Lattice Walks Restricted to the Positive Quarter Plane (and Octant)

Stephen Melczer

Joint work with Alin Bostan, Manuel Kauers, Mireille Bousquet-Mélou and Marni Mishna

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Given: A set of directions
Count: Number of integer lattice walks in the first quadrant using these steps.

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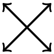
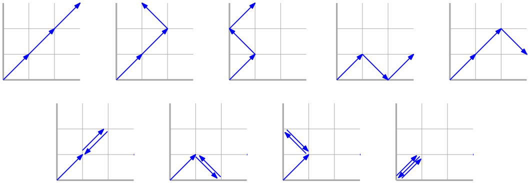
For instance, given the step set $\mathcal{S} = \{NE, SE, NW, SW\}$



there are 9 walks of length 3:



Realistic Goals

Finding a formula to predict the number of walks of length n is too hard in general - we seek **asymptotic estimates**.

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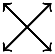
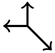
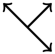
If we can classify the generating function as algebraic or D-Finite (satisfies a nice linear ODE) then we will know the form of its growth.

Classification	Growth	Example
Algebraic	$\frac{c\beta^n n^s}{\Gamma(s+1)}$	
D-Finite	$A\beta^n n^s \log(n)^r$	
Neither	$--$	

Outline of this Talk

Classifying the Generating Function

- Proofs via the ‘Kernel Method’

Asymptotics of D-Finite Walks

- Automatic Guessing of Bostan and Kauers

Extension to 3D

- Reductions to previous cases
- Orbit sum methods

Classifying the Generating Function

History

Of $2^8 = 256$ step sets, we consider only 79 [B.M.&Mishna 2010]. Step sets which are subsets of

correspond to half space problems, which have been previously solved [Banderier&Flajolet 2001].

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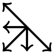
Also, subsets of



will never leave the origin, so these are also not considered.

Tools

The generating function

$$Q(x, y; t) = \sum_{n,i,j \geq 0} q(i, j; n) x^i y^j t^n$$

which counts the number of walks of length n ending at (i, j) satisfies an obvious functional equation.

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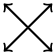
which counts the number of walks of length n ending at (i,j) satisfies an obvious functional equation. For example, with the previous step set

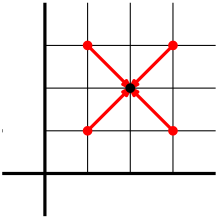


we define the step generating function

$$S(x,y) := xy + \frac{y}{x} + \frac{1}{xy} + \frac{x}{y}$$

A functional equation

$$Q(x,y;t) \quad = \quad 1 + tS(x,y)Q(x,y;t)$$



A functional equation

$$
\begin{aligned}
Q(x,y;t) &= 1 + tS(x,y)Q(x,y;t) \\
&\quad -t\left(\frac{y}{x} + \frac{1}{xy}\right)Q(0,y;t)
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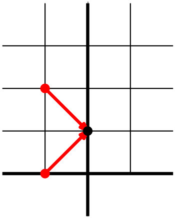

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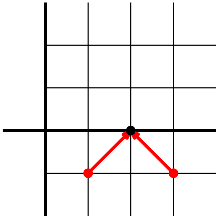

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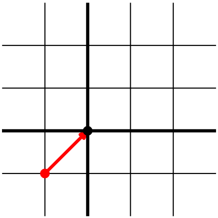

The Kernel Equation

Begin with the functional equation, re-group the terms and multiply by xy to give:

$$\boxed{K(x,y) \cdot xyQ(x,y) = xy - t(y^2+1)Q(y,0) - t(x^2+1)Q(x,0) + tQ(0,0)} \qquad (K)$$

where $K(x,y) = 1 - tS(x,y)$ is called the *kernel* of the walk.

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We now define a group G of bi-rational transformations of the xy-plane which preserves $S(x,y)$ - and thus $K(x,y)$.

The Group of a Walk

To begin, write

$$S(x,y) = \frac{1}{y}A_{-1}(x) + A_0(x) + yA_1(x) = \frac{1}{x}B_{-1}(y) + B_0(y) + xB_1(y).$$

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$$\psi : (x,y) \mapsto \left(\frac{B_{-1}(y)}{xB_1(y)}, y\right) \qquad \tau : (x,y) \mapsto \left(x, \frac{A_{-1}(x)}{yA_1(x)}\right).$$

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In our case, we have

$$\psi : (x,y) \mapsto \left(\frac{1}{x}, y\right) \qquad \tau : (x,y) \mapsto \left(x, \frac{1}{y}\right),$$

so G contains 4 elements: $id, \psi, \tau\psi, \psi\tau\psi$.

Summing the Kernel Equation

Applying the group elements to (K) yields:

$$(K) \qquad K(x,y) \cdot xyQ(x,y) = xy - t(y^2+1)Q(y,0) - t(x^2+1)Q(x,0) + tQ(0,0)$$

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If we take an **alternating sum** of these equalities, **all unknown terms on the RHS cancel!**

Proving D-Finiteness

In 19 of the 23 cases where the group of a walk is finite, we get:

Theorem (Bousquet-Mélou&Mishna 2010)

$$\sum_{g \in G} sign(g) g(xy) Q(g(x,y)) = \frac{1}{K(x,y)} \sum_{g \in G} sign(g) g(xy),$$

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Three of the other four walks use a modified 'half-orbit sum' method, the last (Gessel's walk) is special.

Automatic Asymptotics

How it Works

We calculate lots of terms, then guess a differential equation with certain bounds (implemented in gfun, others).

Given a step set $\mathcal{S}$, we can use a recursive formula to find $q(i, j; n)$ starting with $q(0, 0; 0) = 1$.

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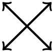
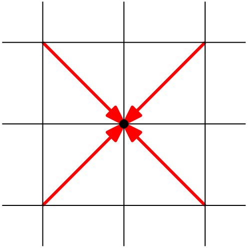
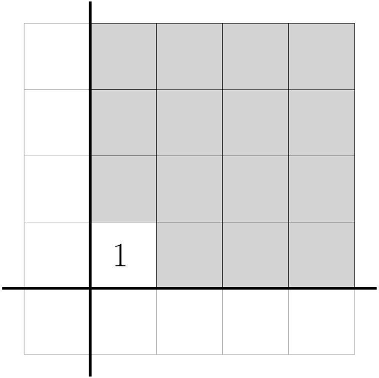
For example, if $\mathcal{S} =$  then

$$\begin{aligned} q(i, j; n) = & q(i-1, j-1; n-1) \\ & + q(i-1, j+1; n-1) \\ & + q(i+1, j-1; n-1) \\ & + q(i+1, j+1; n-1) \end{aligned}$$

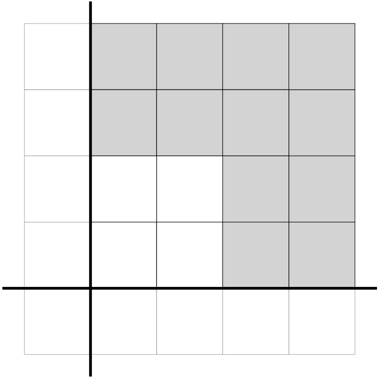


[we only add terms with positive arguments]

Illustration of the Algorithm



$n = 0$



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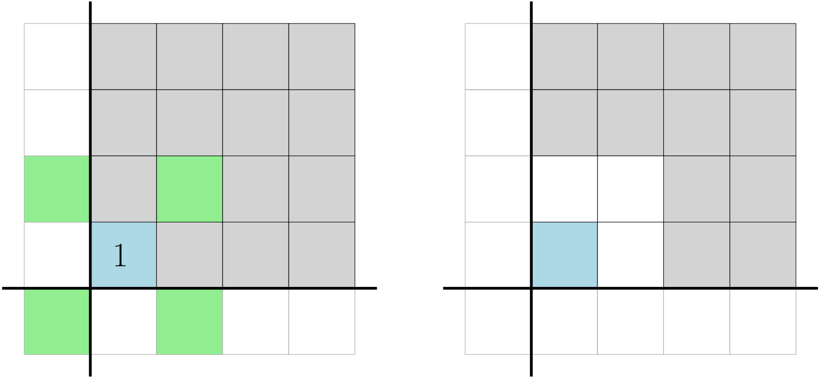

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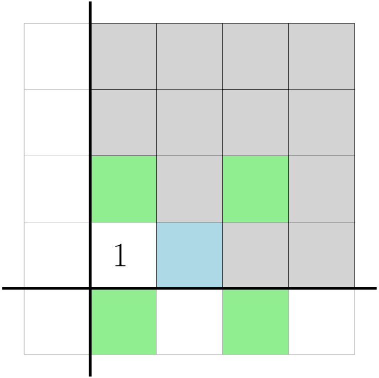

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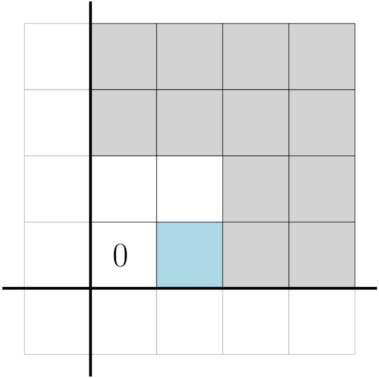

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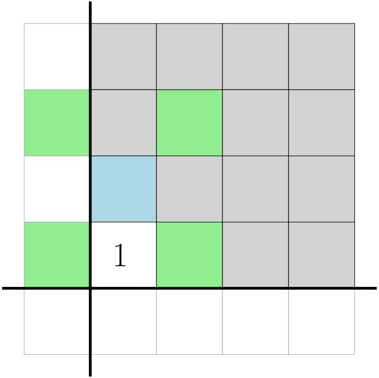

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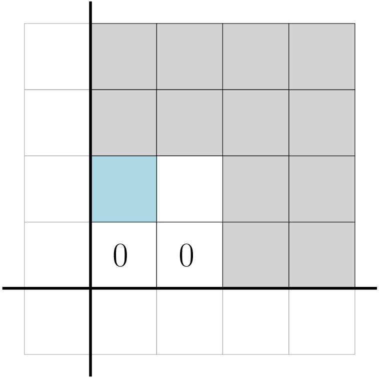

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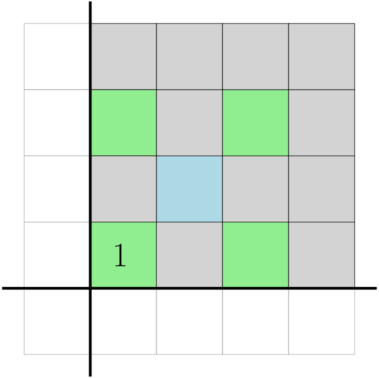

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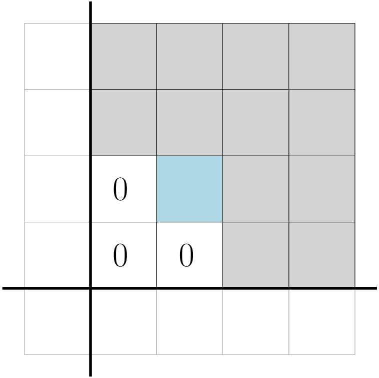

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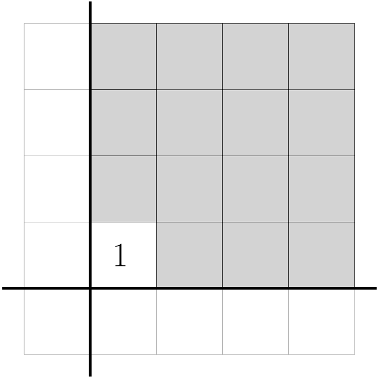
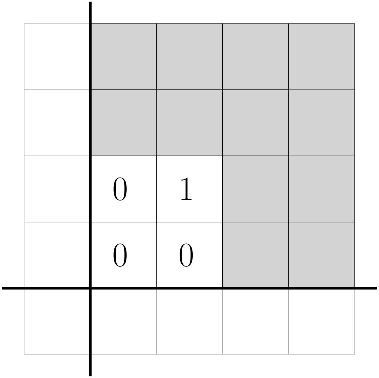


$n = 0$



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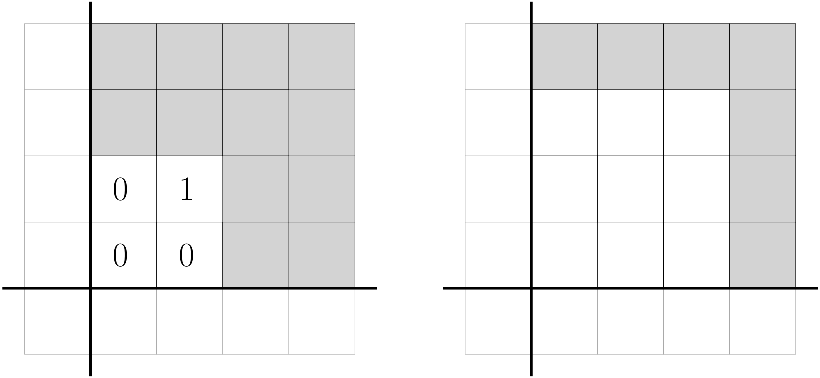
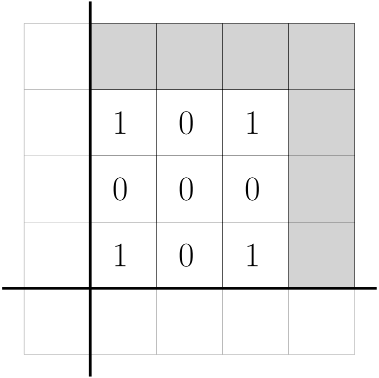
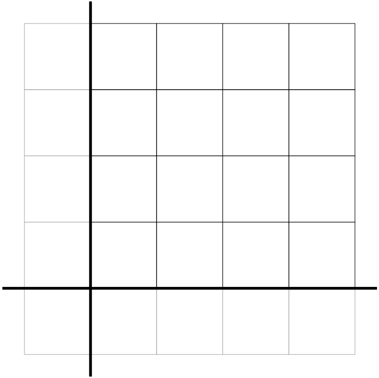


$n = 1$ $\qquad$ $n = 2$

Illustration of the Algorithm



$n = 2$



$n = 3$

Calculating Lots of Terms in C

Algorithm	Time (Modular)	Space (Modular)	Time (Naive)	Space (Naive)
500 Steps 	36 s	4 MB	10 s	48 MB
1000 Steps 	671 s	17 MB	123 s	270 MB
3000 Steps 	59509 s	142 MB	7576 s	5294 MB
500 Steps 	59 s	4 MB	59 s	50 MB
1000 Steps 	1029 s	17 MB	135 s	282 MB
3000 Steps 	89600 s	144 MB	8515 s	5635 MB

We can calculate a sufficient number of terms to guess quickly.

Selected step sets and their asymptotic behaviour

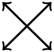
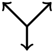
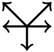
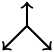
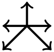
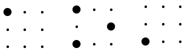
S	c	s	β	Asymptotic Form
	$\frac{2}{\pi}$	-1	4	$\frac{2}{\pi} \cdot \frac{4^n}{n}$
	$\sqrt{\frac{3}{\pi}}$	$-\frac{1}{2}$	3	$\sqrt{\frac{3}{\pi}} \cdot \frac{3^n}{\sqrt{n}}$
	$\frac{\sqrt{5}}{2\sqrt{2\pi}}$	$-\frac{1}{2}$	5	$\frac{\sqrt{5}}{2\sqrt{2\pi}} \cdot \frac{5^n}{\sqrt{n}}$
	$\frac{24\sqrt{2}}{\pi}$	-2	$2\sqrt{2}$	$\frac{24\sqrt{2}}{\pi} \cdot \frac{(2\sqrt{2})^n}{n^2}$
	$\frac{\sqrt{8}(1+\sqrt{2})^{\frac{7}{2}}}{\pi}$	-2	$2(1+\sqrt{2})$	$\frac{\sqrt{8}(1+\sqrt{2})^{\frac{7}{2}}}{\pi} \cdot \frac{(2(1+\sqrt{2}))^n}{n^2}$

Table: Some results of [Bostan&Kauers 2009].

3D Walks

Basic Idea

Now, we look at walks in the xyz-plane restricted to the positive octant.



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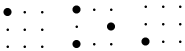

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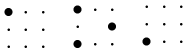

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We focus on the $83,682$ with 5 steps or less. Bostan and Kauers conjectured (up to equivalence) 35 D-Finite steps.

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There are 8 sets with an *infinite group*.

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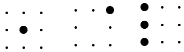

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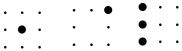

and 'mark' the steps by a, b, c, d, e. Then

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\begin{aligned}
&(x) \qquad & b &\geq (c+d+e) \\
&(y) \qquad & b+c &\geq e \\
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\end{aligned}
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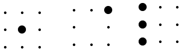

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If (x) is satisfied then (y) must be satisfied!

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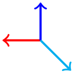

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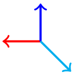

Thus, if

$$R(s,t,u,v,w) = \sum_{a,b,c,d,e \geq 0} r(a,b,c,d,e) s^a t^b u^c v^d w^e$$

$$R'(s,t,u) = \sum_{A,B,C \geq 0} r(A,B,C) s^A t^B u^C$$

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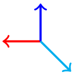

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are the generating functions of these walks marked by the number of each step, we have

$$R(s,t,u,v,w) = R'(s,t,u+v+w).$$

Reduction to 2D

Furthermore, R and R' are algebraic substitutions of $Q(x,y,z;t)$ and $Q'(x,y;t)$:

$$Q(x,y,z;t) = R\left(\frac{t}{z}, txy, \frac{tyz}{x}, \frac{tz}{x}, \frac{tz}{xy}\right)$$

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$Q'(x,1/x;t)$ is known to be algebraic [B.M.&Mishna 2010], thus so is $Q\left(x,y,\frac{1}{xy};t\right)$ for all x,y,t.

Conclusion

Results

2D walks (almost) completely classified

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3D walks classified up to 5 steps

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Keep pushing for larger step sets (more dimensions?)

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3D walks classified up to 5 steps

Keep pushing for larger step sets (more dimensions?)

Develop filters in 3D to apply before guessing

References

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THANK YOU