = \sigma_3 = \cdots, \\
    0 & \text{otherwise,}
\end{cases}
\]

where \( a_\sigma = \text{coeff}_\sigma Z_{\mathcal{A}} \) and \( a_\sigma^* = \text{coeff}_\sigma \Gamma_{\mathcal{A}} \).

For a direct application of (16) and (18), consider the species of endofunctions \( \text{End} = S(\mathfrak{A}) \), where \( S \) is the species of permutations. Taking \( F = S \) and \( R = E \) in (16) and (18), a few computations give

\[
\text{coeff}_\sigma Z_{\text{End}} = \sigma_1^2 \prod_{k \geq 2} (\phi_k^\sigma - k \sigma_k \theta_k^\sigma - 1)
\]

\[
\text{coeff}_\sigma \Gamma_{\text{End}} = \sigma_1^2 (\theta_2^\sigma - 4 \sigma_2 \theta_2^\sigma - 4 \sigma_2 (\sigma_2 - 1) \theta_2^{2\sigma} - 1) \prod_{k \geq 3} (\theta_k^{2\sigma} - k \sigma_k \theta_k^{\sigma - 1}),
\]

where \( \phi_k = \sum_{d|k} d \sigma_d \) and \( \theta_k = \sum_{d|k} (-1)^{(k/d) - 1} d \sigma_d \).

\((\mathbf{R} = \mathbf{S})\). The \( \mathfrak{A}_S \)-structures are called permutation rooted trees. In this case, formula (10) takes the very compact form

\[
\mathfrak{A}_S(x) = \sum_{n \geq 0} a_n x^n = x \frac{1 - \mathfrak{A}_S(x^2)}{1 - \mathfrak{A}_S(x)},
\]

where \( a_0 = 0, a_1 = 1, \) and \( a_{n+1} = (a_1 a_n + a_2 a_{n-1} + \cdots + a_n a_1) - \chi_{\text{even}}(n) a_{n/2} \).

\((\mathbf{R} = \mathbf{E} - E_2)\). A topological tree (also called homeomorphically irreducible tree) is a tree that has no node of degree 2. The species \( \mathcal{A}_{\text{top}} \) of topological trees can be expressed in terms of the species \( \mathcal{A} \) and \( \mathfrak{A} \) through the combinatorial equation

\[
\mathcal{A}_{\text{top}} = \mathcal{A} \left( \frac{X}{1 + X} \right) + \mathcal{X} \mathfrak{A} \left( \frac{X}{1 + X} \right) - X E_2 \left( \mathfrak{A} \left( \frac{X}{1 + X} \right) \right).
\]

This equation gives the formulas

\[
\mathfrak{A}_{\text{top}}(x) = \mathfrak{A} \left( \frac{x}{1 + x}, \frac{x^2}{1 + x^2}, \ldots \right) + x \mathfrak{A} \left( \frac{x}{1 + x}, \frac{x^2}{1 + x^2}, \ldots \right) - \frac{x}{2} \mathfrak{A} \left( \frac{x}{1 + x}, \frac{x^2}{1 + x^2}, \ldots \right) - \frac{x}{2} \mathfrak{A} \left( \frac{x^2}{1 + x^2}, \frac{x^4}{1 + x^4}, \ldots \right),
\]

\[
\mathcal{A}_{\text{top}}(x) = \Gamma_A \left( \frac{x}{1 + x}, \frac{x^2}{1 + x^2}, \ldots \right) + x \Gamma_A \left( \frac{x}{1 + x}, \frac{x^2}{1 + x^2}, \ldots \right) - \frac{x}{2} \Gamma_A \left( \frac{x}{1 + x}, \frac{x^2}{1 + x^2}, \ldots \right) + \frac{x}{2} \Gamma_A \left( \frac{x^2}{1 + x^2}, \frac{x^4}{1 + x^4}, \ldots \right).
\]
5. Related topics

**Theorem 4.** The number of types of rooted plane trees with degree distribution \( \overline{r} = (i_1, i_2, i_3, \ldots) \) is

\[
\frac{1}{n-1} \sum_{p \in \text{supp} \overline{r}} \sum_{d \mid \overline{r} - \overline{r}_p} \phi(d) \left( \frac{n-1}{d} \right) \left( i_1/d, i_2/d, \ldots, (i_p-1)/d, \ldots \right),
\]

where \( \text{supp} \overline{r} = \{ p \mid i_p \neq 0 \} \), \( d \mid \overline{r} \) iff \( \forall p : d \mid i_p \), and \( \overline{r} - \overline{r}_p = (i_1, i_2, \ldots, i_p-1, \ldots) \).

**Theorem 5.** The number of types of bicoloured plane trees with degree distributions \( \overline{r} = (i_1, i_2, i_3, \ldots) \) and \( \overline{r} = (j_1, j_2, j_3, \ldots) \) is

\[
\frac{1}{n} \sum_{p \in \text{supp} \overline{r}} \sum_{d \mid \overline{r}, j \mid \overline{r}} \phi(d) \left( \frac{n}{d} \right) \left( i_1/d, i_2/d, \ldots, (i_p-1)/d, \ldots \right)
\]

\[
+ \frac{1}{m} \sum_{p \in \text{supp} \overline{r}} \sum_{d \mid \overline{r}, j \mid \overline{r}} \phi(d) \left( \frac{n}{d} \right) \left( i_1/d, i_2/d, \ldots, (i_p-1)/d, \ldots \right) \left( \frac{m}{d} \right) \left( j_1/d, j_2/d, \ldots, (j_p-1)/d, \ldots \right)
\]

\[- \frac{n+m-1}{nm} \left( \frac{n}{i_1, i_2, \ldots, \frac{m}{j_1, j_2, \ldots}} \right),
\]

where \( \text{supp} \overline{r} = \{ p \mid i_p \neq 0 \} \), \( d \mid \overline{r} \) iff \( \forall p : d \mid i_p \), and \( \overline{r} - \overline{r}_p = (i_1, i_2, \ldots, i_p-1, \ldots) \).

Let \( B = B(X, Y) \) be the species of rooted trees with internal point of sort \( X \) and leaves of sort \( Y \). This species is characterized by the functional equation

\[
B = Y + X \cdot E^*(B)
\]

where \( E^* = E - 1 \) stands for species characteristic of nonempty sets.

**Theorem 6.** The species \( \mathfrak{A} = \mathfrak{A}(X) \) and \( B = B(X, Y) \) are related by the following combinatorial equation

\[
B = Y - X + \mathfrak{A}(X \cdot E(Y - X))
\]

where \( E = E(X) \) is the species of sets.

Let \( B = B(X, Y) \) be the species of trees with internal point of sort \( X \) and leaves of sort \( Y \).

**Theorem 7.** The species \( \mathcal{A} = \mathcal{A}(X) \) and \( B = B(X, Y) \) are related by the following combinatorial equation

\[
B = (Y - X) + E_2(Y - X) + \mathcal{A}(X \cdot E(Y - X))
\]

where \( E = E(X) \) is the species of sets and \( E_2 = E_2(X) \) is the species of sets of cardinality two.

**Theorem 8.** Let \( U \) be an \( n \)-set, \( \sigma \) be a permutation of \( U \) whose cyclic type is \( (\sigma_1, \sigma_2, \ldots, \sigma_n) \). Then, for \( n \geq 2 \), the expected number of leaves in a random rooted tree (resp. in a random tree) on \( U \) of which \( \sigma \) is an automorphism is given respectively by

\[
\frac{1}{\alpha} \sum_{k=1}^{n} k \sigma_k \left( \sum_{d \mid k} d \sigma_d - k \right) \cdot a_{(\sigma_1, \ldots, \sigma_{n-1}, \ldots, \sigma_n)}
\]

\[
\frac{1}{\alpha} \sum_{k=1}^{n} k \sigma_k \left( \sum_{d \mid k} d \sigma_d - k \right) \cdot \alpha_{(\sigma_1, \ldots, \sigma_{n-1}, \ldots, \sigma_n)}
\]

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where \( a_\sigma \) (resp. \( \alpha_\sigma \)) is the number of rooted trees (resp. trees) of which \( \sigma \) is an automorphism.

**Example.** The expected number of leaves in a random rooted tree with \( \sigma \) as automorphism is

1. \( n(n-1)^{n-1}/n^{n-1} \sim n/e \) if \( \sigma = \text{Id}_n \) (well known),
2. \( \frac{(n-2)^{n-2}}{n^{n-2}} + 2 \) if \( \sigma \) is of type \( (n-2,1,0,\ldots,0) \),
3. \( 97,892,76140 \) if \( n = 186 \) and \( \sigma \) is of type \( (6,1,12,0,0,0,4,3,2,0,0,6) \) (example given after a few seconds, using Maple on a personal computer).

**Bibliography**


