An analogue of Stokes phenomenon for q-difference equations

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Contents

Abstract

In a common work[1] with Jean-Pierre Ramis and Changgui Zhang, we described an analogue of the Stokes phenomenon for linear analytic complex q-di erence equations and used it to get the local analytic classi cation. If time permits, I will also show how it was applied in a common work with J.-P. R. the Galois theory of such equations.

[1] Accepted for publication by Astérisque; meanwhile, see URL http://front.math.ucdavis.edu/0903.0853.

Origin

The program of analytic classification of q-difference equations was first proposed and realized by Birkhoff in 1913 in the context of a unified treatment of the Riemann-Hilbert correspondence for *fuchsian* differential, difference and q-difference equations. The classification program was extended by Birkhoff and Guenter in 1941 for irregular equations, but never pursued:

"Up to the present time, the theory of linear q-difference equations has lagged noticeably behind the sister theories of linear difference and differential equations. In the opinion of the autors, the use of the canonical system, as formulated above in a special case, is destined to carry the theory of q-difference equations to a comparable degree of completeness. This program includes in particular *the complete theory of convergence and divergence of formal series, the explicit determination of the essential transcendental invariants (constants in the canonical form), the inverse Riemann theory both for the neighborhood of $x = \infty$ and in the complete plane (case of rational coefficients), explicit integral representation of the solutions, and finally the definition of q-sigma periodic matrices, so far defined essentially only in the case $n = 1$.* Because of its extensiveness this material cannot be presented here."

G.D. Birkhoff, 1941

Plan

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Local analytic classi cation of irregular equations

Generalities

General notations

$q \in \mathbf{C}$, $|q| > 1$.
For $f \in K := \mathbf{C}(\{z\})$ or $f \in \hat{K} := \mathbf{C}((z))$:

$$\sigma_q f(z) := f(qz).$$

A (complex analytic) linear q-di erence equation writes:

$$f(q^n z) + a_1(z) f(q^{n-1} z) + \cdots + a_n(z) f(z) = 0,$$

where $a_1, \ldots, a_n \in K$, $a_n \neq 0$.

$$\begin{aligned} \text{Encoding:} \quad Lf &= 0, \\ \text{where} \quad L &:= \sigma_q^n + a_1 \sigma_q^{n-1} + \cdots + a_n \in \mathcal{D}_{q,K}, \\ \mathcal{D}_{q,K} &:= K\left\langle \sigma_q, \sigma_q^{-1} \right\rangle, \quad \text{(Ore ring)}, \\ \text{and } a_1, \ldots, a_n &\in K, \ a_n \neq 0. \end{aligned}$$

Formal equation: replace K by $\hat{K}$.

Generalities

Equations, systems, q-difference modules

By vectorialisation the q-difference equation $Lf = 0$ can be turned into a q-difference system:

$$\sigma_q X = AX, A \in \mathrm{GL}_n(K), \text{ where } X = \begin{pmatrix} f \\ \vdots \\ \sigma_q^{n-1} f \end{pmatrix},$$

then into a q-difference module

$$M = (E, \quad), \text{ with } E := K^n, \quad := \quad _A : X \mapsto A^{-1}\sigma_q X.$$

(Compare with vector spaces equipped with a connection.)

Equivalently, M is a left $\mathcal{D}_{q,K}$-module of finite length.

Formal equation: replace K by $\hat{K}$.

Generalities

Analytic, formal classification

Morphisms from $(K^n, \ _A)$ to $(K^n, \ _B)$ correspond to matrices $F \in \mathrm{GL}_n(K)$ such that $(\sigma_q F)A = BF$.
Thus, if $Y = FX$, then $\sigma_q X = AX \Rightarrow \sigma_q Y = BY$.

Local analytic classification: we say that $A \sim B$ if there exists a gauge transformation $F \in \mathrm{GL}_n(K)$ such that:

$$B = F[A] := (\sigma_q F)AF^{-1}.$$

Formal classification: the same with $F \in \mathrm{GL}_n(\hat{K})$.

Generalities

Newton polygon (at 0)

The q-di erence operator P has a *Newton polygon at* 0, which consists in slopes $\mu_1 < \cdots < \mu_k \in \mathbf{Q}$ together with their multiplicities $r_1, \ldots, r_k \in \mathbf{N}^*$. (Precise de nition omitted !)

By the cyclic vector lemma, any q-di erence module can be written $M = \mathcal{D}_{q,K}/\mathcal{D}_{q,K}P$.

Theorem and de nition

Generalities

Fundamental solutions, constants

One can prove that an analytic system $\sigma_q X = AX$, $A \in \mathrm{GL}_n(K)$ always has a fundamental solution:

$$\mathcal{X} \in \mathrm{GL}_n(\mathcal{M}(\mathbf{C}^*, 0)),$$

i.e. uniform in a punctured neighborhood of 0.

Therefore, all uniform meromorphic solutions of $\sigma_q X = AX$ have the form $X = \mathcal{X} C$, where $C \in (\mathcal{M}(\mathbf{C}^*, 0)^{\sigma_q})^n$.
The *field of constants*:

$$\mathcal{M}(\mathbf{C}^*, 0)^{\sigma_q} := \{f \in \mathcal{M}(\mathbf{C}^*, 0) \mid \sigma_q f = f\}$$

can be identi ed with the eld of elliptic functions $\mathcal{M}(\mathbf{E}_q)$,

$$\mathbf{E}_q := \mathbf{C}^*/q^{\mathbf{Z}} \simeq \mathbf{C}/(\mathbf{Z} + \mathbf{Z}\tau), \text{ where } e^{2\mathrm{i}\pi\tau} = q.$$

(Identi cation through the map $x \mapsto z := e^{2\mathrm{i}\pi x}$.)

Generalities

Associated vector bundle

This is for analytic systems (over K). One de nes:

$$F_A^{(0)} := \frac{(\mathbf{C}^*, 0) \times \mathbf{C}^n}{(z, X) \sim (qz, A(z)X)} \longrightarrow \frac{(\mathbf{C}^*, 0)}{z \sim qz} = \mathbf{E}_q.$$

This is a holomorphic vector bundle over the complex torus (or elliptic curve) $\mathbf{E}_q$.

The sheaf of holomorphic solutions of $\sigma_q X = AX$ near 0 is canonically isomorphic to the sheaf of sections of $F_A^{(0)}$

$A \rightsquigarrow F_A^{(0)}$ is a \good" functor for classi cation and for Galois theory (faithful, exact, $\otimes$-compatible).

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Slope filtration

Pure modules, equations, systems

A module with one slope only is called *pure isoclinic*.

Pure isoclinic modules of slope 0 are *fuchsian* modules. They have the shape $(K^n, \ _A)$, with $A \in \mathrm{GL}_n(\mathbf{C})$. Their analytic and formal classification (due to Birkhoff) are the same.

Pure isoclinic modules of slope $\mu \in \mathbf{Z}$ have the shape $(K^n, \ _{z^\mu A})$, with $A \in \mathrm{GL}_n(\mathbf{C})$. Their classification boils down to the fuchsian case.

Pure isoclinic modules of nonintegral slope have been classified by van der Put and Reversat in 2005.

Slope filtration

The canonical filtration

Theorem

Any q-difference module over K admits a unique filtration $(M_{\leq\mu})_{\mu\in\mathbf{Q}}$ such that each $M_{(\mu)} := \frac{M_{\leq\mu}}{M_{<\mu}}$ is pure isoclinic of slope μ. The filtration is functorial and $gr : M \rightsquigarrow \bigoplus M_{(\mu)}$ is a faithful exact $\mathbf{C}$-linear $\otimes$-compatible functor.

Theorem

Over $\hat{K}$, the filtration splits canonically. After formalization (base change $\hat{K} \otimes_K -$), gr becomes isomorphic to the identity functor.

Note that, contrary to the second, the rst theorem has no equivalent in the case of di erential equations: it is a consequence of Adams lemma (existence of an analytic factorisation for q-di erence operators).

Slope filtration

Classification and graduation

A direct sum of pure isoclinic modules is called *pure*.

Corollary

For pure modules, formal and analytic classification are equivalent. Formal classification of an analytic q-difference module M amounts to classification (formal or analytic) of the pure module grM.

We already know:

The formal classi cation, *i.e.* classi cation of pure q-di erence modules.

We want to study:

The analytic classi cation within a formal class, *i.e.* with gr$M$ xed.

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Irregular equations

Isoformal classes

The following de nition is inspired by Babbitt and Varadarajan \Local moduli for meromorphic di erential equations" (Asterisque 169-170).

Fix a pure module $M_0 := P_1 \oplus \cdots \oplus P_k$, where $P_1, \ldots, P_k$ are pure isoclinic with slopes $\mu_1 < \cdots < \mu_k$ and ranks $r_1, \ldots, r_k$.

De ne $\mathcal{F}(M_0) = \mathcal{F}(P_1, \ldots, P_k)$ as the quotient set of pairs (M, u), where $u : \mathrm{gr} M \simeq P_1 \oplus \cdots \oplus P_k$, up to the equivalence relation:

$$(M, u) \sim (M', u') \Longleftrightarrow \exists f : M \to M' \ : \ u = u' \circ \mathrm{gr} f.$$

Example

Two slopes, one level:

$$\mathcal{F}(P_1, P_2) = \mathrm{Ext}(P_2, P_1).$$

Irregular equations

The space of analytic classes

Theorem

One gets an affine space (actually, a scheme) of dimension:

$$\dim \mathcal{F}(P_1, \ldots, P_k) = \sum_{1 \leq i < j \leq k} r_i r_j (\mu_j - \mu_i).$$

(There is a q-Gevrey version.)

This dimension is equal to the *irregularity* of $\mathrm{End}(M_0)$.

From now on,
the slopes will be assumed to be integral:

$$\mu_1, \ldots, \mu_k \in \mathbf{Z}.$$

Irregular equations

Matricial description

A formal class is encoded by $M_0 = (K^n, \quad _{A_0})$, with:

$$A_0 := \begin{pmatrix} z^{\mu_1} A_1 & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & 0 & \dots \\ 0 & \dots & \dots & \dots & \dots \\ \dots & 0 & \dots & \dots & \dots \\ 0 & \dots & 0 & \dots & z^{\mu_k} A_k \end{pmatrix} .$$

An analytic class within $\mathcal{F}(M_0)$ can then be represented by $M := (K^n, \quad _{A})$ with:

$$A = A_U := \begin{pmatrix} z^{\mu_1} A_1 & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & U_{i,j} & \dots \\ 0 & \dots & \dots & \dots & \dots \\ \dots & 0 & \dots & \dots & \dots \\ 0 & \dots & 0 & \dots & z^{\mu_k} A_k \end{pmatrix} ,$$

for some $U := (U_{i,j})_{1 \leq i < j \leq k} \in \prod_{1 \leq i < j \leq k} \mathrm{Mat}_{r_i, r_j}(K)$.

Irregular equations

Birkhoff-Guenther normal form

Using q-Borel transforms one gets an explicit (algorithmic) computation of *Birkhoff-Guenther normal form*:

Theorem

Each class in $\mathcal{F}(P_1, \ldots, P_k)$ admits a unique representative $(K^n, \ {}_{A_U})$ such that each block $U_{i,j}, 1 \leq i < j \leq k$ has coefficients in $\sum\limits_{\mu_i \leq \ell < \mu_j} \mathbf{C} z^\ell$.

Example

If $A_0 := \begin{pmatrix} a & 0 \\ 0 & bz \end{pmatrix}$, $a, b \in \mathbf{C}^*$, then the normal form of $A_u := \begin{pmatrix} a & u \\ 0 & bz \end{pmatrix}$, $u \in K$ is $\begin{pmatrix} a & \mathcal{B}_{q,1} u(a/b) \\ 0 & bz \end{pmatrix}$, where:

$$\mathcal{B}_{q,1}\left(\sum f_n z^n\right) = \sum \frac{f_n}{q^{n(n-1)/2}} z^n.$$

Irregular equations

Formal isomorphism

Call $\mathfrak{G} \subset \mathrm{GL}_n$ the subgroup of matrices:

$$\begin{pmatrix} I_{r_1} & \dots & \dots & \dots \\ \dots & \dots & F_{i,j} & \dots \\ \dots & 0 & \dots & \dots \\ 0 & \dots & \dots & I_{r_k} \end{pmatrix}$$

For all A in the formal class A_0, there is a unique $\hat{F} \in \mathfrak{G}(\hat{K})$ such that $\hat{F}[A_0] = A$; call it $\hat{F}_A$. Then:

$$A \sim A' \Longleftrightarrow \hat{F}_{A'}(\hat{F}_A)^{-1} \in \mathfrak{G}(K).$$

We want to \sum" the divergent series $\hat{F}_A$

Example

$\begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix}$ is an isomorphism from A_0 to A_u if, and only if, $bz\sigma_q f - af = u$. This has a unique formal solution $\hat{f}_u$, and $A_u \sim A_v \Leftrightarrow \hat{f}_u - \hat{f}_v \in K$.

Irregular equations

q-adapted Poincaré asymptotics

There is a q-analogue of Poincare asymptotics with the following features:

Asymptotics for ODE	Asymptotics for q-differences
1. Dynamics is given by the semi-group $:= e^{]-\infty,0]}$.	▶ Dynamics is given by the semi-group $:= q^{-\mathbf{N}}$.
2. -invariants subsets of $(\mathbf{C}^*, 0)$ are sectors.	▶ -invariants subsets of $(\mathbf{C}^*, 0)$ are spiral-like.
3. The horizon $(\mathbf{C}^*, 0)/$ is the circle of directions S^1.	▶ The horizon $(\mathbf{C}^*, 0)/$ is the elliptic curve $\mathbf{E}_q$.
4. Sheaves are de ned over S^1.	▶ Sheaves are de ned over $\mathbf{E}_q$.

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Irregular equations
q-adapted Poincaré asymptotics

We write $\mathcal{A}$ the sheaf of functions with an asymptotic expansion and:

$$_I(M_0) := \mathfrak{G}(\mathcal{A}) \cap \mathrm{Aut}(M_0),$$

the sheaf of automorphisms of M_0 in nitely tangent to identity.

Actually,if $\mathcal{A}_0$ denotes the subsheaf of at functions:

$$_I(M_0) \subset \mathrm{I}_n + \mathrm{GL}_n(\mathcal{A}_0),$$

whence the name.

Irregular equations

Meromorphic summation

The polar divisor of a meromorphic isomorphism $F : A_0 \to A$, is q-invariant near 0, hence de ned over $\mathbf{C}^*/q^{-\mathbf{N}} = \mathbf{E}_q$.

Theorem

There is an explicit finite subset $\ _{A_0} \subset \mathbf{E}_q$ *such that, for all* $\overline{c} \in \mathbf{E}_q \setminus \ _{A_0}$*, and for all* A*, there is a unique meromorphic isomorphism* $F : A_0 \to A_U$ *such that:*

$$\forall 1 \leq i < j \leq k \ , \ div_{\mathbf{E}_q}(F_{i,j}) \geq -(\mu_j - \mu_i)[\overline{-c}]\Big).$$

We write $S_{\overline{c}}\hat{F}_A$ *this* F *and see it as a "resummation of* $\hat{F}_A$ *in the (allowed) direction* $\overline{c} \in \mathbf{E}_q \setminus \ _{A_0}$*". One has moreover:*

$$S_{\overline{c}}\hat{F}_A \sim \hat{F}_A.$$

Irregular equations

Privileged cocycles of $\Lambda_I(M_0)$

We note:

$$S_{\overline{c,d}}\hat{F}_A := (S_{\overline{c}}\hat{F}_A)^{-1}(S_{\overline{d}}\hat{F}_A)$$

Properties:

1. $S_{\overline{c,d}}\hat{F}_A$ is a meromorphic automorphism of M_0.
2. $S_{\overline{c,e}}\hat{F}_A = (S_{\overline{c,d}}\hat{F}_A)(S_{\overline{d,e}}\hat{F}_A)$.
3. $S_{\overline{c,d}}\hat{F}_A - I_n$ is \ at".
4. $\mathrm{div}_{\mathbf{E}_q}((S_{\overline{c,d}}\hat{F}_A)_{i,j}) \geq -(\mu_j - \mu_i)([-\overline{c}] + [-\overline{d}])$.

Thus the $S_{\overline{c,d}}\hat{F}_A$ form a *privileged* cocycle of $_I(M_0)$ for the covering $\mathfrak{U}_{A_0}$ of $\mathbf{E}_q$ made up of the Zariski open subsets $V_{\overline{c}} := \mathbf{E}_q \setminus \{\overline{c}\}$, $\overline{c} \in \mathbf{E}_q \setminus \ _{A_0}$.

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Privileged cocycles of $\Lambda_I(M_0)$

We note:

$$S_{\overline{c,d}}\hat{F}_A := (S_{\overline{c}}\hat{F}_A)^{-1}(S_{\overline{d}}\hat{F}_A)$$

Properties:

1. $S_{\overline{c,d}}\hat{F}_A$ is a meromorphic automorphism of M_0.
2. $S_{\overline{c,e}}\hat{F}_A = (S_{\overline{c,d}}\hat{F}_A)(S_{\overline{d,e}}\hat{F}_A)$.
3. $S_{\overline{c,d}}\hat{F}_A - I_n$ is \ at''.
4. $\mathrm{div}_{\mathbf{E}_q}((S_{\overline{c,d}}\hat{F}_A)_{i,j}) \geq -(\mu_j - \mu_i)([\overline{-c}] + [\overline{-d}])$.

Thus the $S_{\overline{c,d}}\hat{F}_A$ form a *privileged* cocycle of $_I(M_0)$ for the covering $\mathfrak{U}_{A_0}$ of $\mathbf{E}_q$ made up of the Zariski open subsets $V_{\overline{c}} := \mathbf{E}_q \setminus \{\overline{c}\}$, $\overline{c} \in \mathbf{E}_q \setminus \quad _{A_0}$.

Irregular equations

The q-Malgrange-Sibuya theorems

Write $Z^1_{pr}(\mathfrak{U}_{A_0}, \;_I(M_0))$ the space of privileged cocycles.

Theorem

"Meromorphic summation" yields natural isomorphisms:

$$\mathcal{F}(P_1, \ldots, P_k) \simeq Z^1_{pr}(\mathfrak{U}_{A_0}, \;_I(M_0)) \simeq H^1(\mathbf{E}_q, \;_I(M_0)).$$

It is an easy (and pleasant) exercice to compute the dimension of $Z^1_{pr}(\mathfrak{U}_{A_0}, \;_I(M_0))$.

Irregular equations

Dévissage q-Gevrey

\Abelian" case: two slopes $\mu_1 < \mu_2$, one \level" $\delta := \mu_2 - \mu_1$.

Then $_I(M_0)$ is an \elementary" vector bundle of slope δ over $\mathbf{E}_q$:

$$_I(M_0) \simeq (\text{ at bundle of rank } r_1 r_2) \otimes (\text{line bundle of degree } \delta).$$

General case: slopes $\mu_1 < \cdots < \mu_k$, levels $\mu_j - \mu_i, i < j$. The subsheaf $^t_I(M_0)$ made up of F s.t. $F - I_n$ is t- at has only diagonals $\mu_j - \mu_i \geq t$.

$_I(M_0)$ is built from central extensions by elementary bundles $\lambda_I^{(t)}(M_0)$:

$$0 \to \lambda_I^{(t)}(M_0) \to \frac{_I(M_0)}{^{t+1}_I(M_0)} \to \frac{_I(M_0)}{^t_I(M_0)} \to 1.$$

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Irregular equations

Two slopes, one level

$$\mathcal{F}(M_0) \simeq \mathrm{Ext}(P_2, P_1) \simeq \mathrm{Ext}\left(\underline{1}, P_2^\vee \otimes P_1\right) \simeq H^1\left(\mathbf{E}_q, \quad {}_I(M_0)\right).$$

Example

Let $A_0 := \begin{pmatrix} a & 0 \\ 0 & bz^\delta \end{pmatrix} \Longrightarrow \quad {}_{A_0} = \{\overline{c} \in \mathbf{E}_q \mid c^\delta \in q^{\mathbf{Z}} a/b\}$.

Components over $V_{\overline{c}} \cap V_{\overline{d}}$ of cocycles of $Z^1_{pr}(\mathfrak{U}_{A_0}, \quad {}_I(M_0))$ are matrices $S_{\overline{c,d}}\hat{F}_A = \begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix}$, where:

$$\begin{aligned} f(z) &= \frac{g(z)}{\theta_q(z/c)^\delta \theta_q(z/d)^\delta}, \\ g \in \mathcal{O}(\mathbf{C}^*) \text{ s.t. } \sigma_q g &= (a/b)(z/cd)^\delta g. \end{aligned}$$