Symbolic Integration in Differential Fields

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Motivation

Parameter integrals

- Evaluate:

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satisfies

$$J_n'(y) = -J_{n+1}(y) + \tfrac{n}{y}J_n(y)$$

$$J_{n+2}(y) - \tfrac{2(n+1)}{y}J_{n+1}(y) + J_n(y) = 0$$

$$J_n''(y) + \tfrac{1}{y}J_n'(y) + (1 - \tfrac{n^2}{y^2})J_n(y) = 0$$

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Given f , find g s.t.

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Transfer this to a relation of corresponding integrals

$$c_0 \int_a^b f_0 dx + \dots + c_m \int_a^b f_m dx = g(b) - g(a)$$

Input functions

D-finite setting

- differential field (K, D), $a_0, \dots, a_d \in K$
- extend K by t s.t. $a_d D^d t + \dots + a_0 t = 0$: $V = \mathrm{span}_K \{D^j t\}_j$

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- iterate: t_i s.t. $Dt_i = a_{i,d_i} t_i^{d_i} + \dots + a_{i_0}$, $a_{i,j} \in K(t_1, \dots, t_{i-1})$,

$$F = K(t_1, \dots, t_n)$$

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D-finite setting

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- Find $g \in V$ s.t.
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differential field setting

- Given (F, D) and $f \in F$
- Find an elementary extension (E, D) of (F, D) and $g \in E$ s.t.
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Admissible fields

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3. for each t_i and $F_{i-1} := C(t_1, \dots, t_{i-1})$ either
    1. t_i is a Liouvillian monomial over F_{i-1}, i.e., either
        1. $Dt_i \in F_{i-1}$ (primitive), or
        2. $\frac{Dt_i}{t_i} \in F_{i-1}$ (hyperexponential); or

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    2. there is a $q \in F_{i-1}[t_i]$ with $\deg(q) \geq 2$ such that
        1. $Dt_i = q(t_i)$ and
        2. $Dy = q(y)$ does not have a solution $y \in \overline{F_{i-1}}$.

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Generalization to 2-dim systems

$$\begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}' = \begin{pmatrix} a_{11}(x) & a_{12}(x) \\ a_{21}(x) & a_{22}(x) \end{pmatrix} \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix} + \begin{pmatrix} b_1(x) \\ b_2(x) \end{pmatrix}$$

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orthogonal polynomials, associated Legendre functions, complete elliptic integrals, Bessel/Struve/Anger/Weber/Lommel functions, Airy/Scorer functions, Whittaker functions, hypergeometric functions, Heun functions, Mathieu functions, etc.

Field extension for the integral

Elementary extensions

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- θ_i is algebraic over E_{i-1}, or
- $D\theta_i = \frac{Da}{a}$ for some $a \in E_{i-1}$ ($\theta_i = \ln(a)$), or
- $\frac{D\theta_i}{\theta_i} = Da$ for some $a \in E_{i-1}$ ($\theta_i = \exp(a)$)

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Elementary integral

We say that $f \in F$ has an elementary integral over (F, D) if there exists an elementary extension (E, D) of (F, D) and $g \in E$ s.t.

$$f = Dg$$

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Parametric elementary integration

Given: an admissible differential field (F, D) and $f_0, \ldots, f_m \in F$

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Given: an admissible differential field (F, D) and $f_0, \ldots, f_m \in F$

Compute: $(c_0, \ldots, c_m) \in Const(F)^{m+1}$ and

g from some elementary extension of (F, D) with

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Given: an admissible differential field (F, D) and $f_0, \dots, f_m \in F$

Compute: a vector space basis of all $(c_0, \dots, c_m) \in Const(F)^{m+1}$ s.t. $c_0 f_0 + \dots + c_m f_m$ has an elementary integral over (F, D) and corresponding g's from some elementary extension of (F, D) with

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Previous results

- Risch 1969, Mack 1976: complete for elementary (F, D)
- Singer et al. 1985: complete for Liouvillian (F, D)
- Bronstein 1990/97: parts for general (F, D)

Result

Theorem

Let $(F, D) = (C(t_1, \ldots, t_n), D)$ be an admissible differential field

Then we can solve the parametric elementary integration problem over (F, D).

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Let $(F, D) = (C(t_1, \dots, t_n), D)$ be an admissible differential field with the restriction that for any two non-Liouvillian monomials t_i, t_j, $i < j$, none of the monomials $t_{i+1}, \dots, t_{j-1}$ is allowed to be hyperexponential.

Then we can solve the parametric elementary integration problem over (F, D).

Decision procedure

Recursive reduction algorithm

integrands from $K(t_n) = C(t_1, \ldots, t_n)$

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2. Residue Criterion for computing elementary extensions

$$\sum_{i=1}^{m} \sum_{Q_i(\alpha)=0} \alpha \log(S_i(\alpha, t_n))$$

where $Q_i \in C[z]$ and $S_i \in K[z, t_n]$

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3. compute degree bound and coefficients by solving auxiliary problems in K
4. remaining integrands are from K, reduce elementary integration over $K(t_n)$ to elementary integration over K

Recursive call with integrands from $K = C(t_1, \dots, t_{n-1})$ to find elementary integrals over K

Auxiliary problem

parametric linear ODEs

Given: a differential field (F, D) and $a_0, \dots, a_{d-1}, f_0, \dots, f_m \in F$

Compute: a vector space basis of all $(g, c_0, \dots, c_m) \in F \times C^{m+1}$ such that

$$D^d g + a_{d-1} D^{d-1} g + \dots + a_0 g = c_0 f_0 + \dots + c_m f_m$$

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Previous results

- Abramov 1989: complete for rational functions
- Singer 1991: complete for Liouvillian functions
- Bronstein 1992: partial generalization
- Fredet 2004: partial generalization

Examples of indefinite integrals

- Using $F = \mathbb{Q}(x, \ln(1-x), \mathrm{Li}_2(x), \mathrm{Li}_3(x))$ we find

$$\int \frac{\mathrm{Li}_3(x) - x\mathrm{Li}_2(x)}{(1-x)^2}\,dx =$$

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- Using $F = \mathbb{Q}(\pi, n)\left(x, \frac{J_{n+1}(x)}{J_n(x)}, J_n(x), \frac{Y_n(x)}{J_n(x)}\right)$ we find

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Our algorithm finds

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$$I(n+1) - \tfrac{n}{n+1} I(n) = \tfrac{i}{(n+1)(2n+1)\pi}$$

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Solution:

$$I(n) = -\frac{1}{4n} + \frac{i}{n\pi} \sum_{k=1}^{n} \frac{1}{2k-1}$$

Example: Binet-like integral

$$B(\sigma) := \int_0^1 \underbrace{\left(\frac{1}{\ln(x)} + \frac{1}{1-x}\right)^2 x^\sigma}_{=:f(\sigma,x)} \, dx \qquad \text{for } \sigma > -1$$

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$$\frac{\partial^2 f}{\partial\sigma^2}(\sigma+1,x) - \frac{\sigma+1}{\sigma}\frac{\partial^2 f}{\partial\sigma^2}(\sigma,x) = \frac{d}{dx}\left(-\frac{\ln(x)^2}{\sigma(1-x)} - \frac{2\ln(x)}{\sigma+1} + \frac{x}{\sigma+2} - \frac{\sigma^2+1}{\sigma(\sigma+1)^2}\right)x^{\sigma+1}$$

$$\Delta_\sigma^2 \frac{\partial f}{\partial\sigma}(\sigma,x) - \frac{1}{\sigma+3}\Delta_\sigma^2 f(\sigma,x) = \frac{d}{dx}\left(\frac{(1-x)^2}{(\sigma+3)\ln(x)} + \frac{\ln(x)}{\sigma+1} - \frac{2x}{\sigma+2} + \frac{2(\sigma^2+3\sigma+1)}{(\sigma+1)^2(\sigma+3)}\right)x^{\sigma}$$

Example: Binet-like integral

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Our algorithm finds

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Integrating over $(0,1)$ yields the relations

$$\begin{aligned} B''(\sigma+1)-\frac{\sigma+1}{\sigma}B''(\sigma) &= -\frac{2}{\sigma(\sigma+1)^2(\sigma+2)} \\ \Delta_\sigma^2 B'(\sigma)-\frac{1}{\sigma+3}\Delta_\sigma^2 B(\sigma) &= -\frac{2}{(\sigma+1)^2(\sigma+2)(\sigma+3)} \end{aligned}$$

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We use these to find the evaluation

$$B(\sigma) = \sigma\psi(\sigma+1) - 2\ln\Gamma(\sigma+1) + (\sigma+1)\ln(\sigma+1) - 2\sigma + \ln(2\pi) - \frac{3}{2}$$