10

Holonomic Symmetric Functions

Dominique Gouyou-Beauchamps
LRI, Orsay

[summary by Philippe Flajolet]

By suitably defining a notion of holonomy for symmetric functions in infinitely many variables, Gessel [3] derives or rederives the holonomic character of generating functions for structures as diverse as: *k–regular graphs; Young tableaux of fixed height k; permutations with longest increasing subsequence of length k; square integer matrices with row sum and column sum k; $k \times n$ latin rectangles.*

Each of these problems represents a highly non trivial enumeration problem that is far from being amenable to direct symbolic methods (except for simple boundary cases).

Recall that a univariate function is *holonomic* if it satisfies a linear differential equation with polynomial coefficients (this is also called D–finite). Accordingly, its coefficients, in turn named holonomic, satisfy recurrences with polynomial coefficients (this is also called P–recursive).

Symmetric functions. We work with infinitely many indeterminates $x_1, x_2, \ldots$, and we are concerned with particular *symmetric functions*.

(a). If $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ is a partition, then define the *monomial* symmetric function

$$m_\lambda = \sum_{i_1, i_2, \ldots, i_k} x_{i_1}^{\lambda_1} x_{i_2}^{\lambda_2} \cdots x_{i_k}^{\lambda_k}, \tag{1}$$

the sum being extended to all distinct indices. For instance, if $\lambda = (3,1,1) \equiv (3,1^2)$, then $m_\lambda = \sum x^3 yz$.

(b). The *elementary* symmetric functions are

$$e_r = \sum_{i_1 < i_2 < \cdots < i_r} x_{i_1} x_{i_2} \cdots x_{i_r}, \tag{2}$$

so that

$$E(t) := \sum_{r \geq 0} e_r t^r = \prod_{i \geq 1} (1 + x_i t). \tag{3}$$

This is extended to partitions indices: $e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_k}$. For instance, $e_{(3,3,1,1,1)} = e_3^2 e_1^3$.

(c). The *complete* symmetric functions are

$$h_r = \sum_{i_1 \leq i_2 \leq \cdots \leq i_r} x_{i_1} x_{i_2} \cdots x_{i_r}, \tag{4}$$

so that

$$H(t) := \sum_{r \geq 0} h_r t^r = \prod_{i \geq 1} (1 - x_i t)^{-1}. \tag{5}$$

One similarly defines $h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_k}$.
(d). The *power* symmetric functions are

$$p_r = \sum_i x_i^r \tag{6}$$

so that

$$P(t) := \sum_r p_r t^r = \sum_i \frac{t x_i}{1 - t x_i}. \tag{7}$$

Again the definition extends to $p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_k}$.
Each of the sets $\{e_\lambda\}$, $\{h_\lambda\}$, $\{m_\lambda\}$, $\{p_\lambda\}$ constitutes a basis for symmetric functions. Formulæ for changing bases derive from the generating functions (3,5,7).

Operations. (a). One classically defines a *scalar product* with orthogonality properties

$$(P_1): \quad \langle m_\lambda, h_\mu \rangle = \delta_{\lambda,\mu}; \qquad (P_2): \quad \langle p_\lambda, p_\mu \rangle = z_\lambda \delta_{\lambda,\mu}, \tag{8}$$

where $z_\lambda = 1^{r_1} 2^{r_2} \cdots k^{r_k} r_1! r_2! \cdots r_k!$ if $\lambda = (1^{r_1} 2^{r_2} \cdots k^{r_k})$.
(b). The *internal* (or *inner*) product is defined by

$$p_\lambda * p_\mu = \delta_{\lambda,\mu} z_\lambda p_\lambda. \tag{9}$$

It is like the scalar product except that it keeps track of variables.
(c). Finally, there is an operation of *composition* (also known as *plethysm*) written $f(g)$ and defined by $p_m(p_n) = p_{mn}$.

The example of regular graphs. The scalar product permits us to extract coefficients: If f is symmetric, then the coefficient of $x_{i_1}^{\lambda_1} \cdots x_{i_k}^{\lambda_k}$ in f is up to normalization the coefficient of m_λ in f which equals [by (P_1) of Eq. (8)] $\langle f, h_\lambda \rangle$. Given that f is symmetric, it can be expressed in terms of the power functions p_r, as well as h_λ. The computation of the coefficient then reduces to the computation of a scalar product which [by (P_2) of Eq. (8)] is a modified Hadamard product of two functions.
Let us work out the example of 2–*regular graphs*. The function

$$G(x;t) = \prod_{i<j} (1 + x_i x_j t) \tag{10}$$

is a generating function in *infinitely many variables* that encodes all the graphs over a denumerable collection of vertices. (Vertices are labelled and vertex i is represented by the indeterminate x_i. Thus $x_i x_j$ encodes the edge connecting i and j.) The additional variable t records the number of edges. It is a simple exercise to express G in terms of the $p_n = \sum_i x_i^n$:

$$\log G = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{t^n}{n} \left(\sum_{i<j} x_i^n x_j^n \right),$$

so that

$$G(x;t) = \exp\left(\frac{1}{2} \sum_{n=1}^{\infty} \frac{t^n}{n} (p_n^2 - p_{2n}) \right). \tag{11}$$

The number R_n of labelled 2–regular graphs on n nodes is given by

$$R_n t^n = [x_1^2 x_2^2 \cdots x_n^2] G(x;t).$$

From the definition of m_λ, we have $[x_1^2 \cdots x_n^2] m_{(2^n)} = n!$, thus

$$R_n t^n = n! \cdot [\text{coeff. of } m_{(2^n)}]\ G(x;t),$$

and by orthogonality [see (P_1)]:

$$R_n \frac{t^n}{n!} = \langle G(x;t), h_{(2^n)} \rangle = \langle G(x;t), h_2^n \rangle. \qquad (12)$$

Quantity h_2 is itself expressible in the p's: $h_2 = \frac{1}{2}p_1^2 + \frac{1}{2}p_2$. Thus:

$$R_n \frac{t^n}{n!} = \langle G(x;t), (\frac{1}{2}p_1^2 + \frac{1}{2}p_2)^n \rangle. \qquad (13)$$

In this scalar product, only the part of G that involves p_1 and p_2 counts. In other words,

$$R_n \frac{t^n}{n!} = \langle \exp(\frac{t}{2}(p_1^2 - p_2) + \frac{t^2}{4}p_2^2), (\frac{1}{2}p_1^2 + \frac{1}{2}p_2)^n \rangle. \qquad (14)$$

This already provides a combinatorial expression (a sum of finite "rank" in the sense of Comtet [2, p. 216]) for R_n upon expanding and using (P_2). The inner product can also be formulated as a Hadamard product with respect to $\{p_1, p_2\}$ and t. One finally needs to generate the coefficients z_λ with is achieved by means of a further Hadamard product with

$$\Phi(p_1)\Phi(2p_2) \qquad \text{where} \qquad \Phi(z) = \sum_{n=0}^{\infty} n! z^n.$$

Summarizing, we have obtained[3]

$$\sum_n R_n \frac{t^n}{n!} = \left[\exp(\frac{t}{2}(p_1^2 - p_2) + \frac{t^2}{4}p_2^2) \odot_{t,p_1,p_2} \frac{1}{1 - t(\frac{1}{2}p_1^2 + \frac{1}{2}p_2)} \odot_{p_1,p_2} \Phi(p_1)\Phi(2p_2) \right]_{p_1,p_2 \mapsto 1}. \qquad (15)$$

The process is of course fully general.

Theorem 1 *For any fixed k, the exponential generating function of k–regular graphs is holonomic and expressible as a Hadamard product*

$$\exp(E(t;\mathbf{p})) \odot_{t,\mathbf{p}} R(\mathbf{p}) \odot_{\mathbf{p}} \prod_{j=1}^{k} \Phi(jp_j) \Bigg|_{\mathbf{p} \mapsto \mathbf{1}},$$

for some effectively computable polynomial E and rational function R.

Notice that the number of r–regular labelled graphs on n vertices satisfies (see Bollobás's book [1, II.4]):

$$R_n^{(k)} \sim \sqrt{2} e^{-(k^2-1)/4} \left(\frac{k^{k/2}}{e^{k/2} k!} \right)^n n^{kn/2}.$$

[3] The enumeration of 2–regular graphs is accessible to symbolic methods and it is easily found directly (see [2, p. 272]) that the exponential generating function is

$$\frac{e^{-t/2 - t^2/4}}{\sqrt{1-t}}.$$

Coefficient extraction. The example of regular graphs points to a general methodology by which coefficients of sufficiently "regular" monomials inside huge symmetric functions are extracted using a combination of change of bases and inner or scalar products. This extends techniques used earlier by Read, Goulden and Jackson and others. A typical statement is:

Theorem 2 *The coefficient $[x_1 x_2 \cdots x_n]$ in the symmetric function f is computable by its generating function,*

$$\sum_{n=0}^{\infty} \{[x_1 x_2 \cdots x_n]\, f\}\, \frac{X^n}{n!} = f(X, 0, 0, 0, \ldots),$$

where $f(p_1, p_2, \ldots)$ is the expression of f in terms of the p's.

This is Theorem 1 of [3]. A similar theorem gives the bivariate generating function for the coefficient of $x_1 \cdots x_m x_{m+1}^2 \cdots x_{m+n}^2$ in f (Theorem 2 of [3]) or $[x_1 \cdots x_m x_{m+1}^3 \cdots x_{m+n}^3]\, f$ (Theorem 3 of [3]), and so on.
There are applications to the area of order patterns (up and down patterns, increasing subsequences) in permutations with repetitions of various types. For instance, the symmetric generating function of up–and–down sequences is seen to be (use inclusion–exclusion)

$$E = 1/\sum_{r=0}^{\infty}(-1)^r h_{2r},$$

from which counting results derive for alternating permutations of multisets that involve the secant numbers.

Holonomy. A function in the finite set of variables x is holonomic if and only if the vector space spanned over $\mathbb{C}(x)$ by its derivatives is finite dimensional. We shall say that a symmetric function f is holonomic (or D–finite) if and only if, as a function of the *infinite* set p's, it is holonomic in any finite combination of the p's. Most standard holonomy closure properties carry over to this definition provided infinite operations are carefully avoided.
Holonomic symmetric functions appear to be closed under inner product, scalar product with mild restrictions, and certain forms of plethysm. The proofs present no difficulty but have to relie on the closure of standard holonomic functions under Hadamard products (or diagonals), which is Lipshitz's "hard" theorem.
This concept constitutes the natural abstract setting in which results of the previous sections can be cast.

Extensions. Gessel's paper is very rich. It contains a discussion, based on Schur functions, of Young tableaux of bounded height. (The Schur function s_λ is defined as

$$s_\lambda = \det(h_{\lambda_i - i + j})_{1 \le i,j \le n},$$

where we take $h_n = 0$ for $n < 0$.) It continues with symmetric functions in two sets (or more!) of variables, with applications to longest increasing subsequences in permutations.
The following theorem is notable:

Theorem 3 *(i). The exponential generating function of the Young tableaux of height at most k is expressible in terms of a determinant of Bessel functions.*
(ii). The doubly exponential generating function for permutations with longest increasing subsequence of length at most k is expressible in terms of a determinant of Bessel functions.

Note: It was proved by Regev using the hook formula of tableaux and the Selberg integral that the number of permutations of n with longest increasing subsequence of length at most k satisfies asymptotically

$$L_n^{(k)} \sim \gamma_k k^{2n}, \qquad \gamma_k \in \mathbb{R}.$$

Direct asymptotics. Direct asymptotic approximations may well result from symmetric functions, after it is realized how different variables induce weights of different orders of growth. An example suggested by Gessel is the result of Everett and Stein: the number of $n \times n$ non–negative matrices with every row and column sum k is $\langle h_k^n, h_k^n \rangle$, and it satisfies asymptotically

$$M_n^{(k)} \sim \frac{(kn)!}{(k)!^{2n}} e^{(k-1)^2/2}.$$

References

[1] B. Bollobás. *Random Graphs.* Academic Press, 1985.

[2] L. Comtet. *Advanced Combinatorics.* Reidel, Dordrecht, 1974.

[3] I. M. Gessel. Symmetric functions and P–recursiveness. *Journal of Combinatorial Theory,* Series A, 53:257–285, 1990.