# Exponentially-improved asymptotic solutions of ordinary differential equations

*Adri Olde Daalhuis*
CWI, Amsterdam

June 14, 1993

[summary by Bruno Salvy[1]]

**Abstract**

Re-expansions are found for the optimal remainder terms in the well-known asymptotic series solutions of homogeneous linear differential equations of the second order in the neighbourhood of an irregular singularity of rank one. The re-expansions are in terms of generalized exponential integrals and have greater regions of validity than the original expansions, as well as being considerably more accurate and providing a smooth interpretation of the Stokes phenomenon. They are also of strikingly simple form. Also found are explicit asymptotic expansions for the higher coefficients of the original asymptotic solutions.

The object of study is a linear ordinary differential operator

$$L = \frac{d^2}{dz^2} + f(z)\frac{d}{dz} + g(z),$$

where the coefficients $f(z)$ and $g(z)$ have Laurent expansions in a non-empty neighbourhood of infinity

$$f(z) = \sum_{s\geq 0} \frac{f_s}{z^s}, \quad g(z) = \sum_{s\geq 0} \frac{g_s}{z^s},$$

and infinity is an irregular singular point of rank 1, which means that the asymptotic expansion of two distinct solutions are given by

$$w_1 \sim e^{\lambda_1 z} z^{\mu_1} \sum_{s\geq 0} \frac{a_{s,1}}{z^s}, \quad |\operatorname{Arg}[(\lambda_2-\lambda_1)z]| \leq 3\pi/2-\delta,$$
$$w_2 \sim e^{\lambda_2 z} z^{\mu_2} \sum_{s\geq 0} \frac{a_{s,2}}{z^s}, \quad |\operatorname{Arg}[(\lambda_1-\lambda_2)z]| \leq 3\pi/2-\delta,$$

where $\delta$ is an arbitrary small real number.

It is known that these expansions are in general divergent, and several methods to get numerical values out of them have been studied. The most widely known is *summation to the least term* where one adds up the contributions from $a_{s,i}/z^s$ until they start increasing. The main result of this paper is an asymptotic expansion of the remainder term which makes it possible to get even more precision than summation to the least term.

The exponents $\lambda_1$, $\lambda_2$ are roots of the characteristic equation

$$\lambda^2 + f_0\lambda + g_0 = 0,$$

and by changing $z$ into $z/(\lambda_2-\lambda_1)$ one may assume $\lambda_2-\lambda_1 = 1$.

[1]This very concise summary was written several months after the talk without very precise notes. It should be viewed mainly as a pointer to the reference [1].

Since $e^{2i\pi\mu_1} w_1(ze^{-2i\pi})$ has the same asymptotic expansion as $w_1$ and is also a solution of the operator, one obtains the following *connection formulæ*:

$$w_1(z) = e^{2i\pi\mu_1} w_1(ze^{-2i\pi}) + C_1 w_2(z),$$
$$w_2(z) = e^{-2i\pi\mu_2} w_2(ze^{2i\pi}) + C_2 w_1(z).$$

We can now state the main result of [1].

THEOREM 1. *Define $R_n^{(1)}(z)$ and $R_n^{(2)}(z)$ by*

$$w_1 = e^{\lambda_1 z} z^{\mu_1} \sum_{s=0}^{n-1} \frac{a_{s,1}}{z^s} + R_n^{(1)}(s), \quad w_2 = e^{\lambda_2 z} z^{\mu_2} \sum_{s=0}^{n-1} \frac{a_{s,2}}{z^s} + R_n^{(2)}(s),$$

*where $n = |z| + \alpha$ and $\alpha$ is bounded. Then*

$$R_n^{(1)}(z) = (-1)^{n-1} i e^{(\mu_2-\mu_1)\pi i} e^{\lambda_2 z} z^{\mu_2} \left\{ C_1 \sum_{s=0}^{m-1} (-1)^s a_{s,2} \frac{F_{n+\mu_2-\mu_1-s}(z)}{z^s} + R_{m,n}^{(1)}(z) \right\},$$
$$R_n^{(2)}(z) = (-1)^{n} i e^{(\mu_2-\mu_1)\pi i} e^{\lambda_1 z} z^{\mu_1} \left\{ C_2 \sum_{s=0}^{m-1} (-1)^s a_{s,1} \frac{F_{n+\mu_1-\mu_2-s}(ze^{-\pi i})}{z^s} + R_{m,n}^{(2)}(z) \right\},$$

*where $m$ is an arbitrary fixed non-negative integer, and for large $|z|$*

$$R_{m,n}^{(1)}(z) = O(e^{-|z|-z} z^{-m}), \qquad |\operatorname{Arg} z| \le \pi,$$
$$R_{m,n}^{(1)}(z) = O(z^{-m}), \qquad \pi \le |\operatorname{Arg} z| \le \frac{5}{2}\pi - \delta,$$
$$R_{m,n}^{(2)}(z) = O(e^{-|z|+z} z^{-m}), \qquad 0 \le \operatorname{Arg} z \le 2\pi,$$
$$R_{m,n}^{(2)}(z) = O(z^{-m}), \qquad -\frac{3}{2}\pi + \delta \le \operatorname{Arg} z \le 0 \text{ and } 2\pi \le \operatorname{Arg} z \le \frac{7}{2}\pi - \delta,$$

*uniformly with respect to* $\operatorname{Arg} z$ *in each case.*

In this theorem $F$ denotes the following generalized exponential integral

$$F_p(z) = \frac{e^{-z}}{2\pi} \int_0^\infty \frac{e^{-zt} t^{p-1}}{1+t} dt.$$

Note that the coefficients that appear in the expansion are precisely the coefficients of the expansions of $w_1$ and $w_2$. This is related to the following older theorem of Olver.

THEOREM 2. *Let $m$ be an arbitrary fixed non-negative integer. Then as $s \to \infty$,*

$$a_{s,1} = (-1)^s \frac{e^{(\mu_2-\mu_1)\pi i}}{2\pi i} \left\{ C_1 \sum_{j=0}^{m-1} (-1)^j a_{j,2} \Gamma(s+\mu_2-\mu_1-j) + \Gamma(s+\mu_2-\mu_1-m) O(1) \right\},$$
$$a_{s,2} = -\frac{1}{2\pi i} \left\{ C_2 \sum_{j=0}^{m-1} a_{j,1} \Gamma(s+\mu_1-\mu_2-j) + \Gamma(s+\mu_1-\mu_2-m) O(1) \right\}.$$

## Bibliography

[1] Daalhuis (A. B. Olde) and Olver (F. W. J.). – Exponentially-improved asymptotic solutions of ordinary differential equations II: irregular singularities of rank one. – Preprint, 1993.