

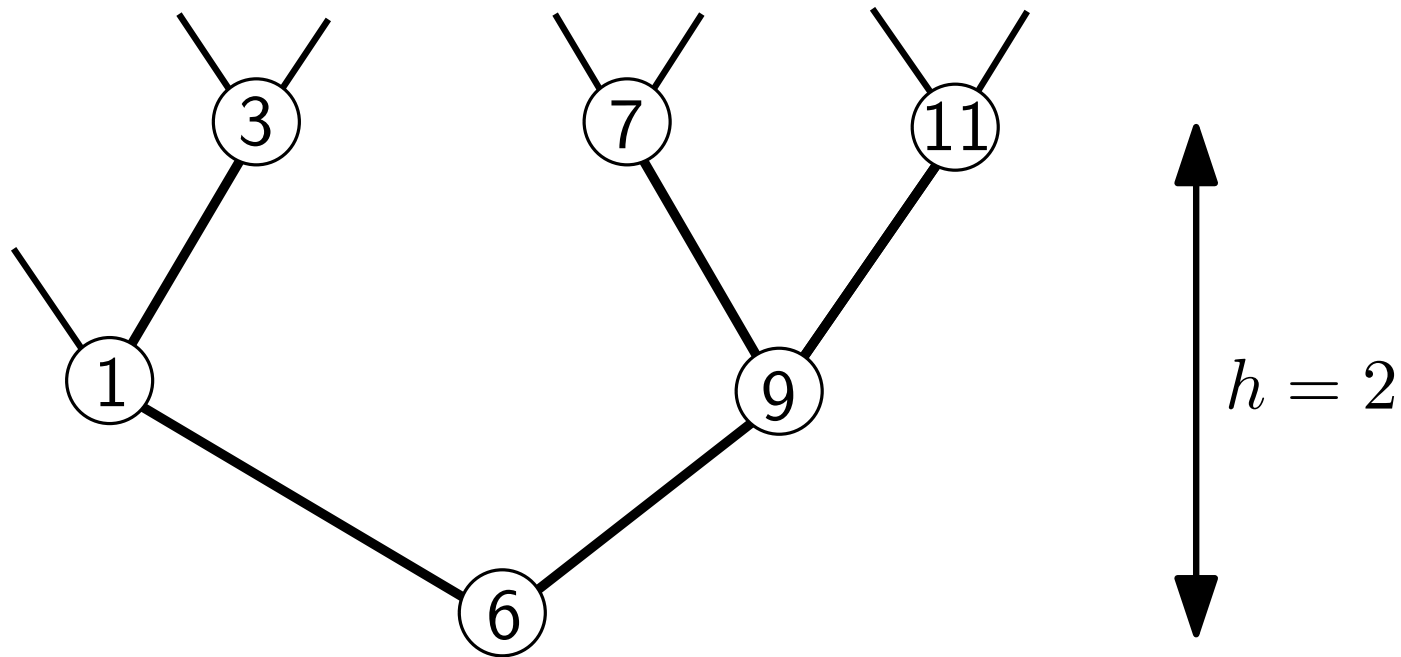
On the number of intervals in the Tamari lattices

Éric Fusy (LIX)

Joint work with M. Bousquet-Mélou (LaBRI) and
L.F. Préville-Ratelle (UQAM)

Binary search trees

A binary search tree (BST) is a data structure to store (comparable) items

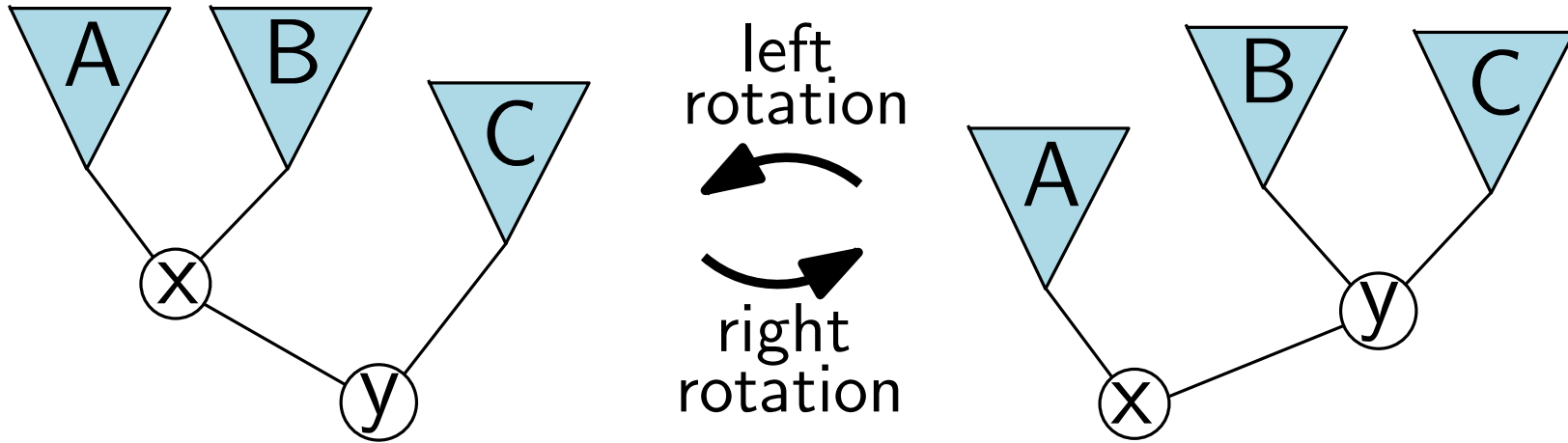
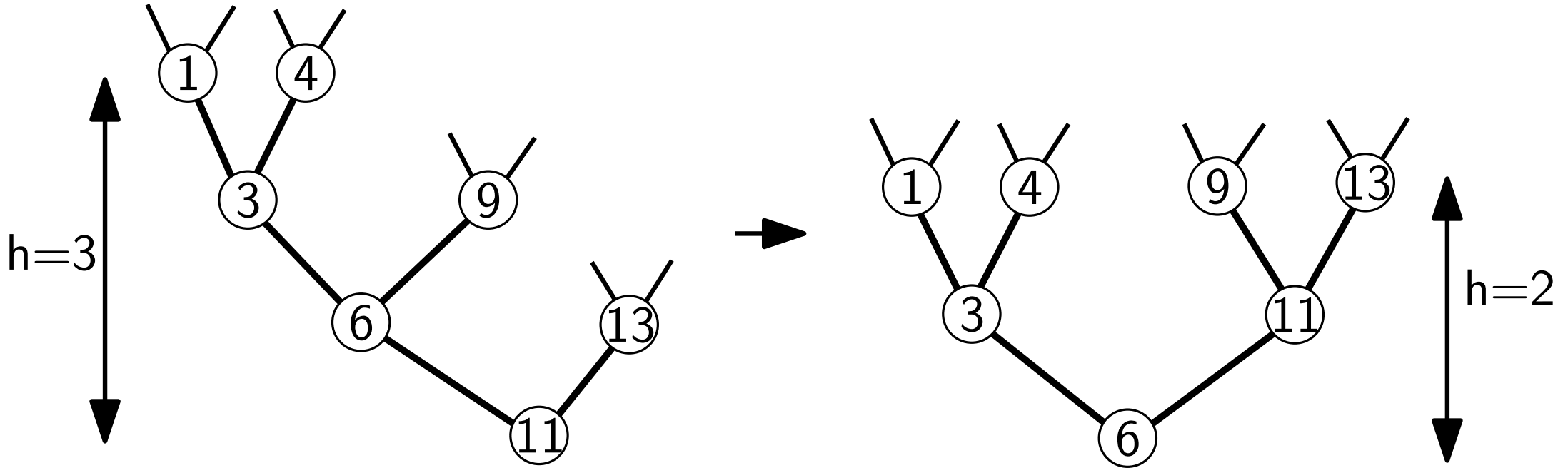


Items are in increasing order “from left to right” in the BST

Insertion, deletion, search of an item are done in time $O(h)$

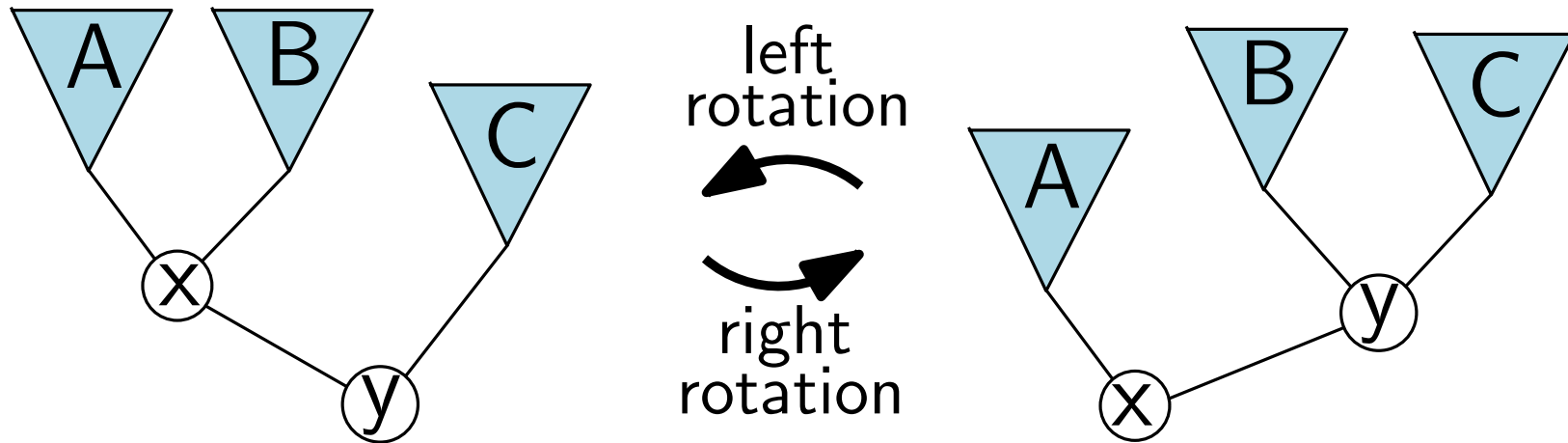
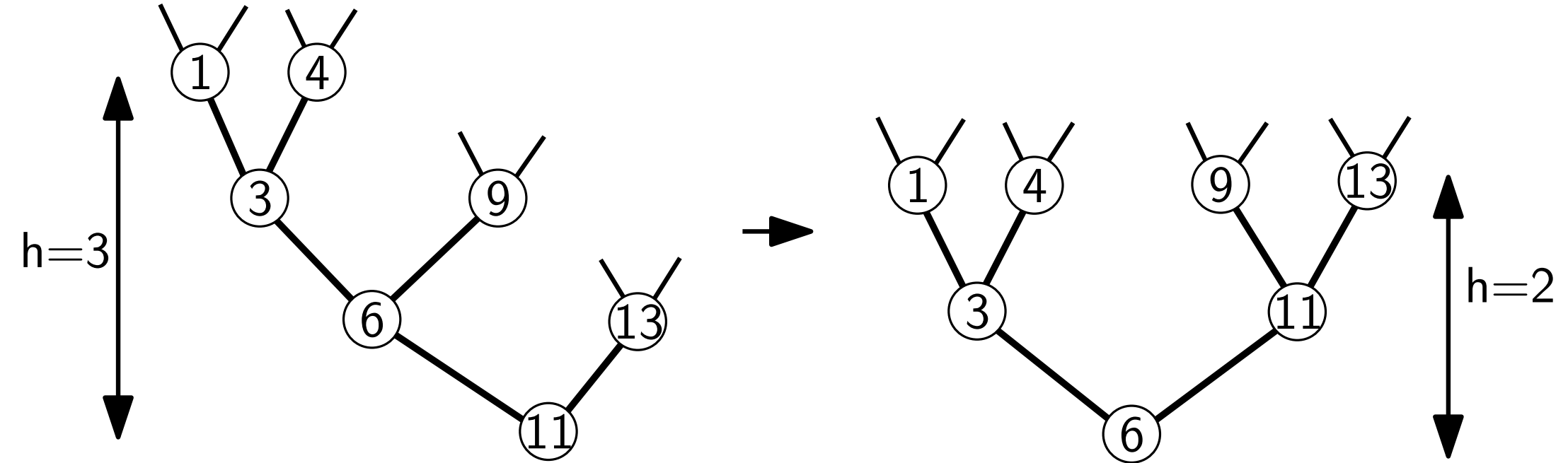
Balancing a BST

Rotation operations can be used to decrease the height



Balancing a BST

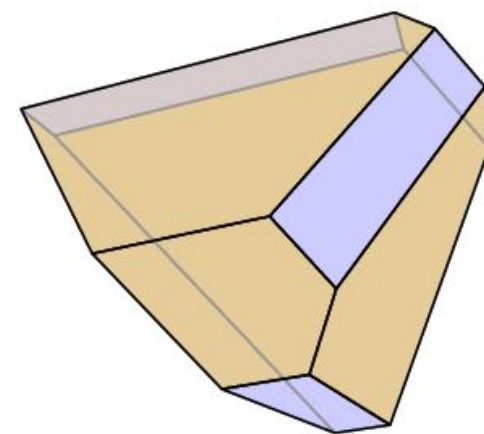
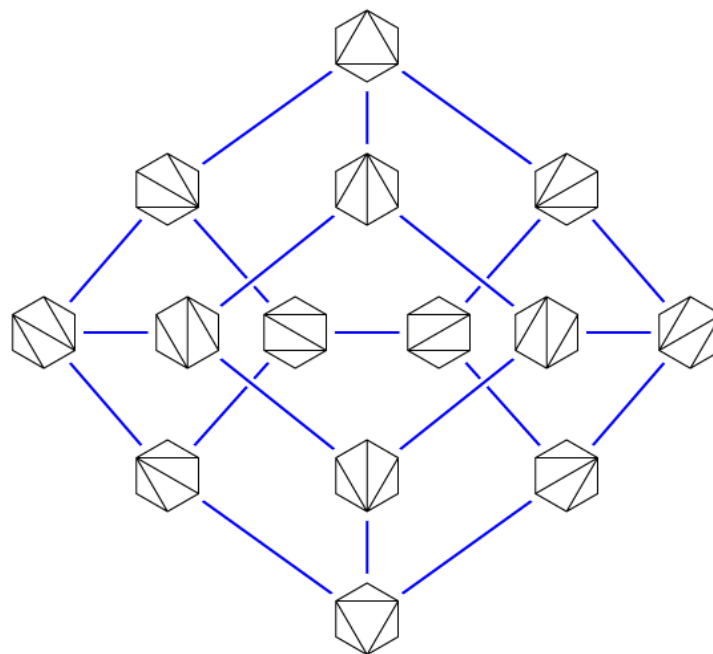
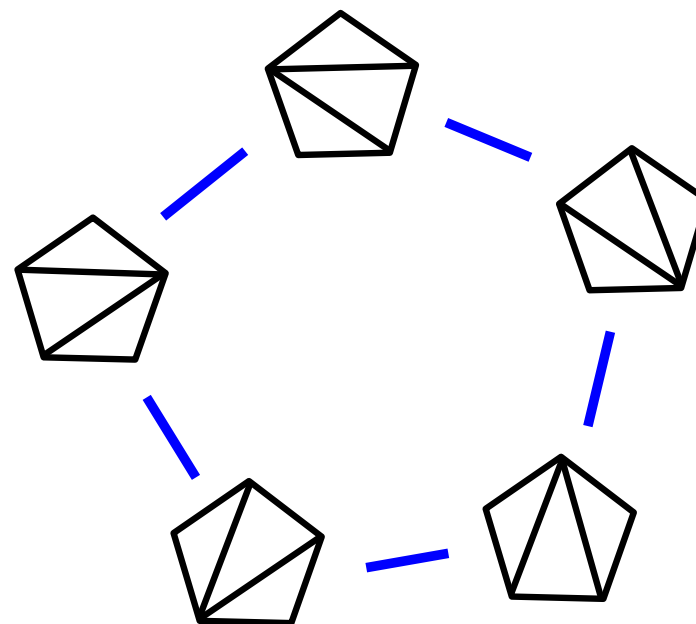
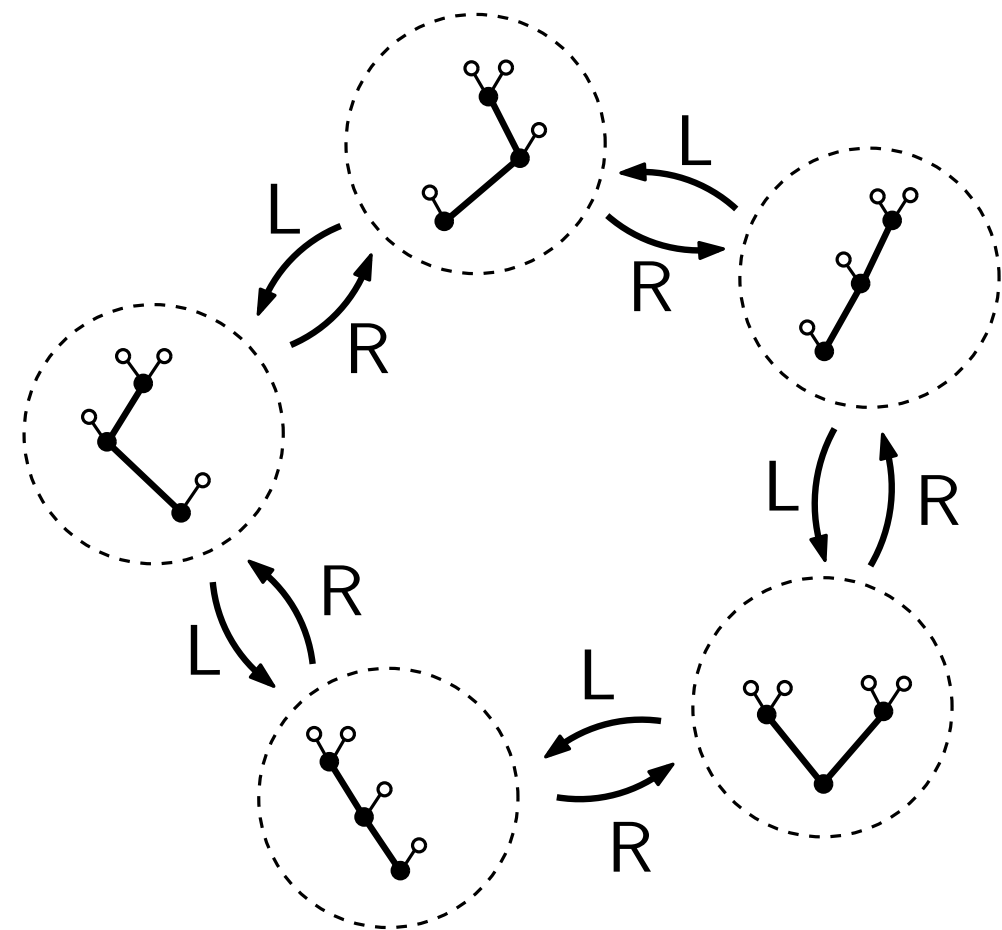
Rotation operations can be used to decrease the height



\Rightarrow efficient data structures (AVL): maintain height in $O(\log(n))$

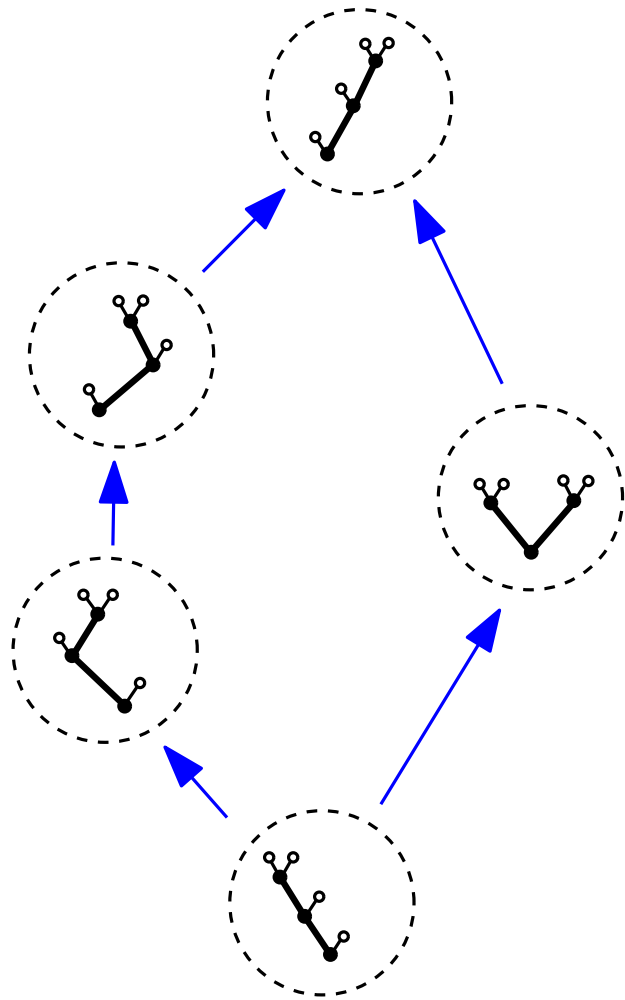
The adjacency graph for rotation-relations

of the associahedron

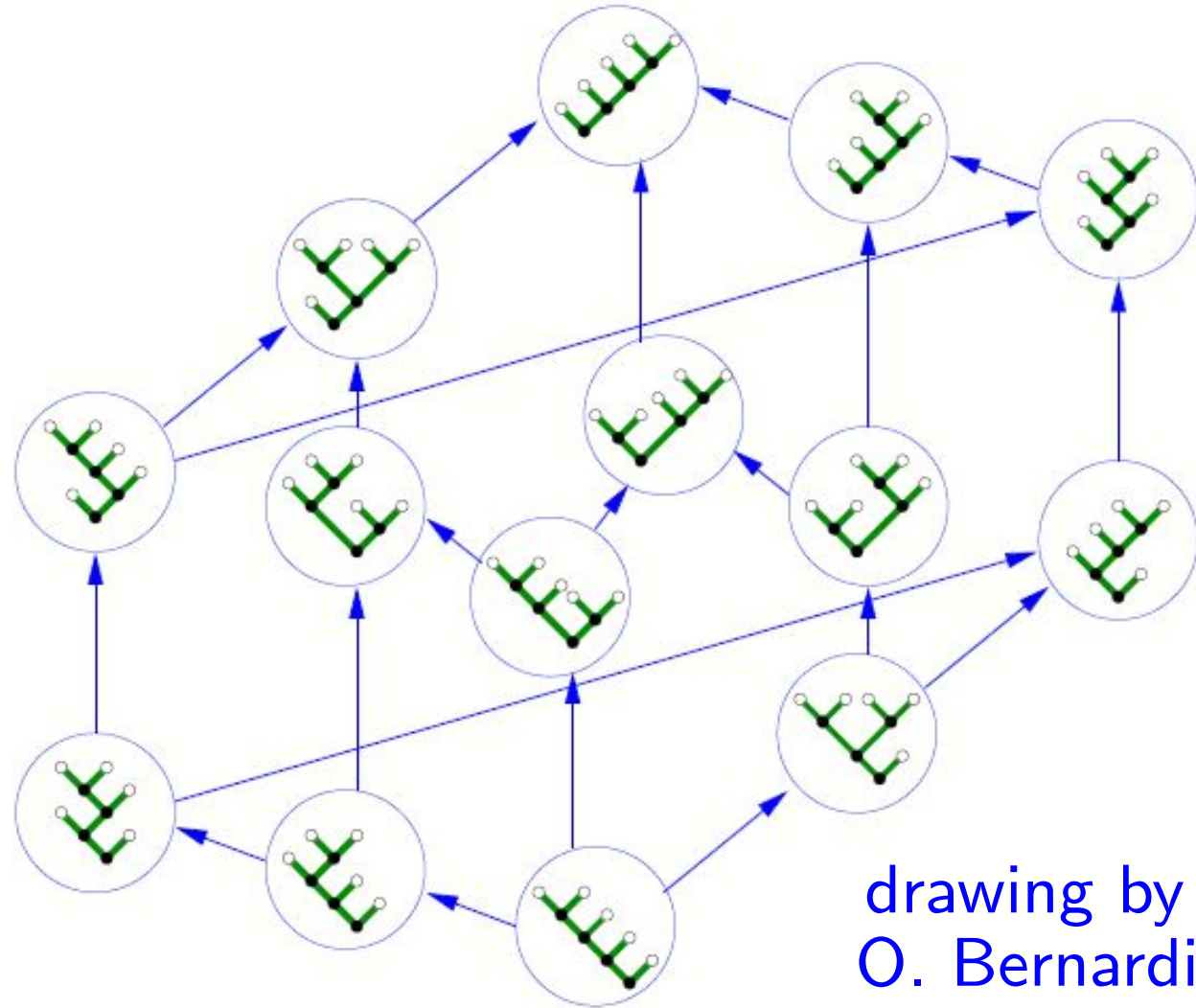


The Tamari lattice

The Tamari lattice \mathcal{T}_n is the partial order on binary trees with n nodes where the covering relation corresponds to right rotation



$n=3$

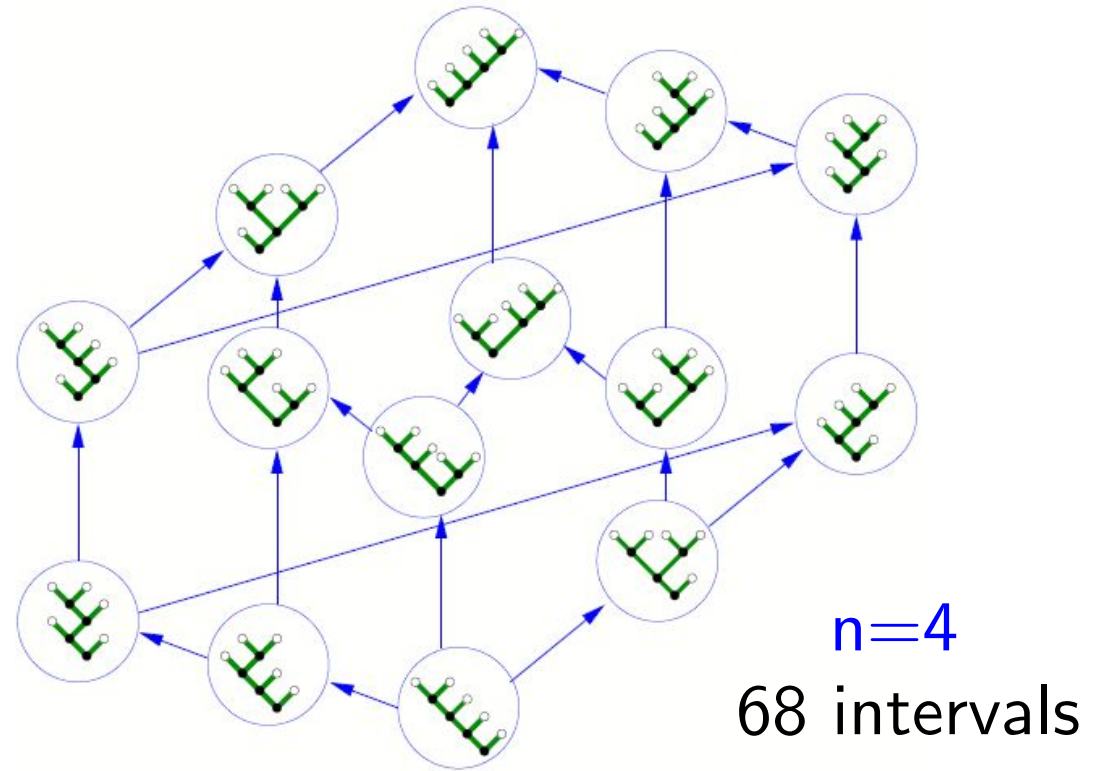
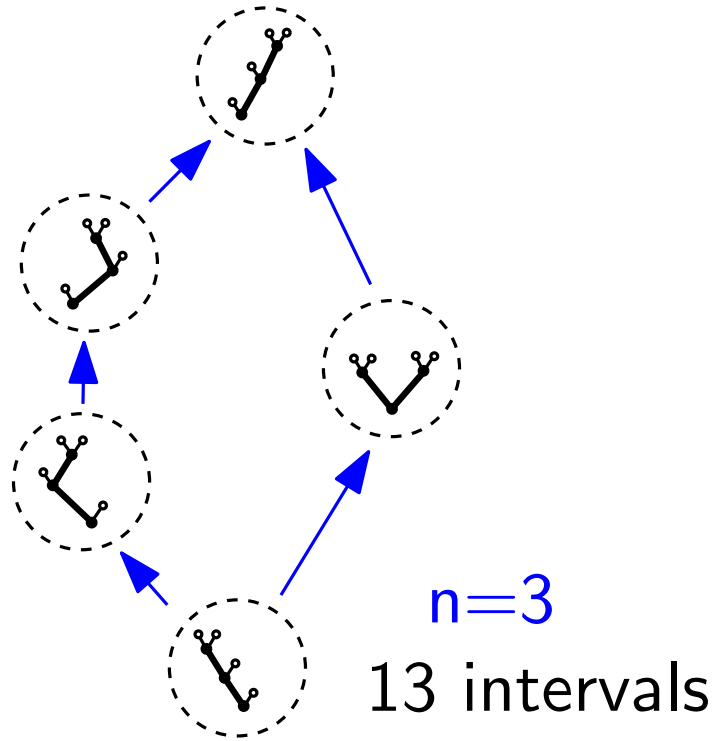


$n=4$

drawing by
O. Bernardi

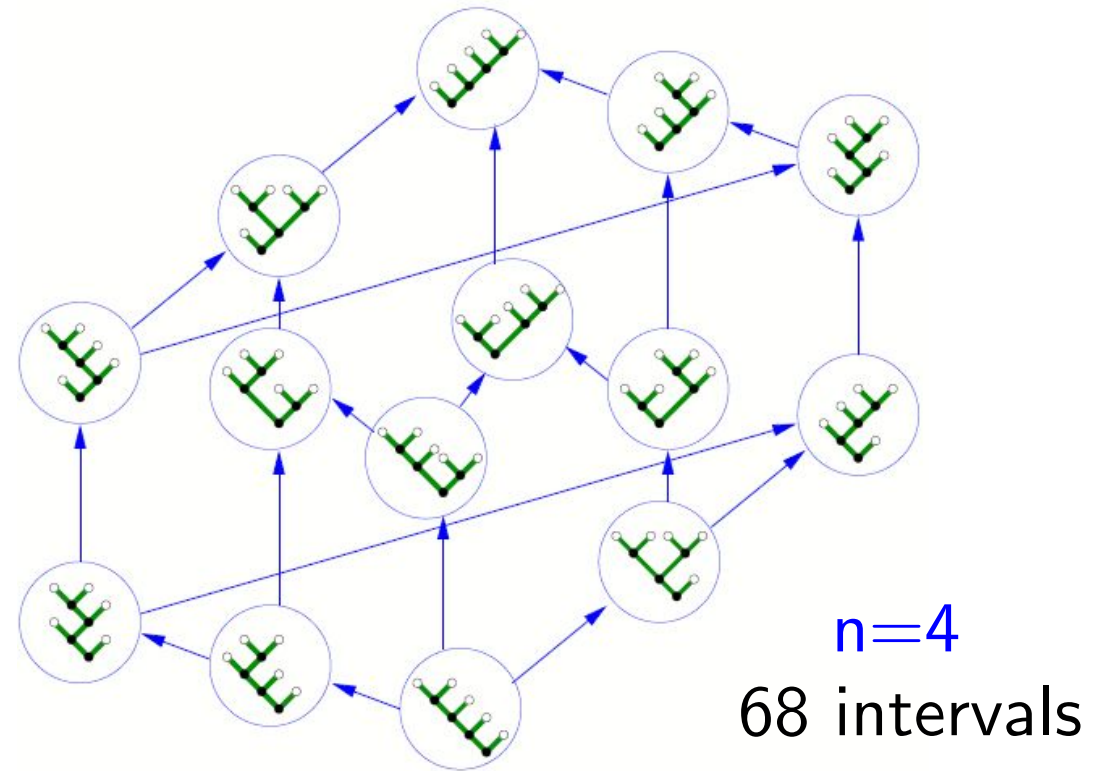
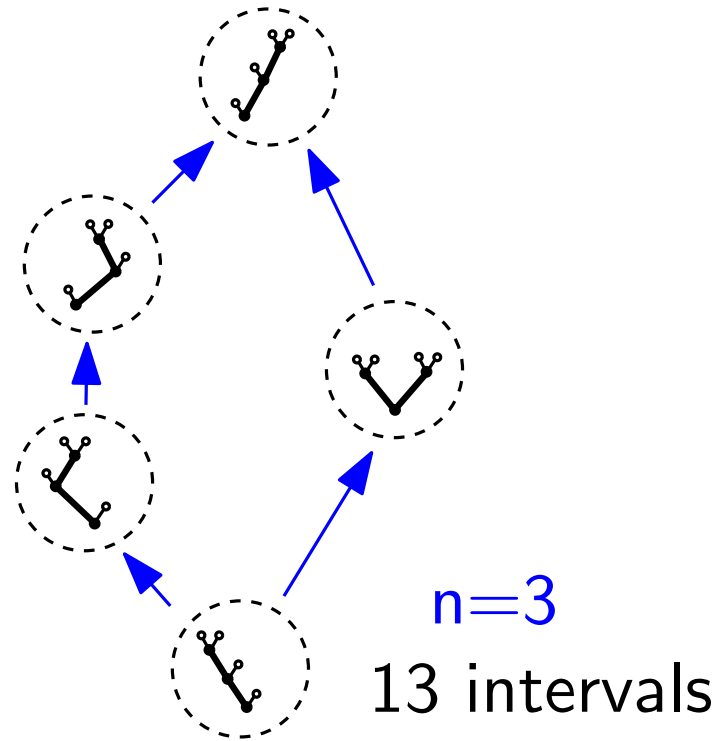
Enumeration of intervals in the Tamari lattice

An interval in \mathcal{T}_n is a pair (t, t') such that $t \leq t'$



Enumeration of intervals in the Tamari lattice

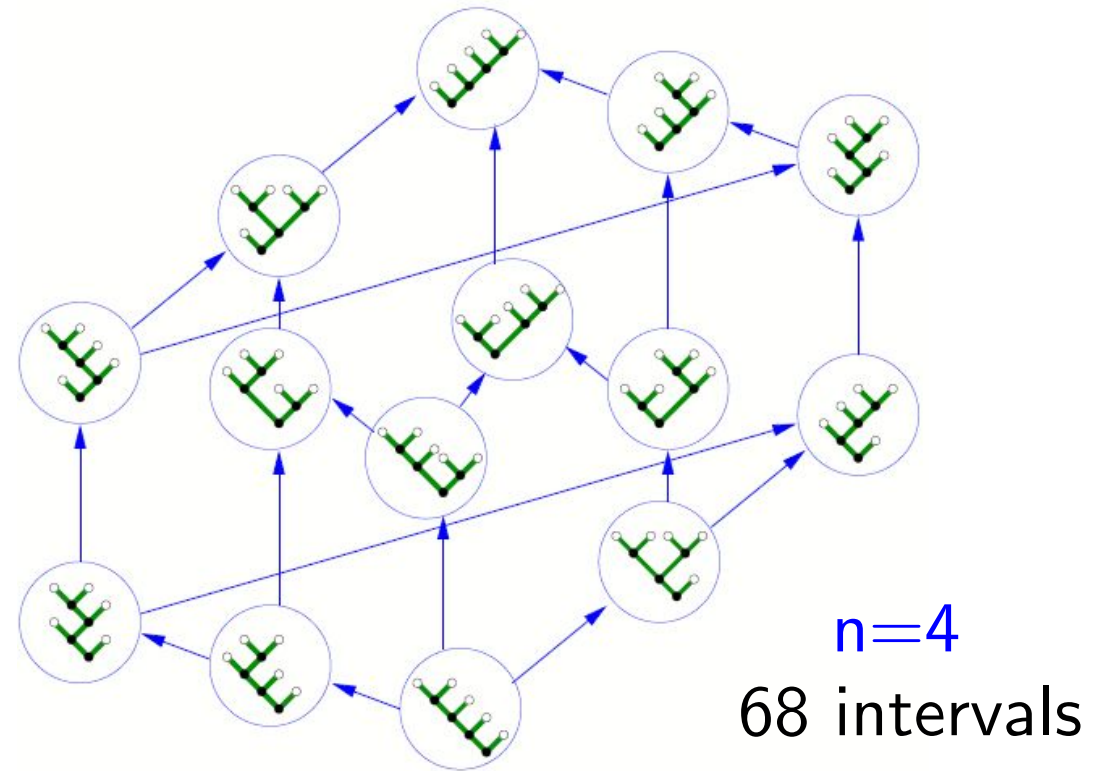
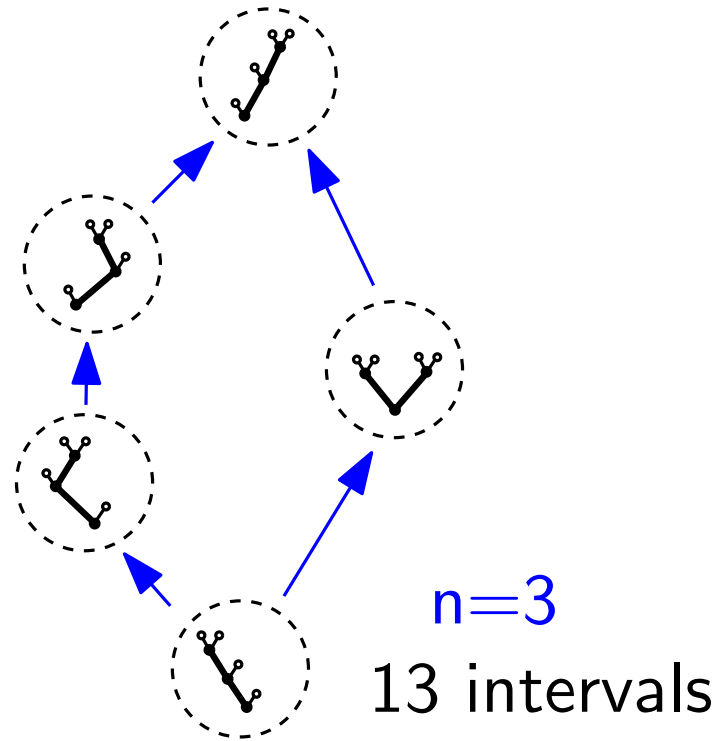
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Theorem [Chapoton'06]: there are $\frac{2}{n(n+1)} \binom{4n+1}{n-1}$ intervals in \mathcal{T}_n

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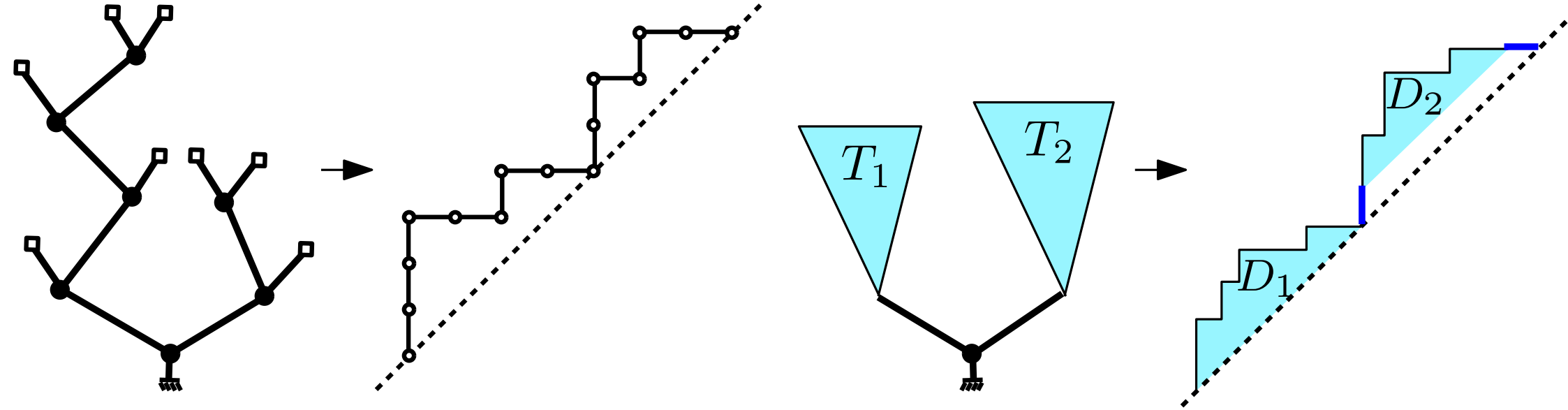


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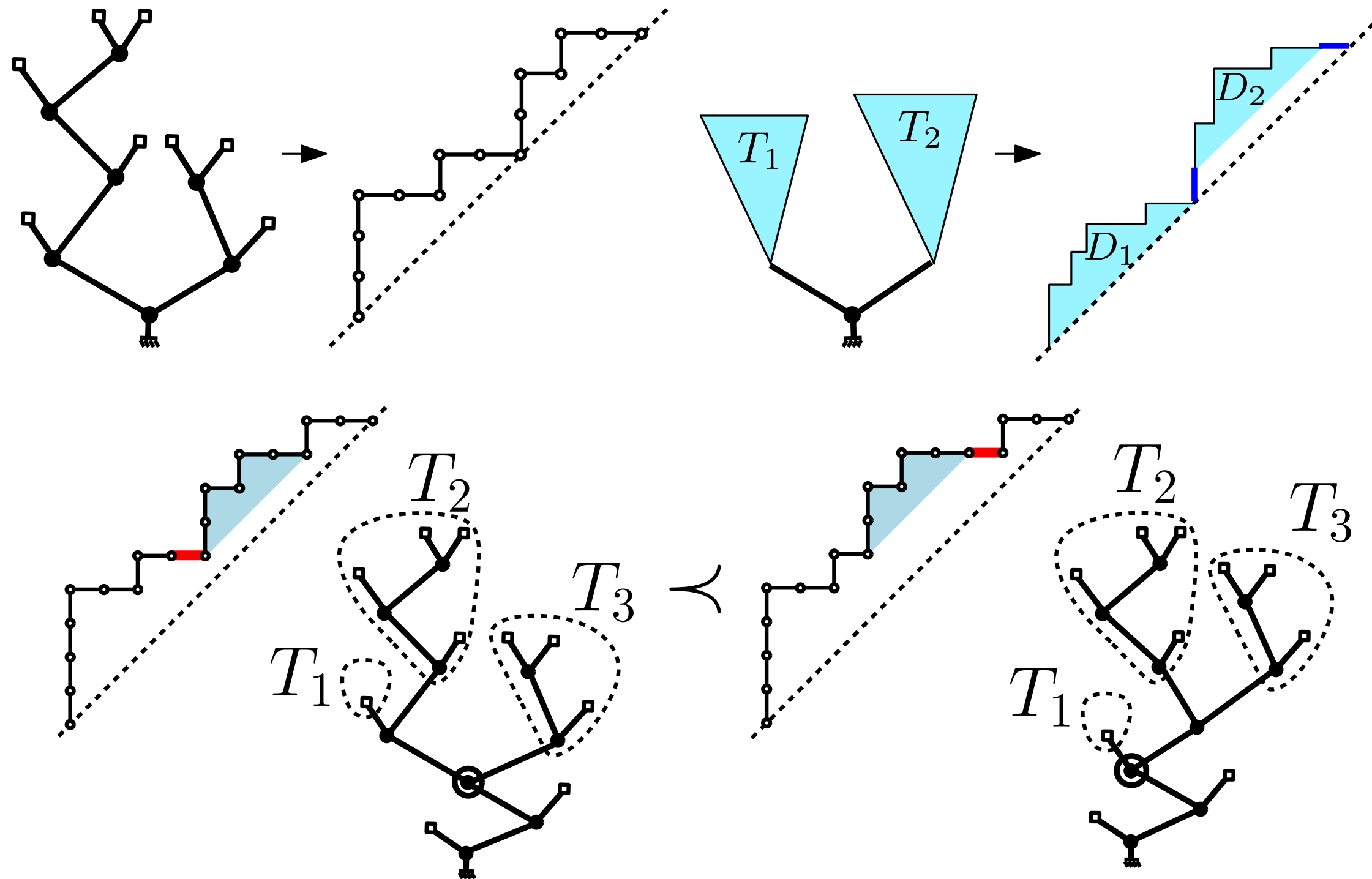
Talk overview:

- 3 proofs of Chapoton's result
- generalization to so-called " m -Tamari" lattices

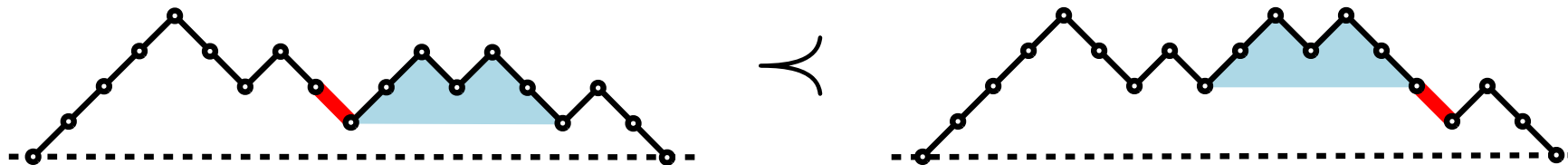
Reformulation of the Tamari lattice for Dyck paths



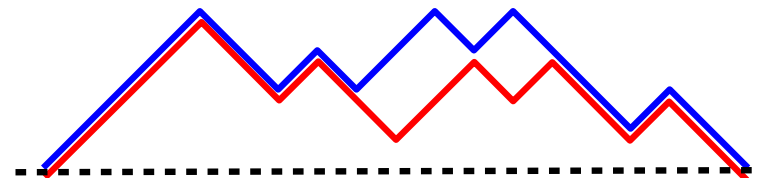
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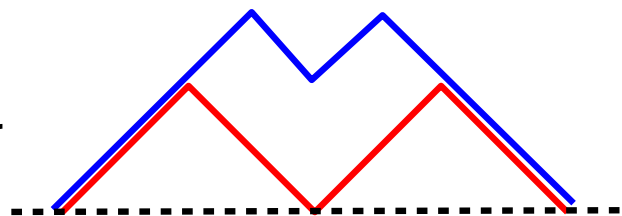
Reformulation of the Tamari lattice for Dyck paths



Rk: if $t \leq t'$ in \mathcal{T}_n , then t is below t'
the converse is not true !

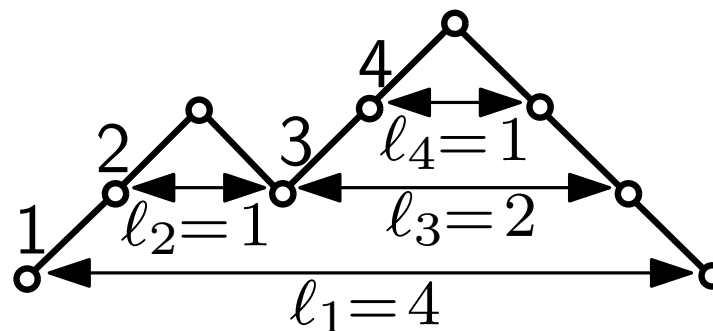


Q: How to test if a pair



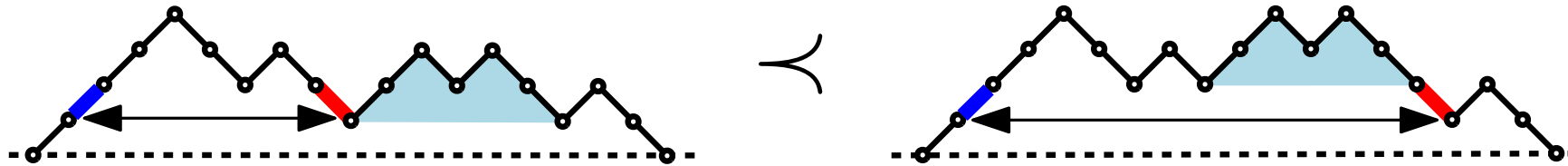
is an interval in \mathcal{T}_n ?

Length-vector L_D of D :

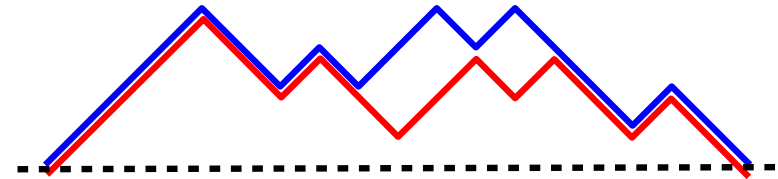


$$L_D = (4, 1, 2, 1)$$

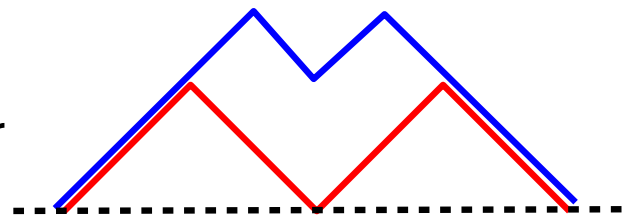
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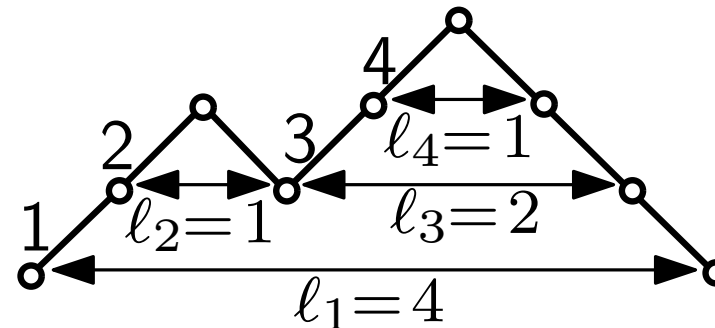
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the converse is not true !



Q: How to test if a pair is an interval in \mathcal{T}_n ?



Length-vector L_D of D :

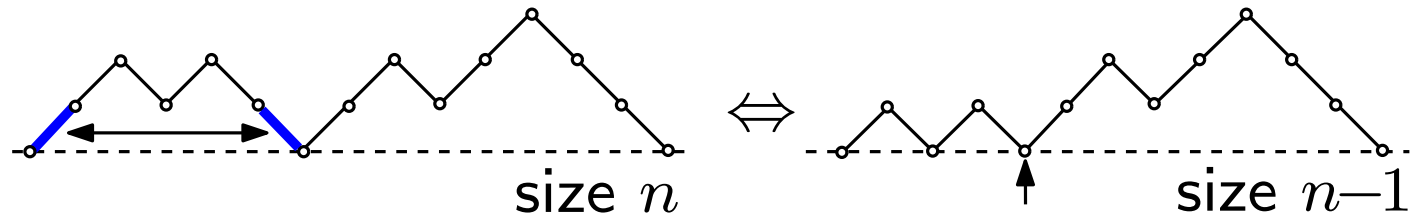


$$L_D = (4, 1, 2, 1)$$

Lem: $D \leq D'$ in \mathcal{T}_n iff $L_D \leq L_{D'}$

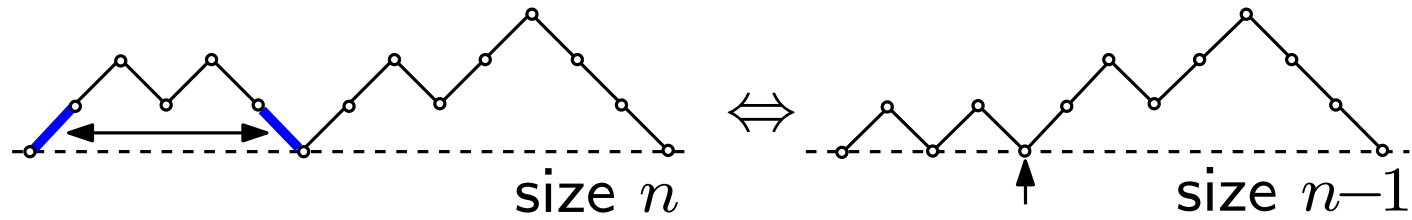
Recursive method for intervals in the Tamari lattice

- Reduction of a Dyck path:



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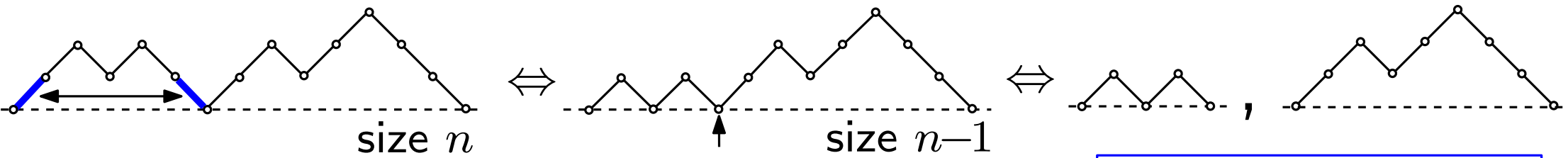
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(removes 1st component in length-vector)

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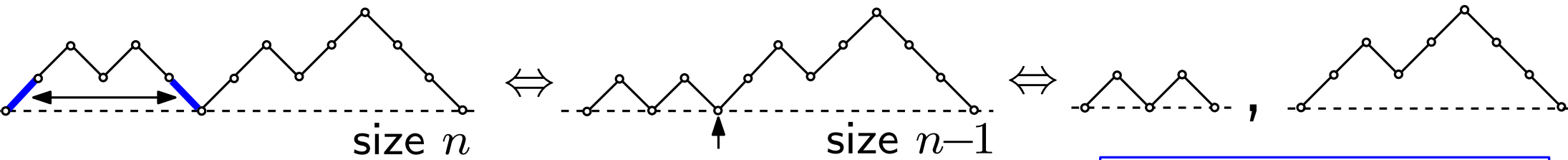


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$$D(t) = 1 + tD(t)^2$$

Recursive method for intervals in the Tamari lattice

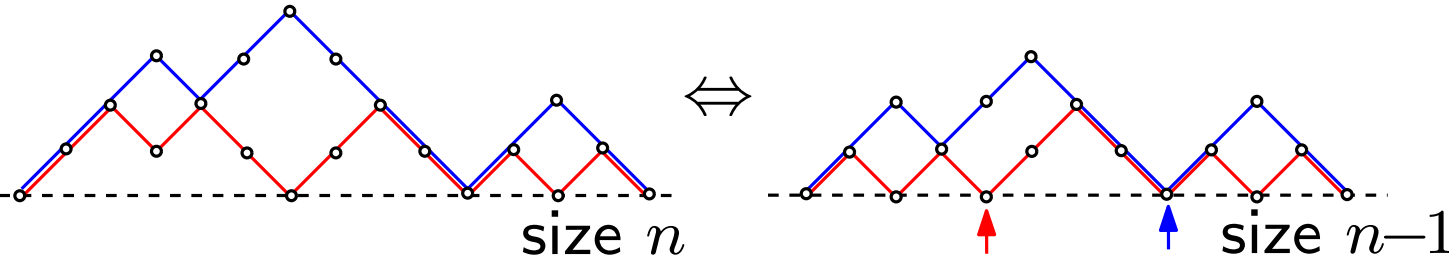
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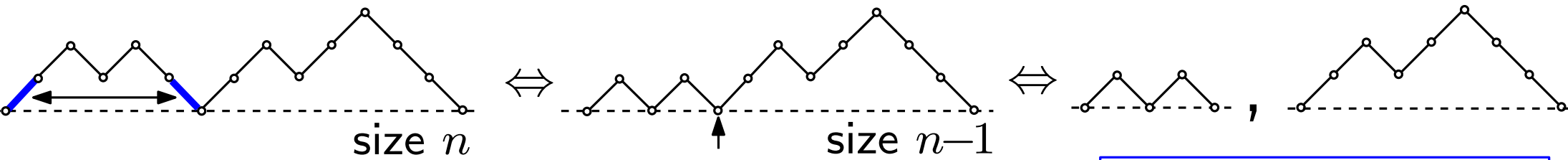
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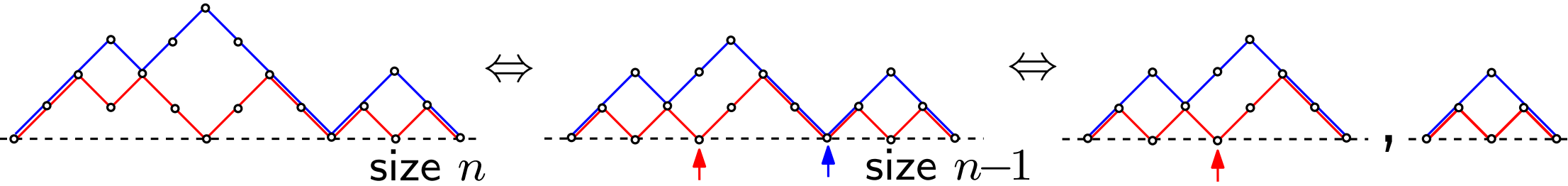
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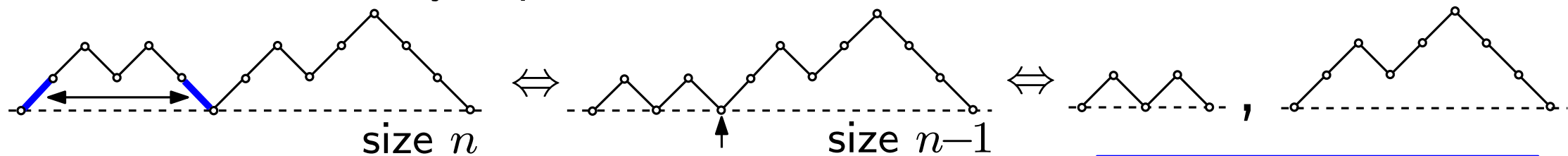
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Recursive method for intervals in the Tamari lattice

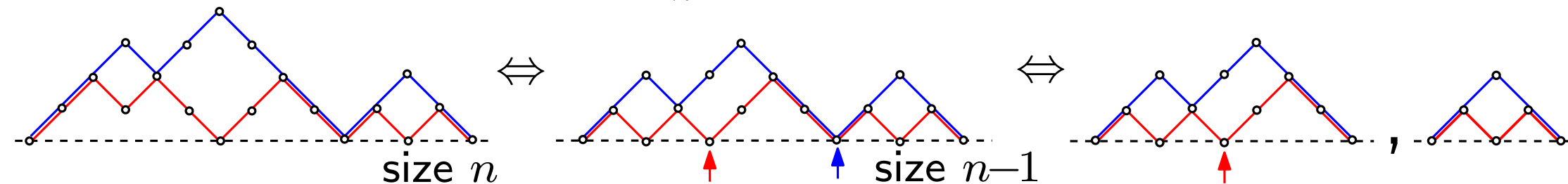
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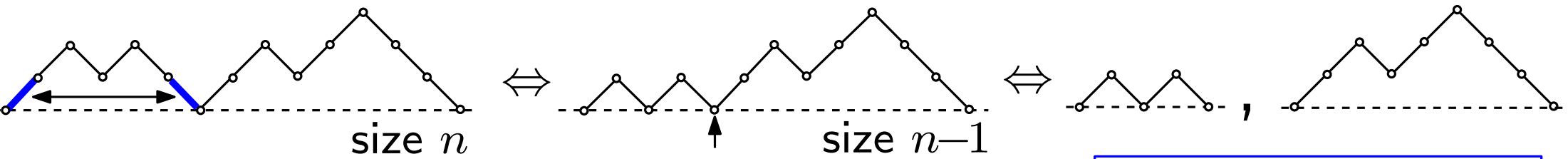


Let $a_{n,i} = \#$ (intervals in \mathcal{T}_n with i bottom contacts)

Let $F(t, x) := \sum_{n,i} a_{n,i} t^n x^i$. Then $F(t, 1) = 1 + t \cdot F_x(t, 1)F(t, 1)$

Recursive method for intervals in the Tamari lattice

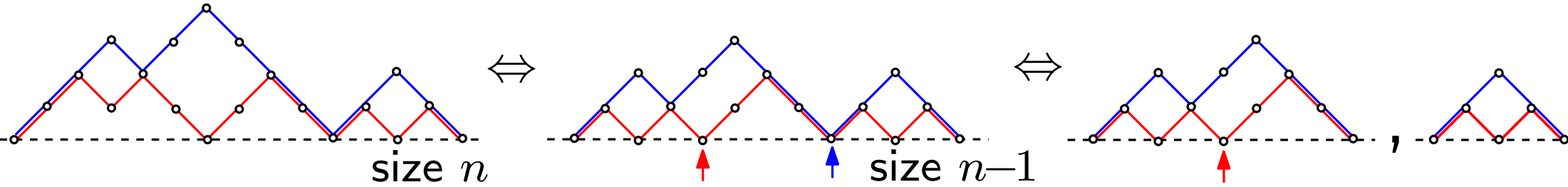
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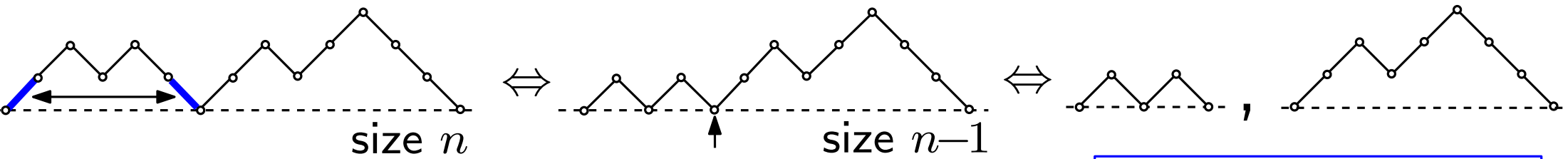
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More generally:

$$F(t, x) = x + t \cdot \text{subs}(x^i = x + \dots + x^i, F(t, x)) \cdot F(t, x)$$

Recursive method for intervals in the Tamari lattice

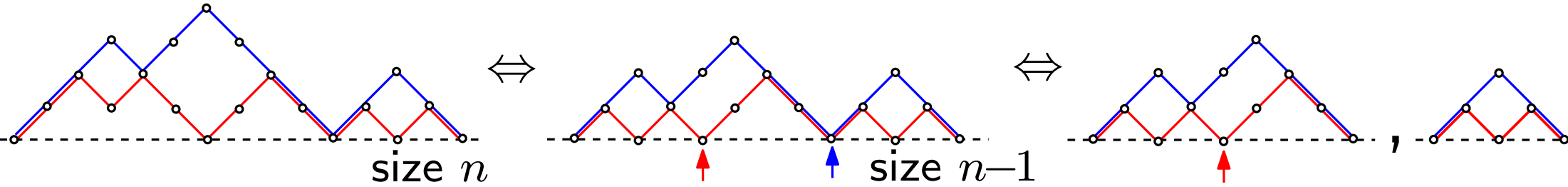
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$$F(t, x) = x + txF(t, x) \frac{F(t, x) - F(t, 1)}{x - 1}$$

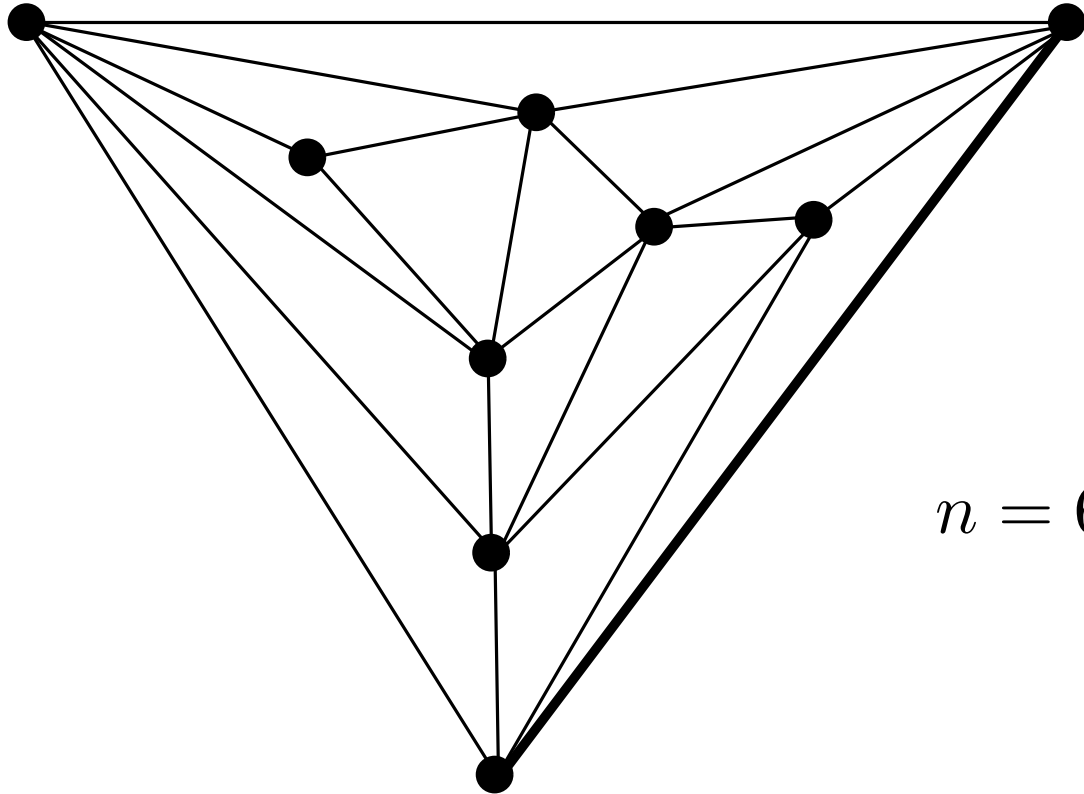
Hence the number of intervals in \mathcal{T}_n is $[t^n]F(t, 1)$, with

$$F(t, x) = x + txF(t, x) \frac{F(t, x) - F(t, 1)}{x - 1}$$

We present 3 methods for finding $[t^n]F(t, 1)$ from the equation above:

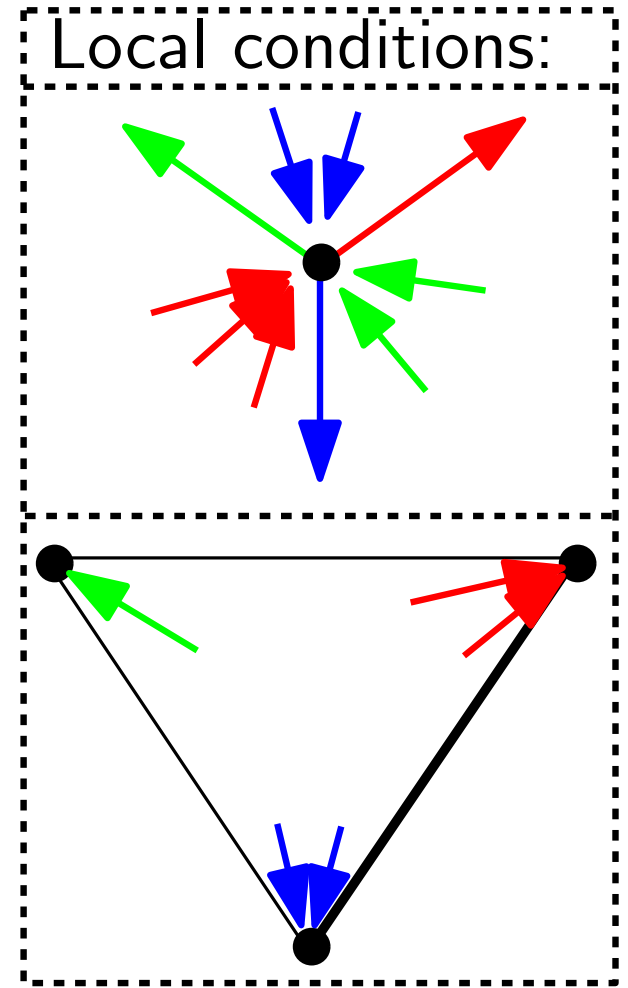
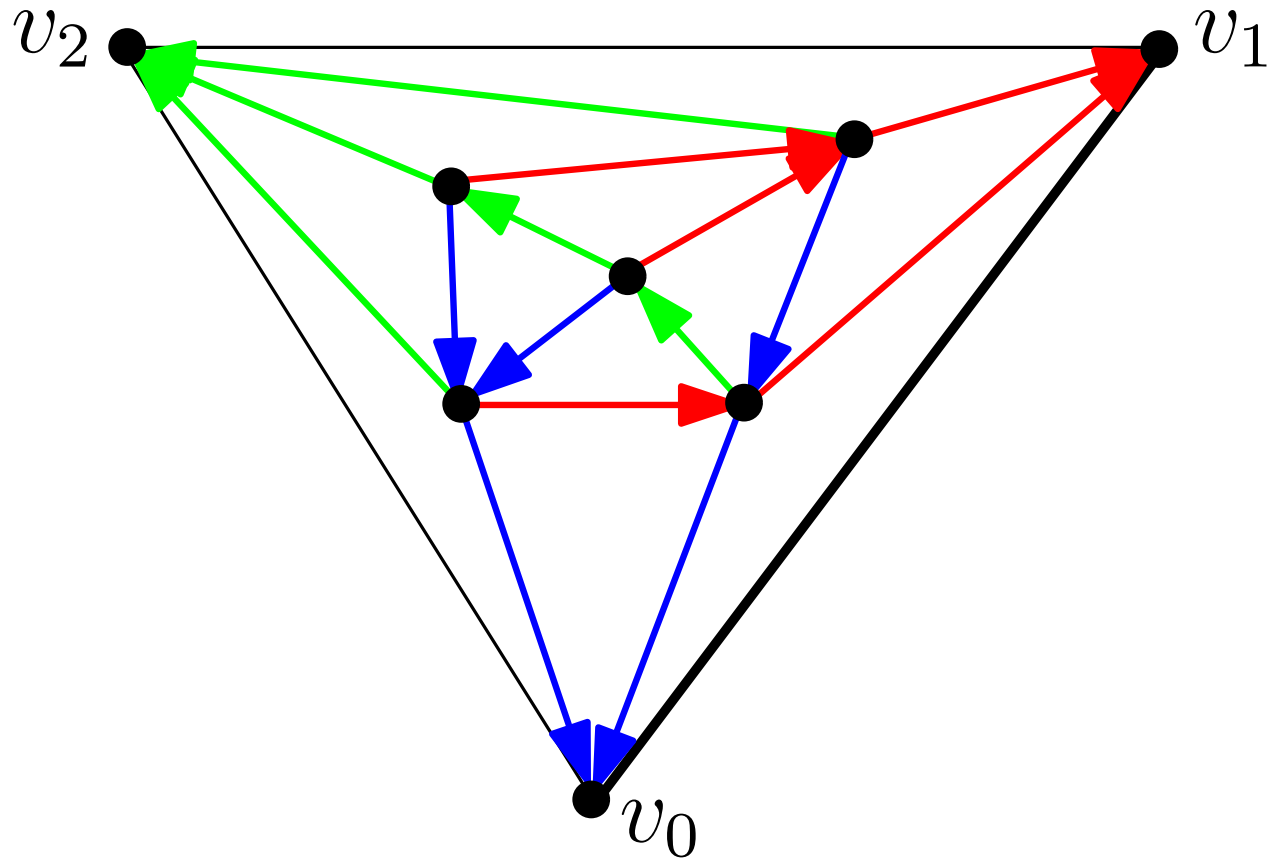
- bijective
- Solve the equation using the quadratic method
- Solve the equation by guessing/checking

Triangulations

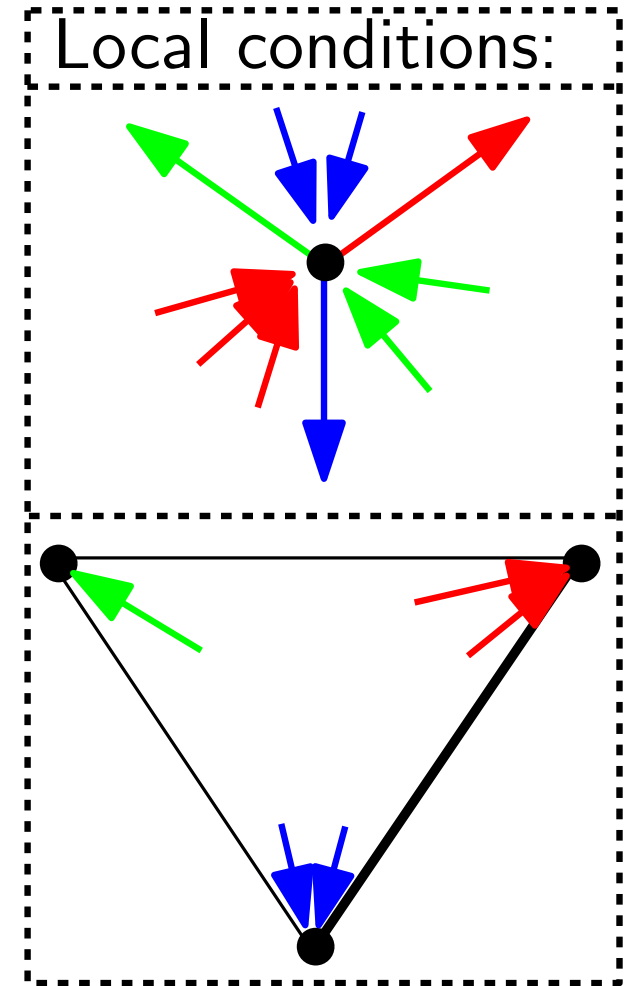
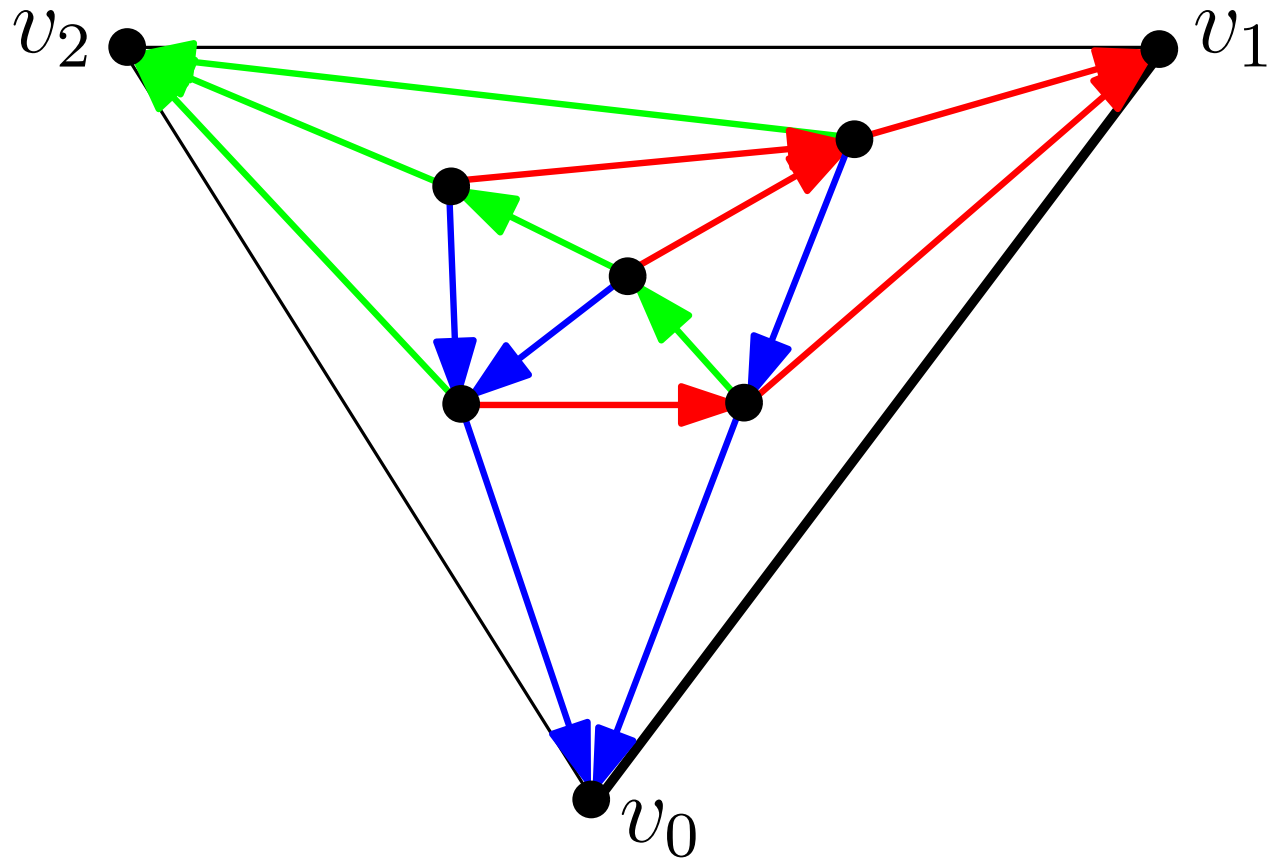


$n = 6$ internal vertices

Schnyder woods



Schnyder woods

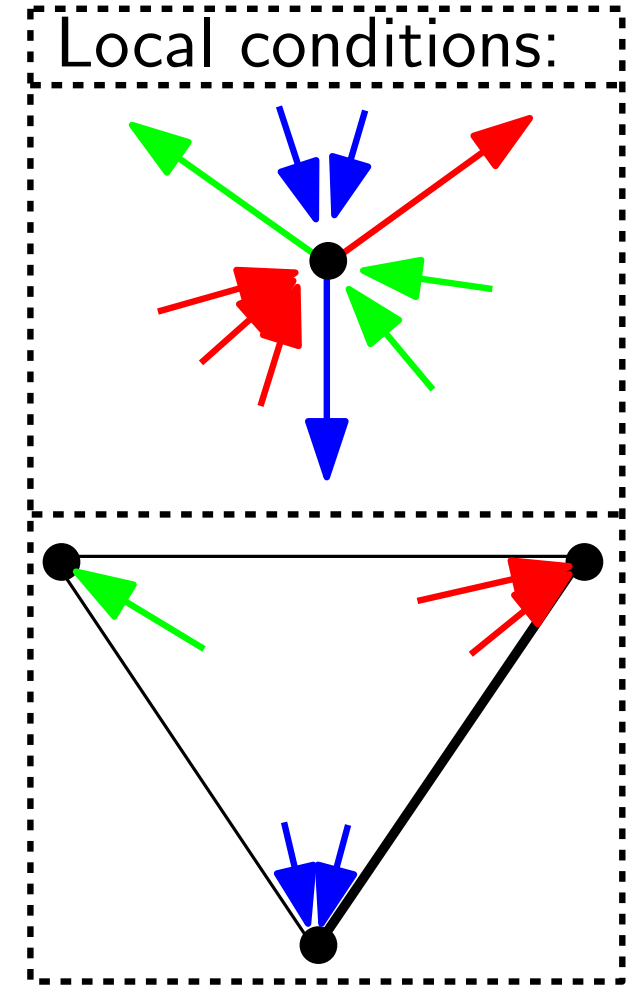
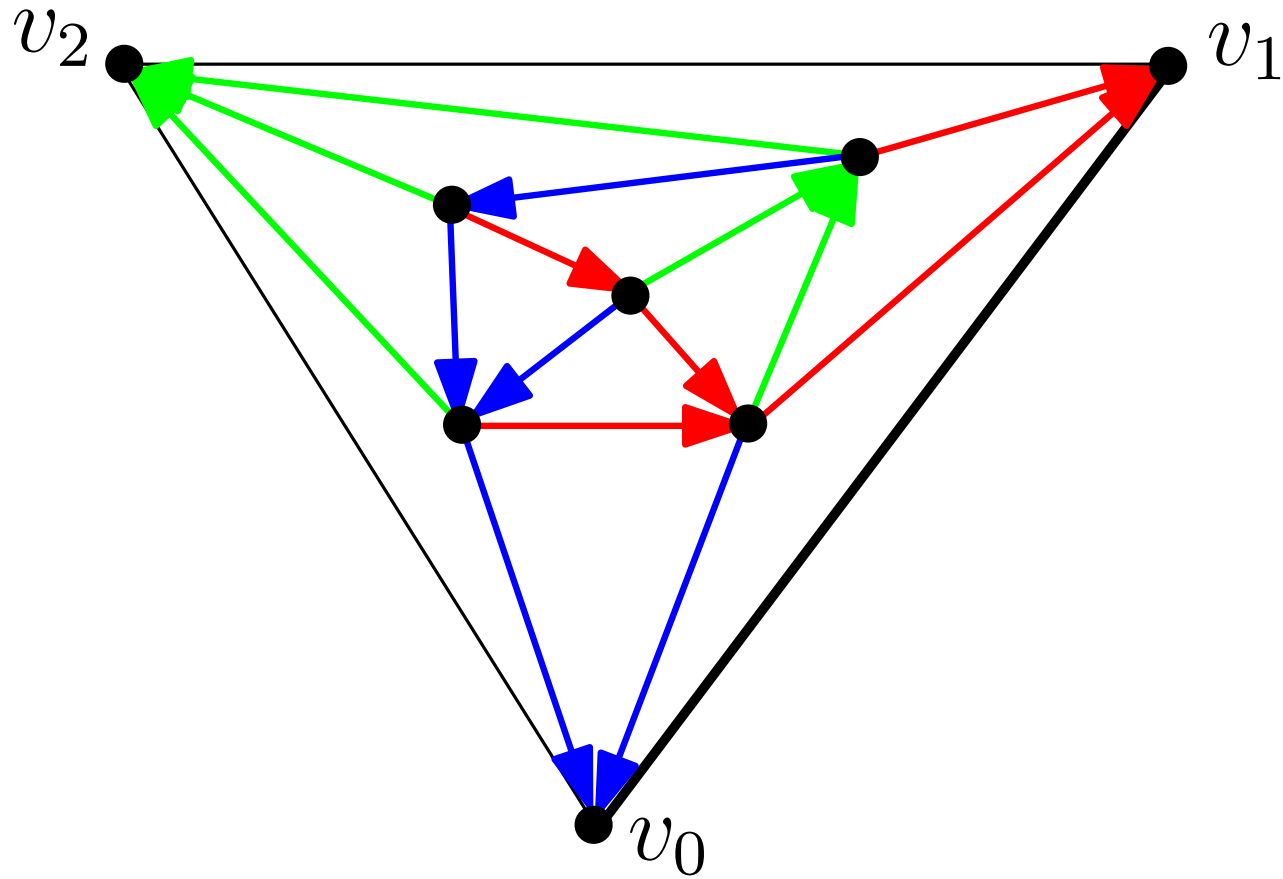


The Schnyder wood is called minimal if it has no clockwise circuit

Theo [Schnyder'90, Brehm'03]:

Any triangulation has a unique minimal Schnyder wood

Schnyder woods

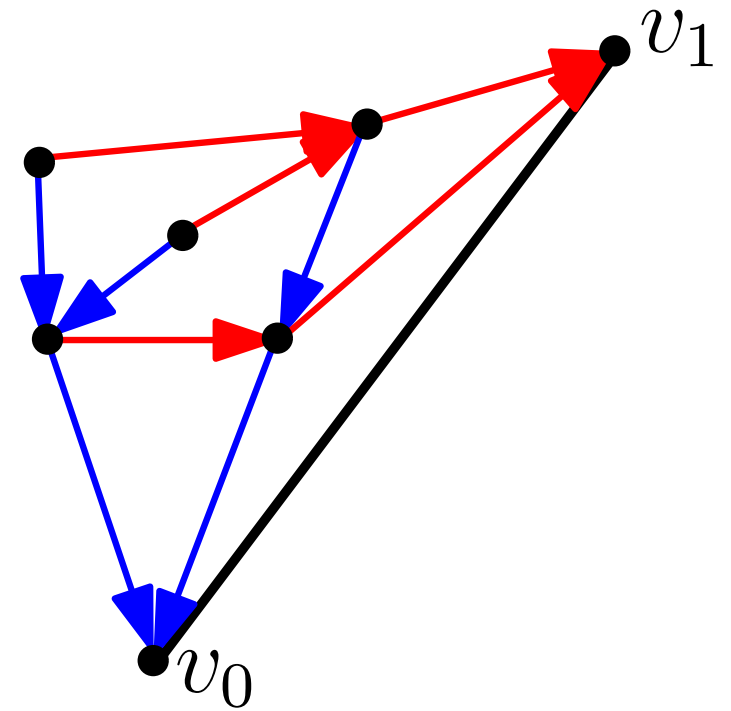
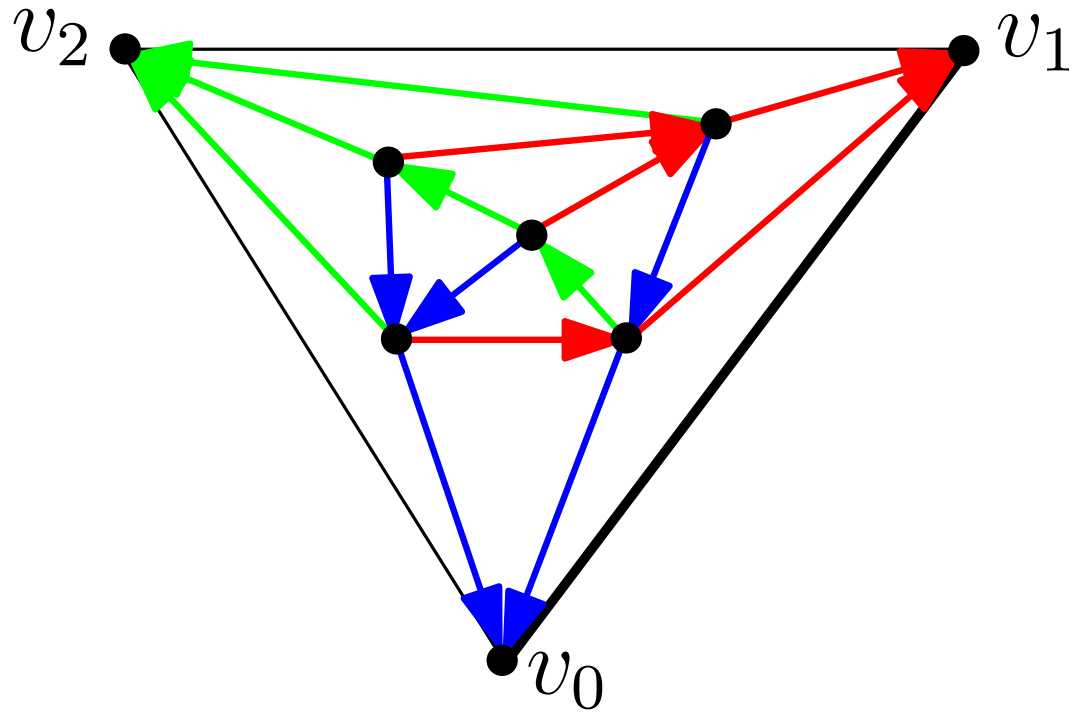


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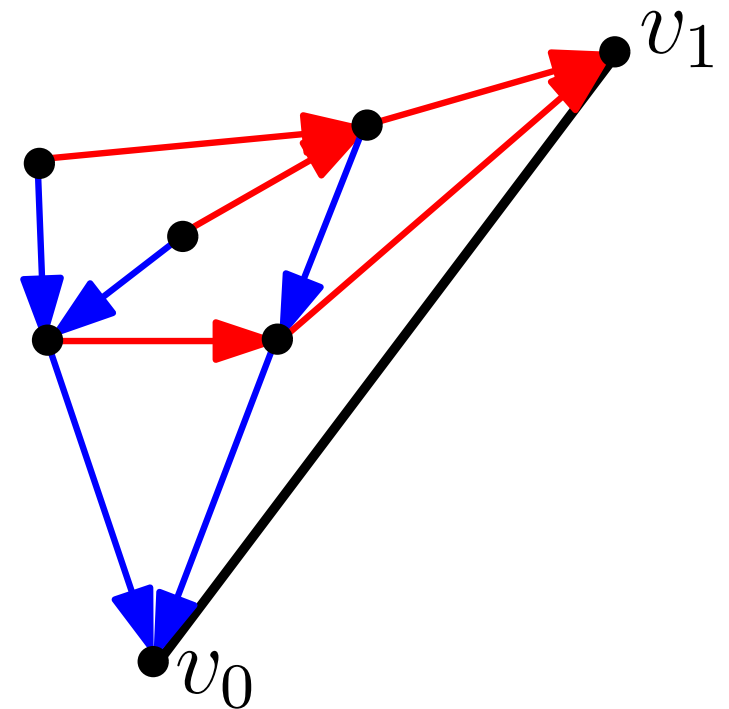
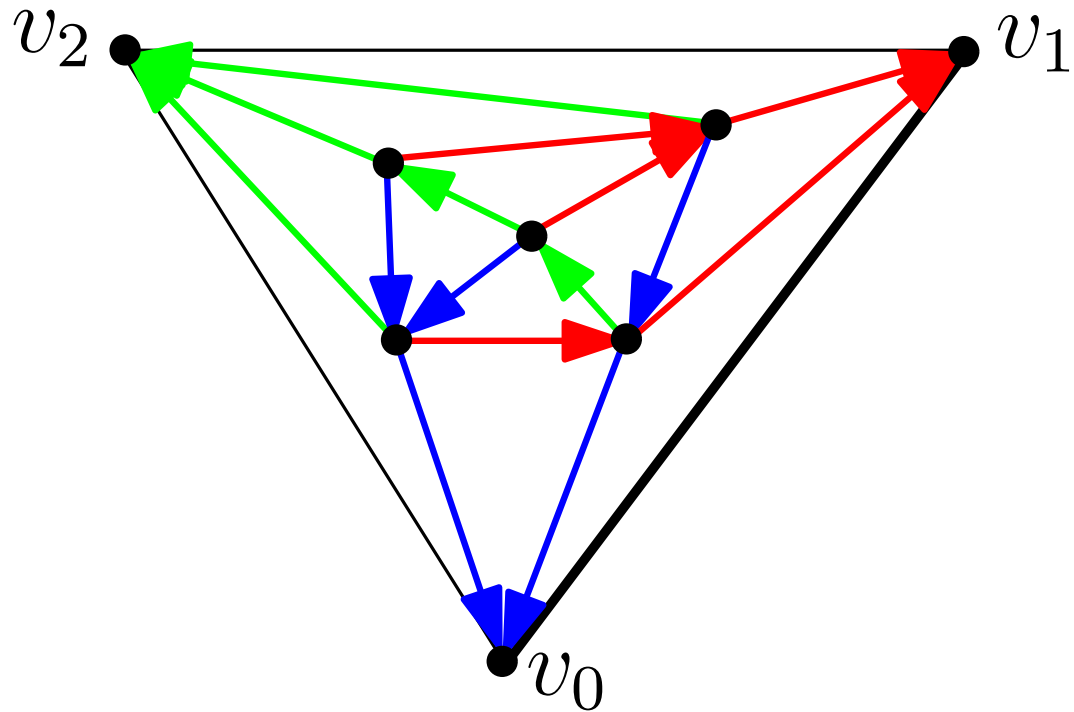
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The red-blue induced structure



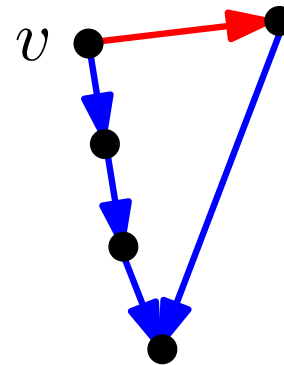
There is no loss of information in deleting the green edges

The red-blue induced structure



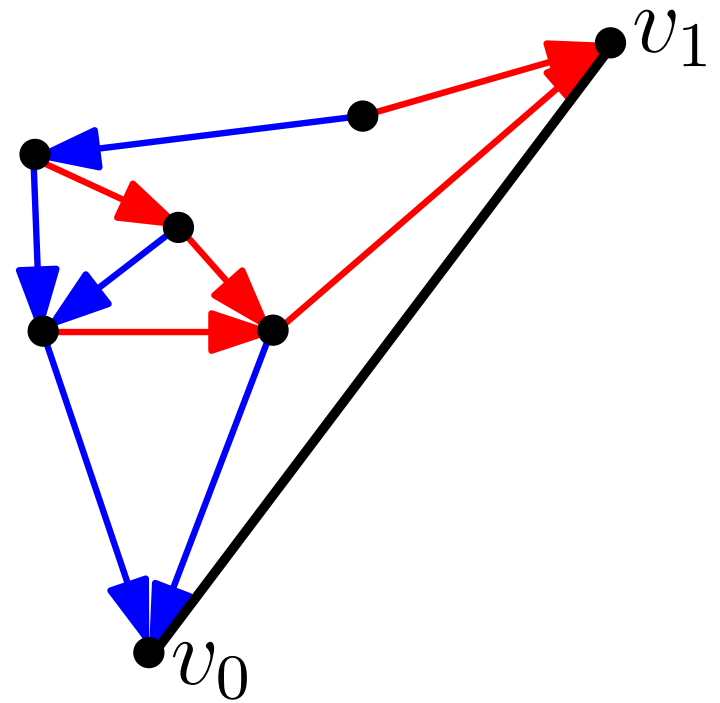
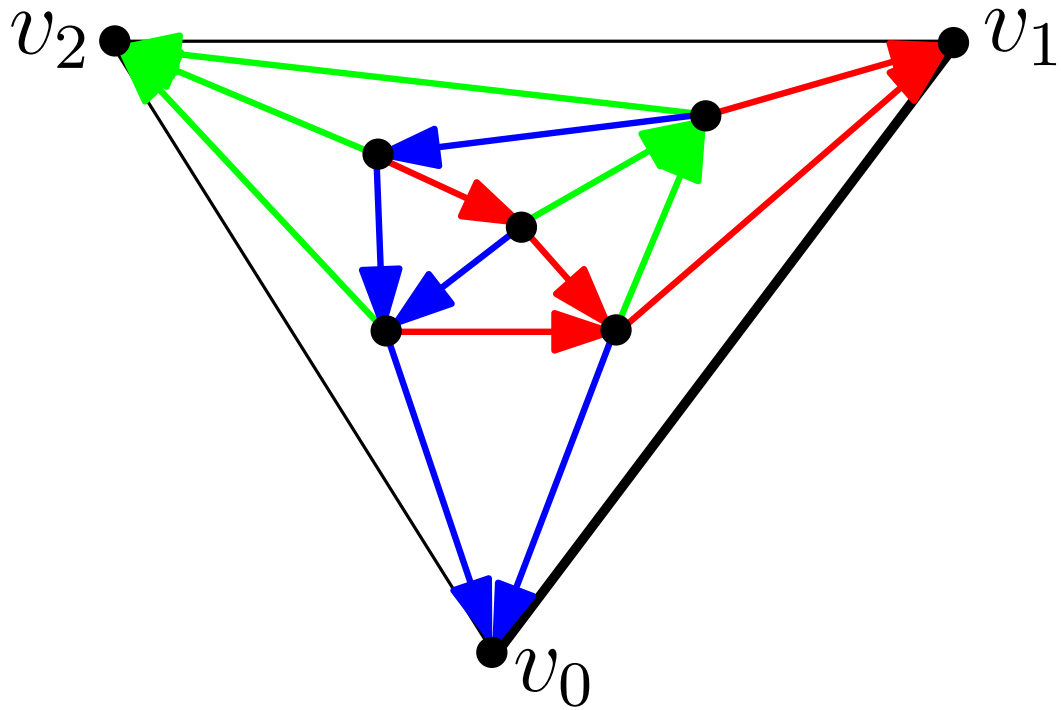
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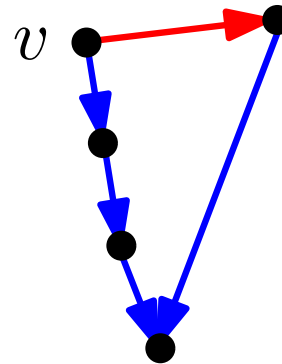
$\forall v$ interval vertex
the red parent of
the blue parent is
a blue ancestor

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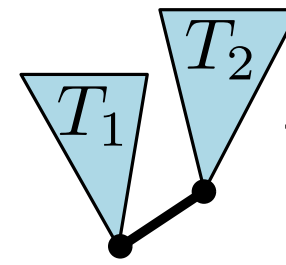
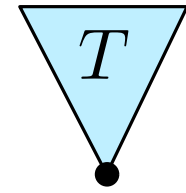
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Decomposing a minimal red-blue structure

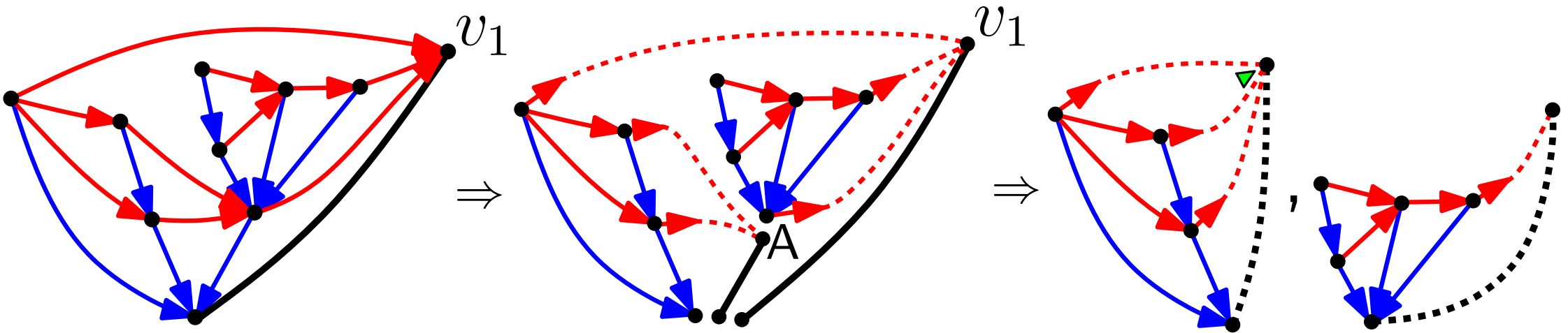
cf L.-F. Prévaille-Ratelle. Idea: apply



to the blue tree

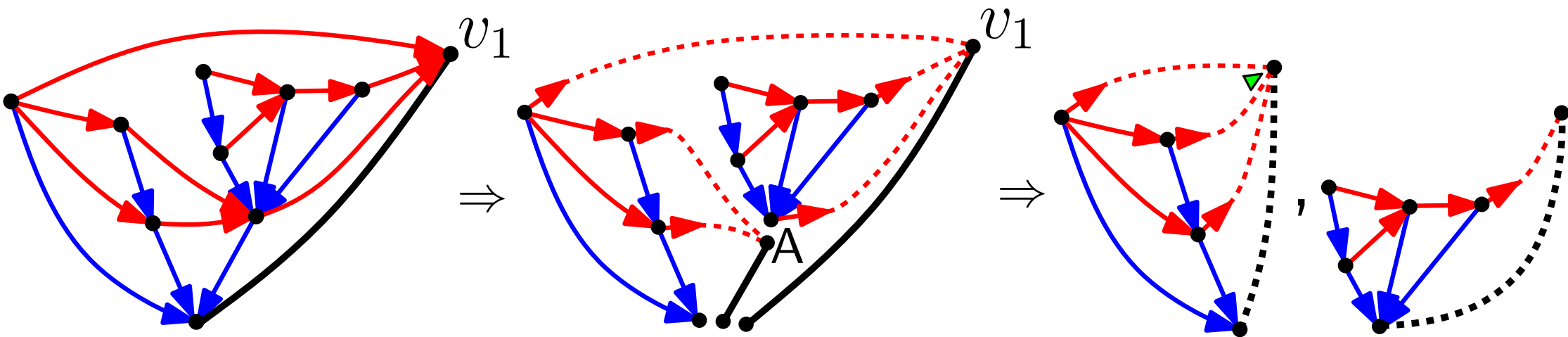
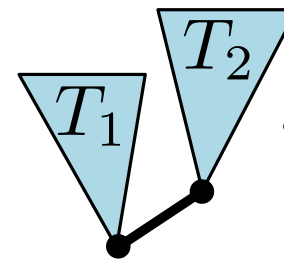
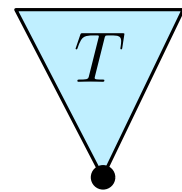
Decomposing a minimal red-blue structure

cf L.-F. Prévaille-Ratelle. Idea: apply $T \rightarrow T_1 T_2$ to the blue tree



Decomposing a minimal red-blue structure

cf L.-F. Prévaille-Ratelle. Idea: apply T to the blue tree



Let $a_{n,i} = \#(\text{ triangulations with } n + 3 \text{ vertices, } \deg(v_1) = i + 1)$

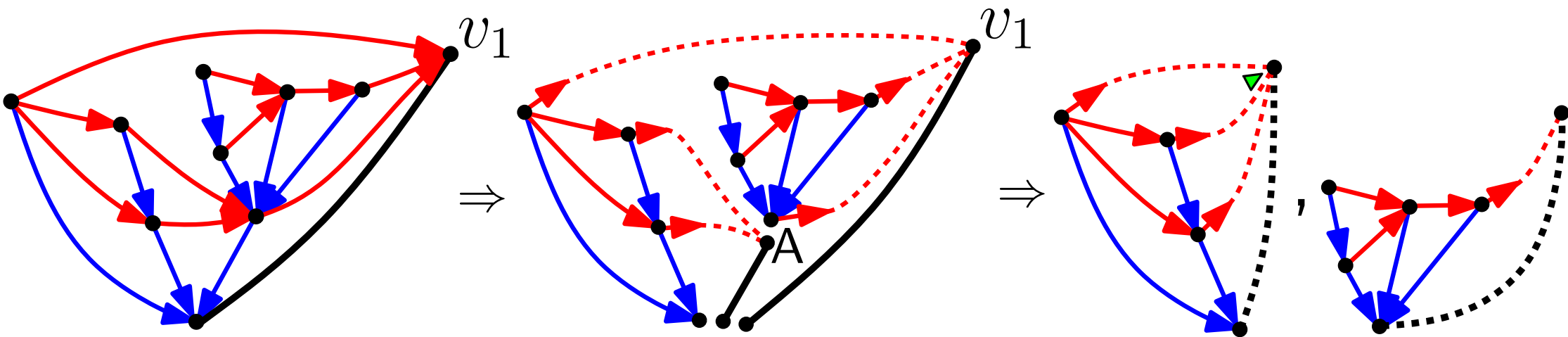
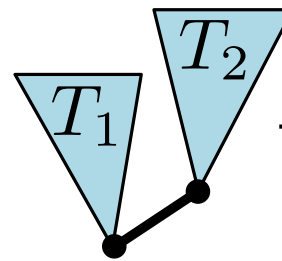
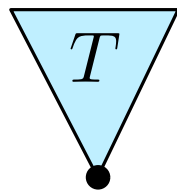
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Then

$$F(t, x) = x + txF(t, x) \frac{F(t, x) - F(t, 1)}{x - 1}$$

Decomposing a minimal red-blue structure

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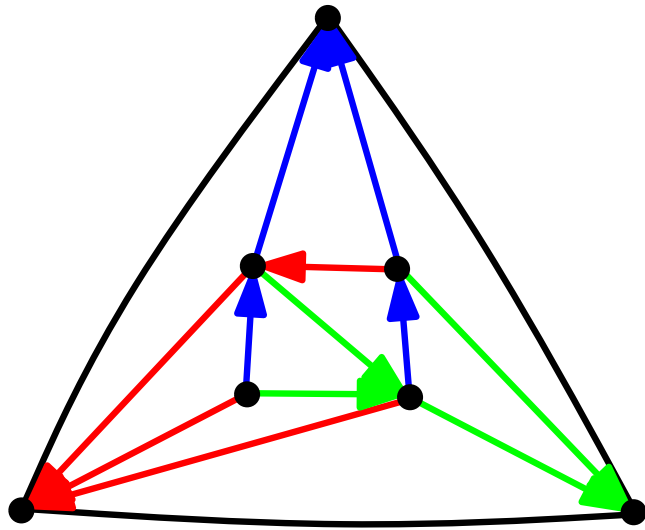
$$F(t, x) = x + txF(t, x) \frac{F(t, x) - F(t, 1)}{x - 1}$$

$\Rightarrow \#(\text{intervals in } \mathcal{T}_n) = \#(\text{ triangulations } n \text{ internal vertices})$

(also direct bijection in [Bernardi, Bonichon'09])

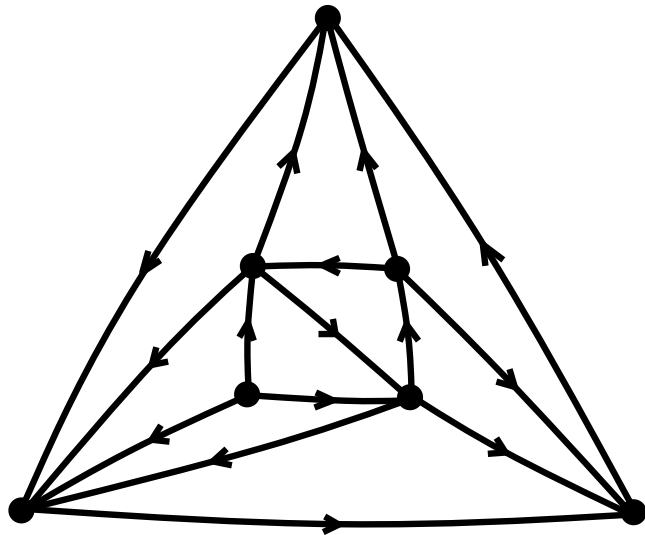
Bijection counting of triangulations

Moreover, minimal Schnyder woods give a bijection to count triangulations



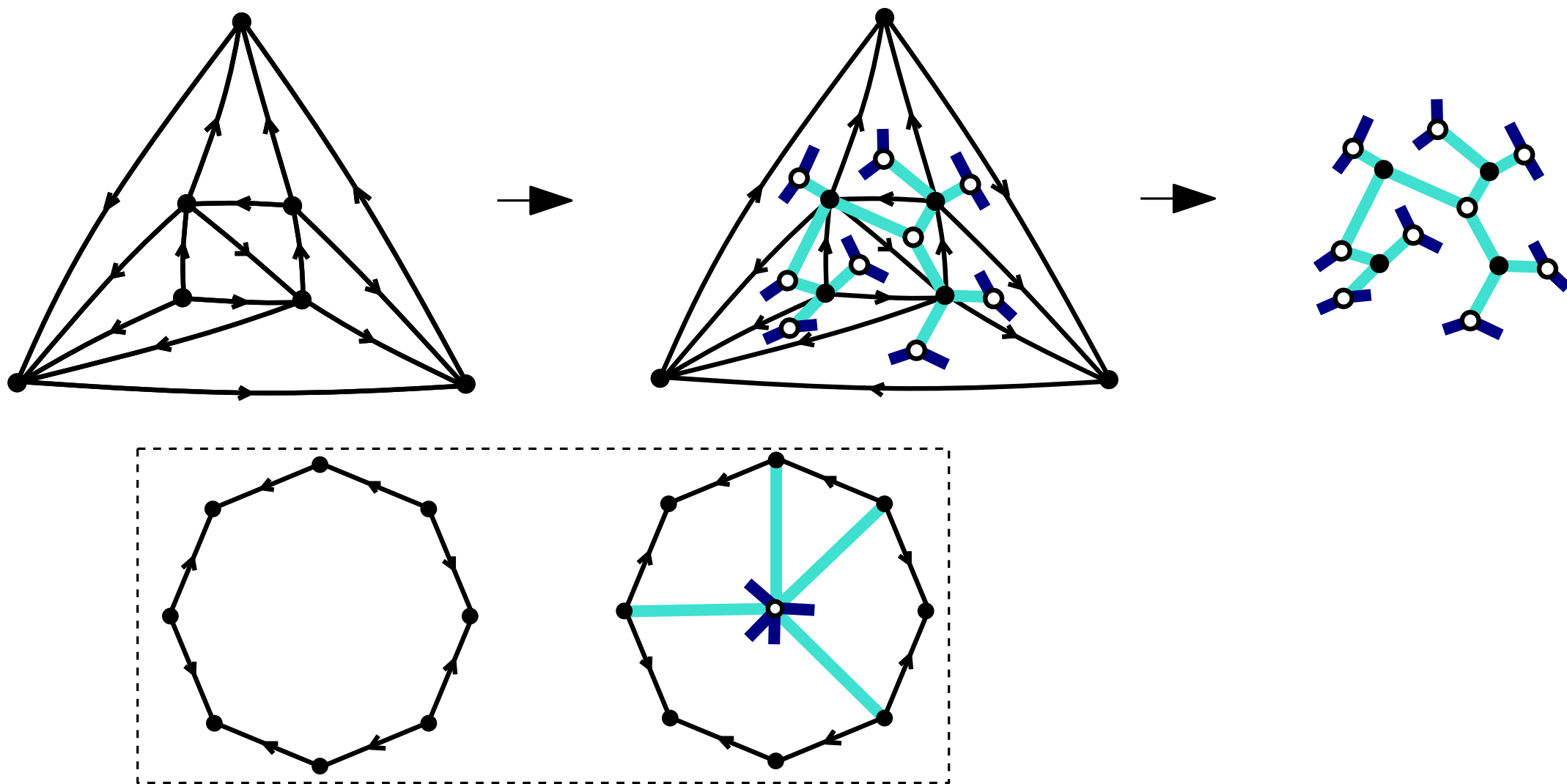
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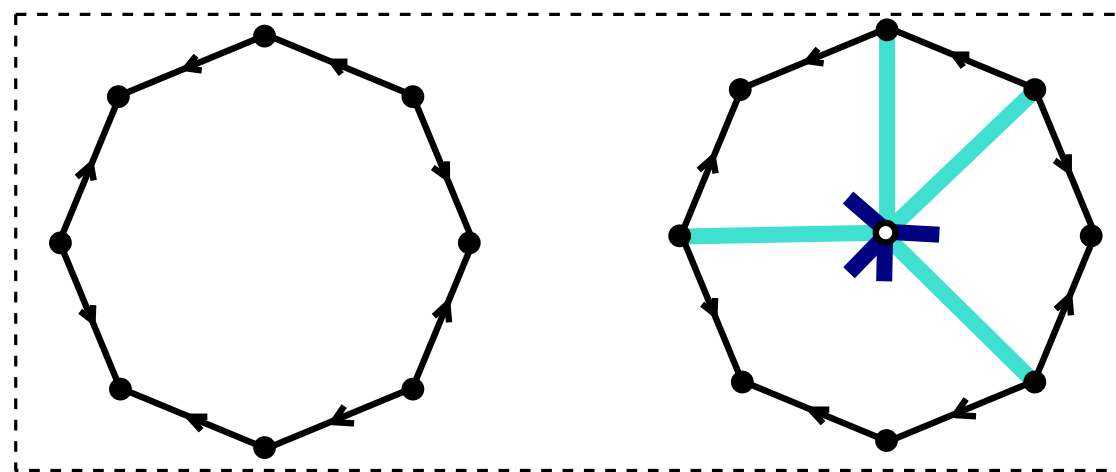
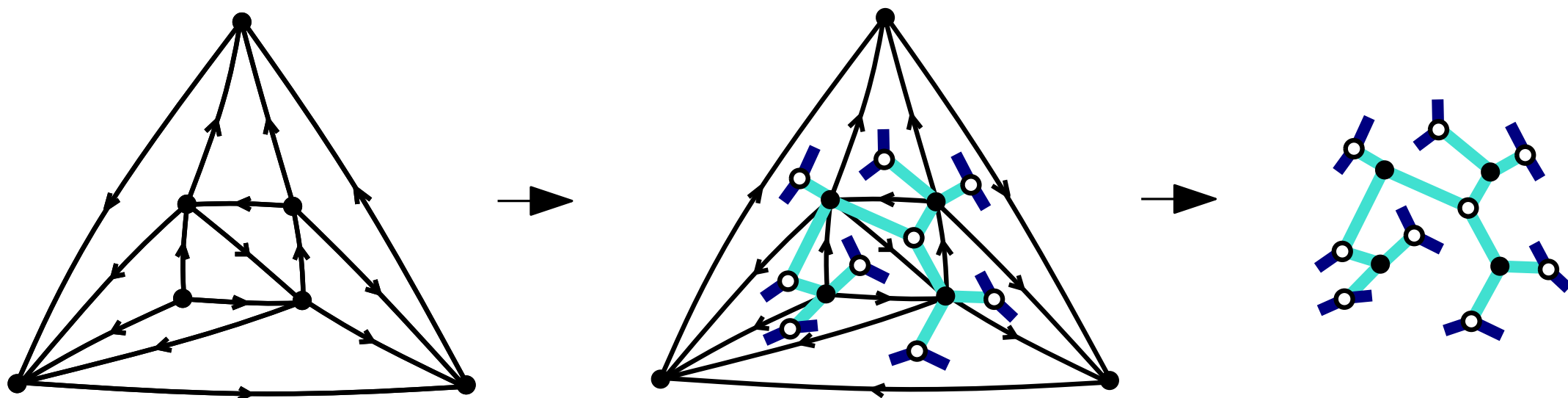
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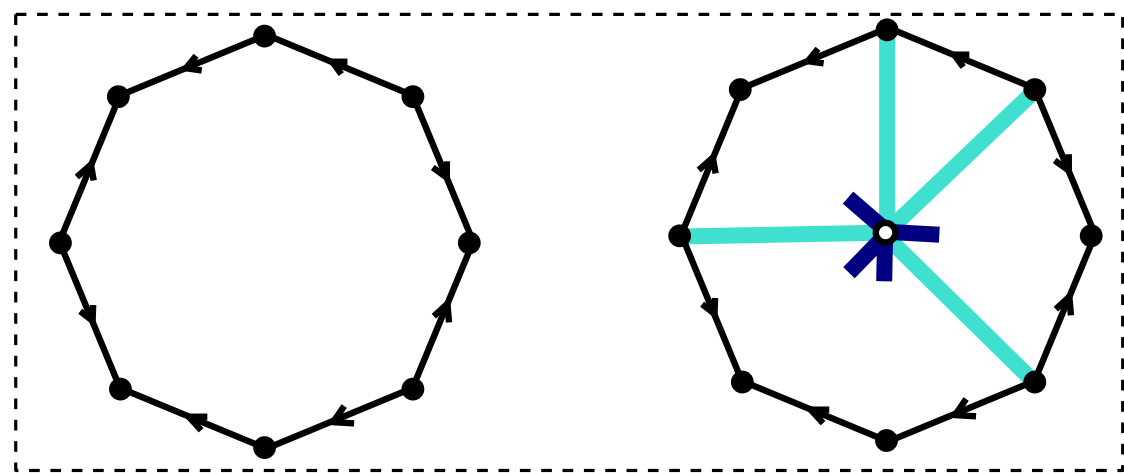
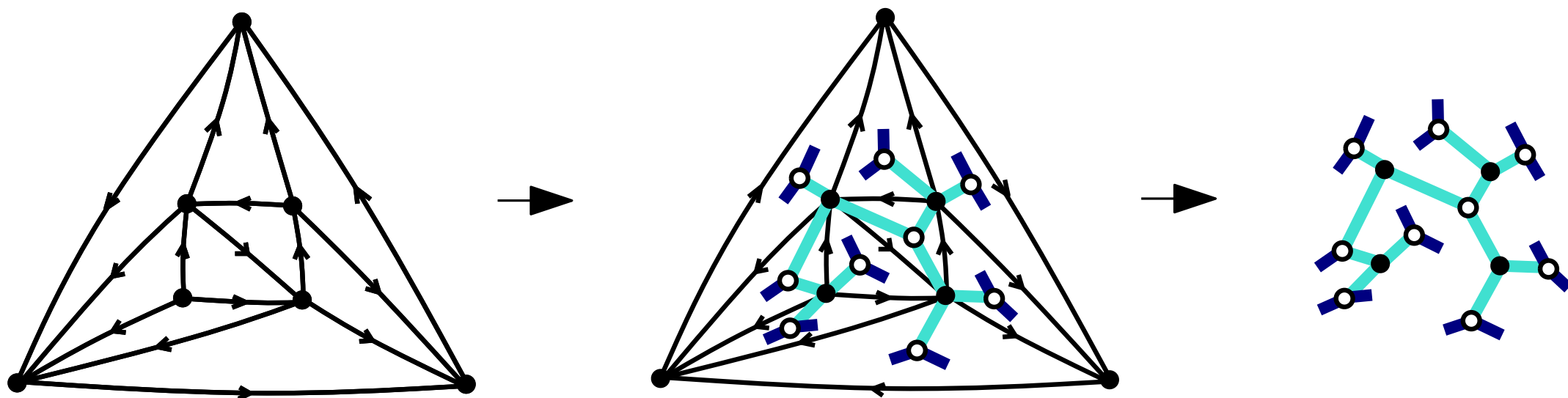
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Method to solve directly

$$F(t, x) = x + txF(t, x) \frac{F(t, x) - F(t, 1)}{x - 1} \quad (\text{E})$$

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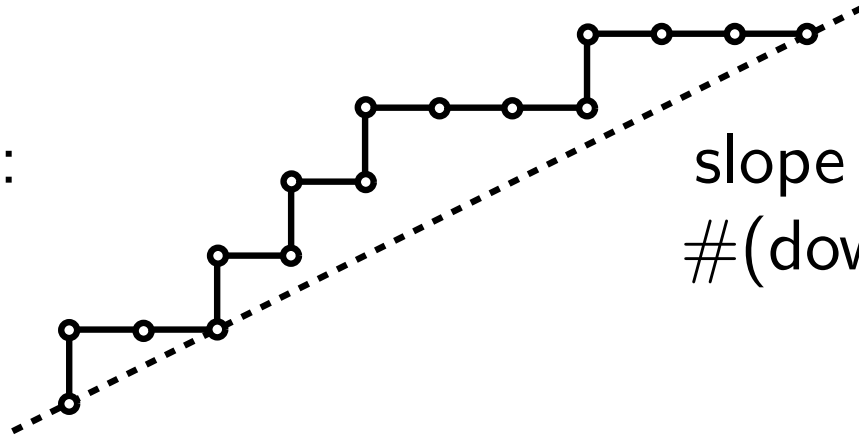
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Generalization to “ m -Tamari lattices”, for any $m \geq 1$

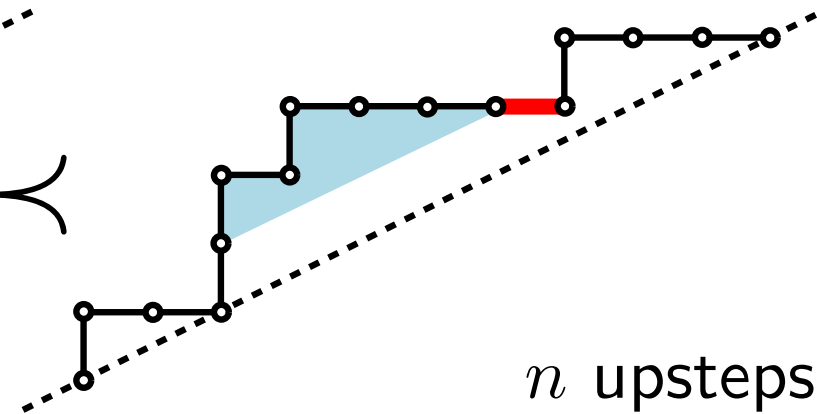
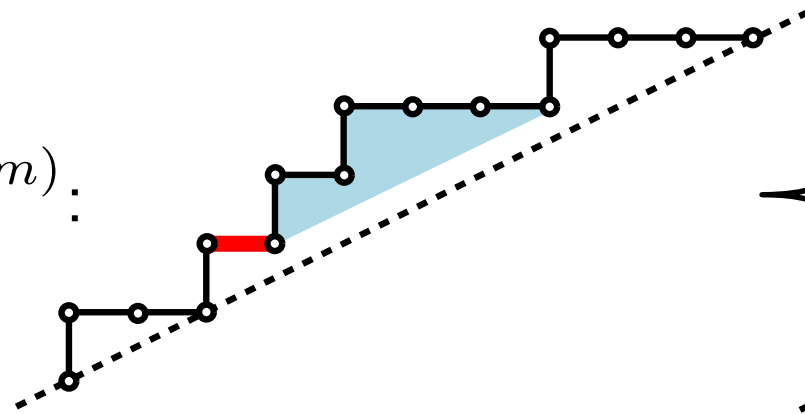
m -Dyck path:



slope = $1/m$

$\#(\text{downsteps}) = m \cdot \#(\text{upsteps})$

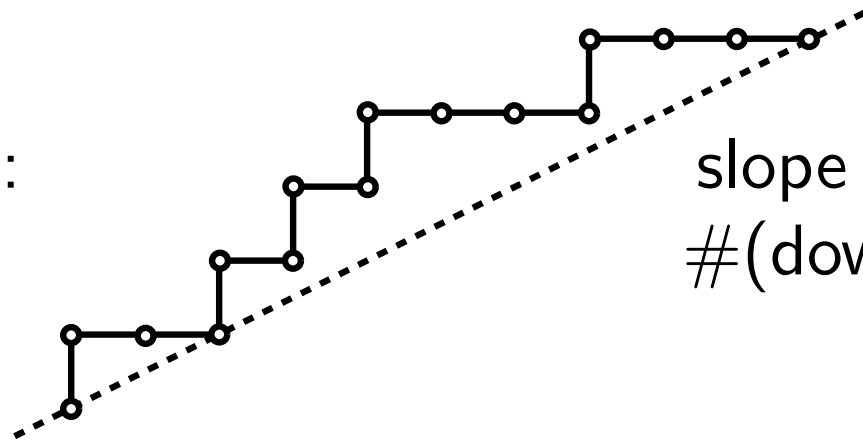
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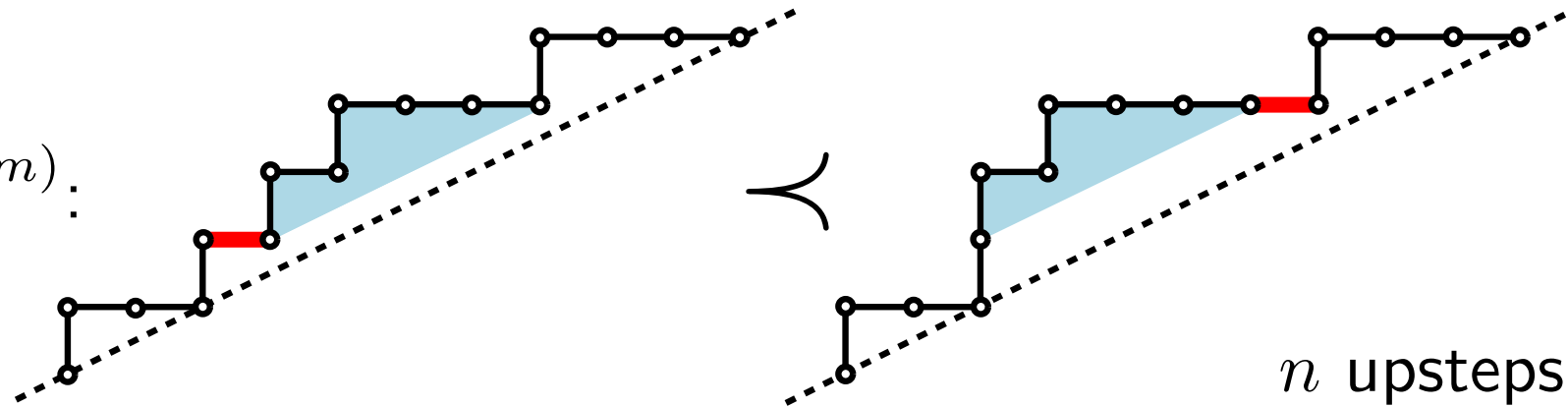
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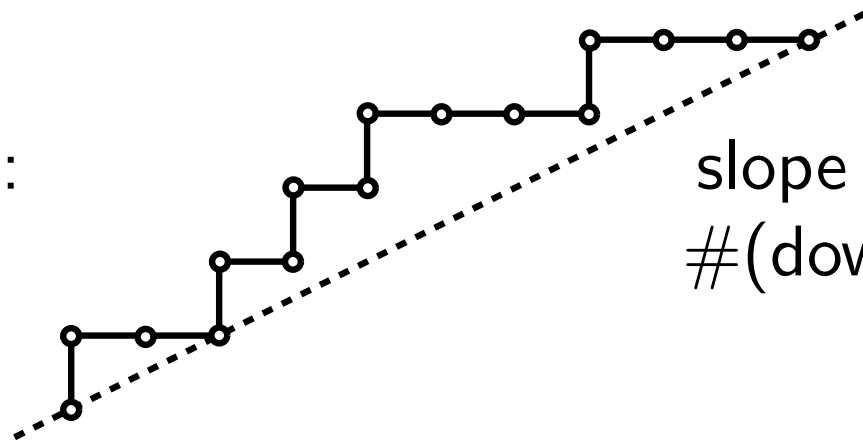
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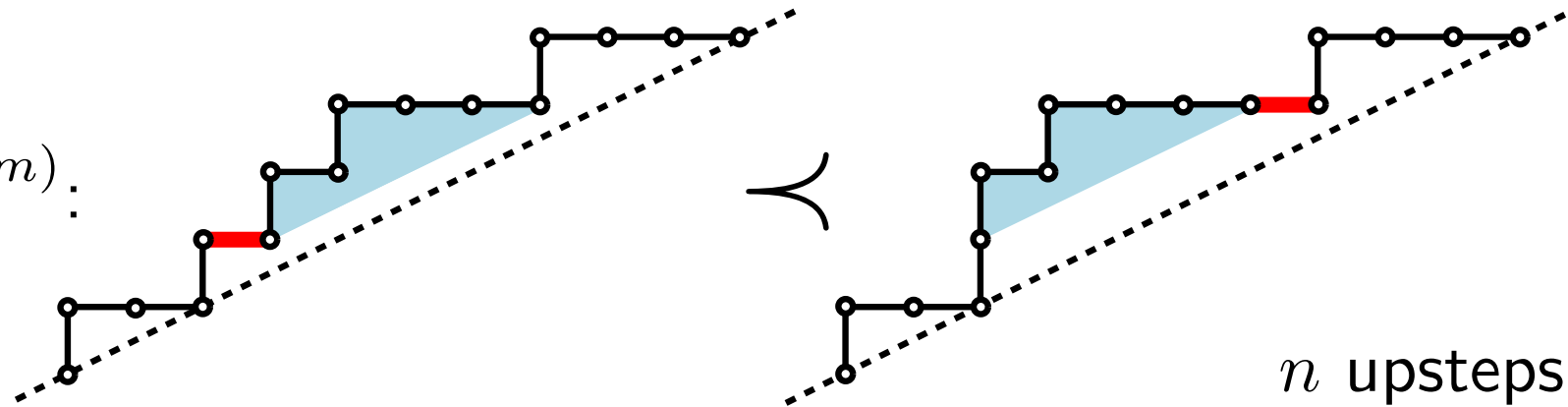
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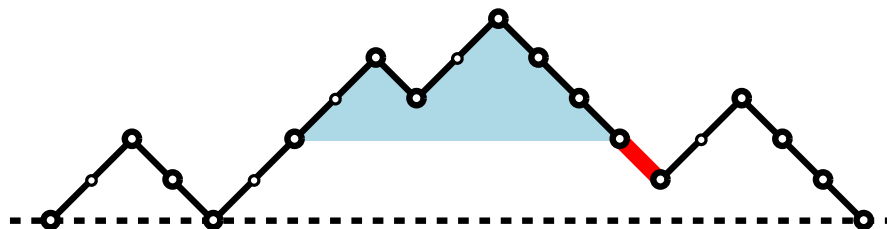
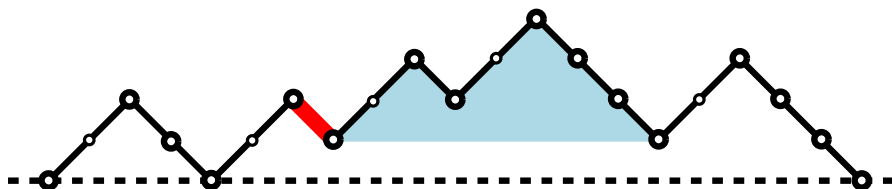
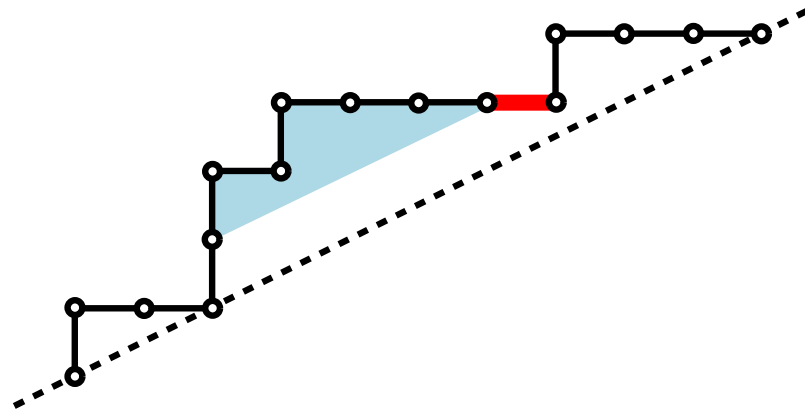
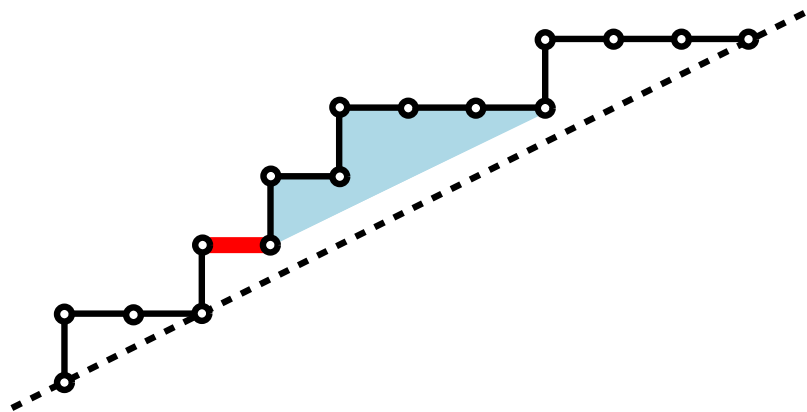


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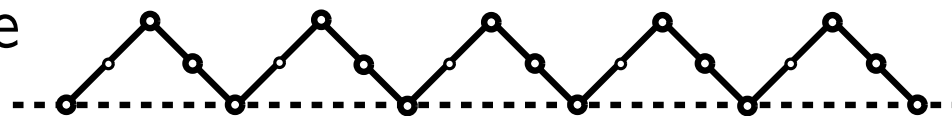
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Now proved in [Bousquet-Mélou, F, Préville-Ratelle'11]

A slight reformulation



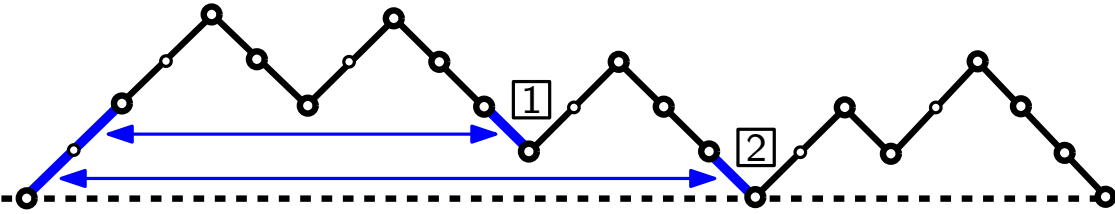
$\mathcal{T}_n^{(m)} \simeq$ sublattice of paths above



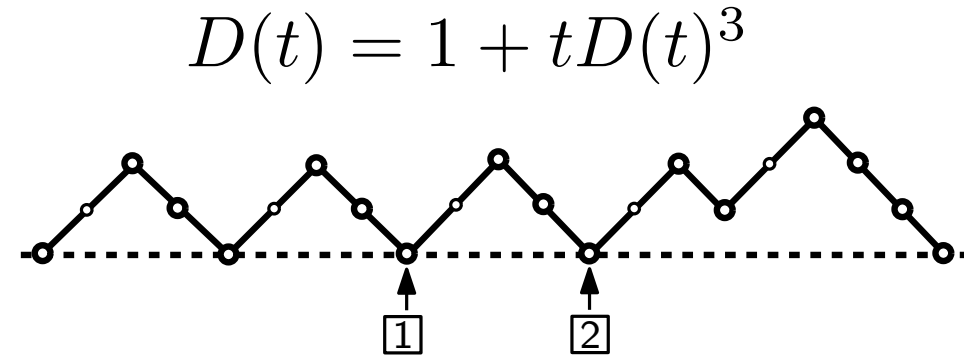
in \mathcal{T}_{nm}

The functional equation for $m \geq 1$

- Reduction of one path:

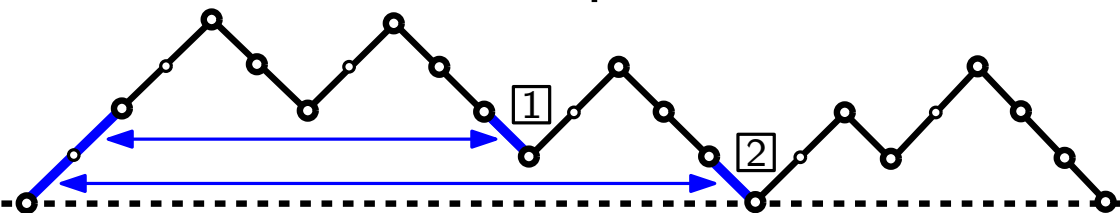


\Rightarrow

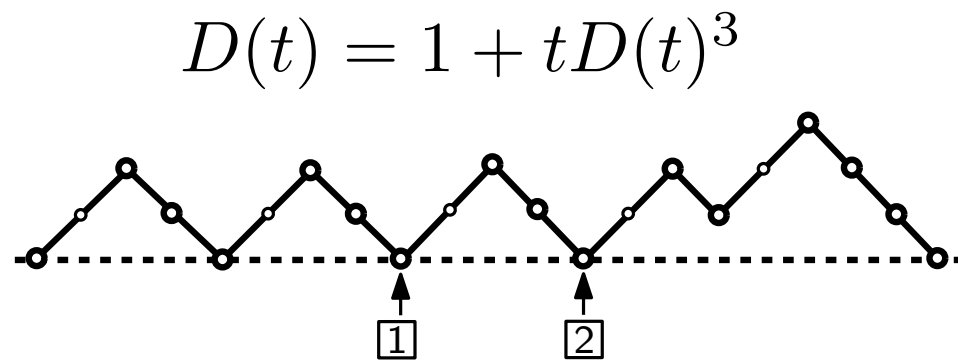


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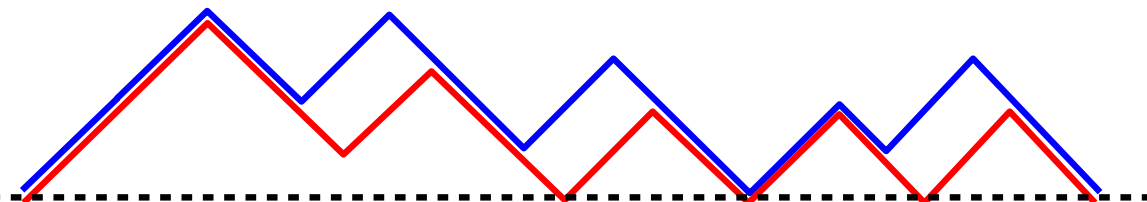
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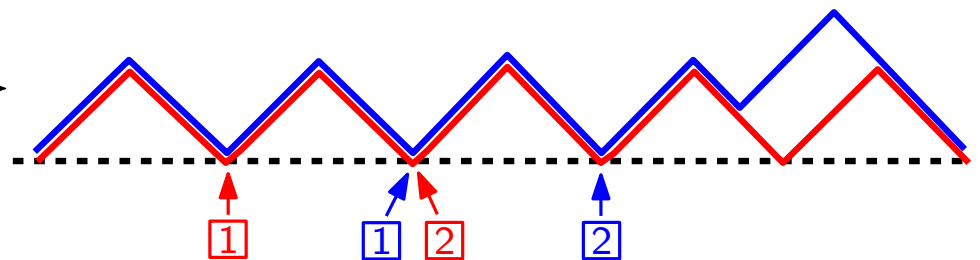
\Rightarrow



- Reduction of an interval:

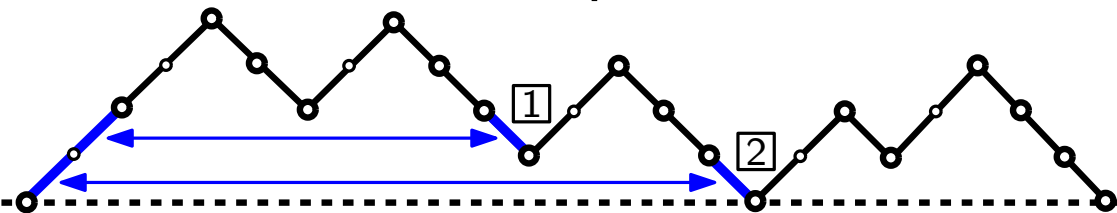


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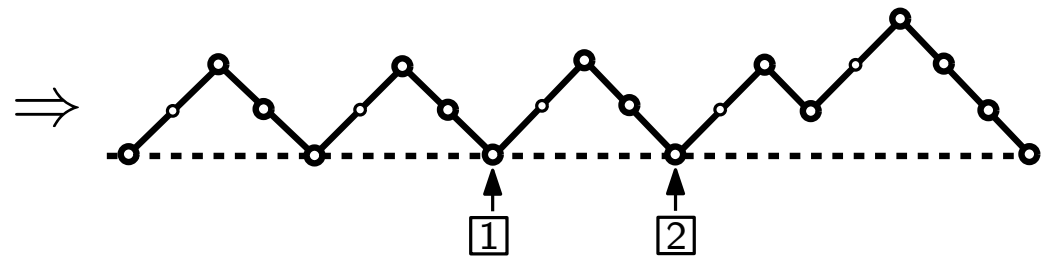


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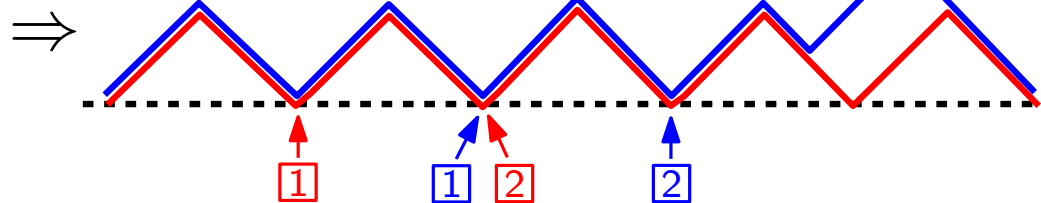
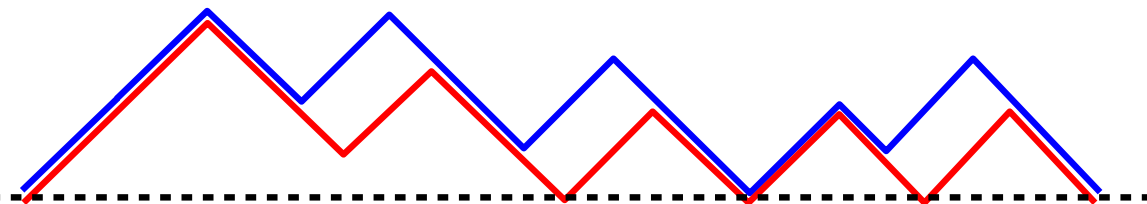
- Reduction of one path:



$$D(t) = 1 + tD(t)^3$$



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- Functional equation:

Let $a_{n,i} = \#(\text{intervals in } \mathcal{T}_n^{(m)} \text{ with } i \text{ bottom contacts})$

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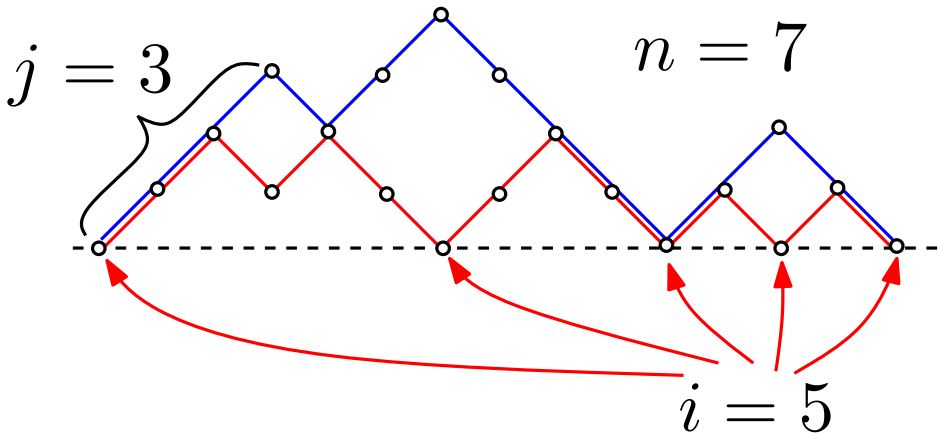
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Lagrange inv. formula \Rightarrow

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Other results



Can express trivariate generating function:

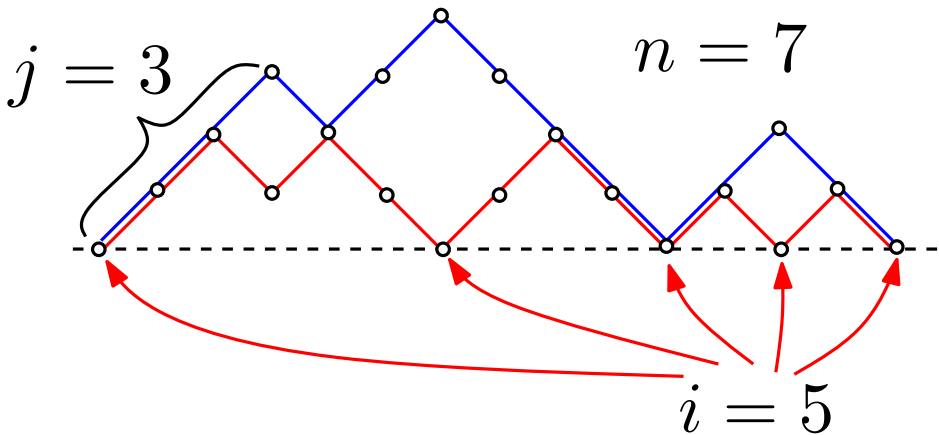
$$F(t, x, y) = \sum_{n,i,j} a_{n,i,j} t^n x^i y^j$$

Surprising symmetry in x and y

change var.: $t = z(1 - z)^{m^2+2m}$, $x = \frac{1 + u}{(1 + zu)^{m+1}}$, $y = \frac{1 + v}{(1 + zv)^{m+1}}$.

$$yF = \frac{(1 + u)(1 + zu)(1 + v)(1 + zv)}{(u - v)(1 - zuv)(1 - z)^{m+2}} \cdot \left(\frac{(1 + u)}{(1 + zu)^{m+1}} - \frac{(1 + v)}{(1 + zv)^{m+1}} \right)$$

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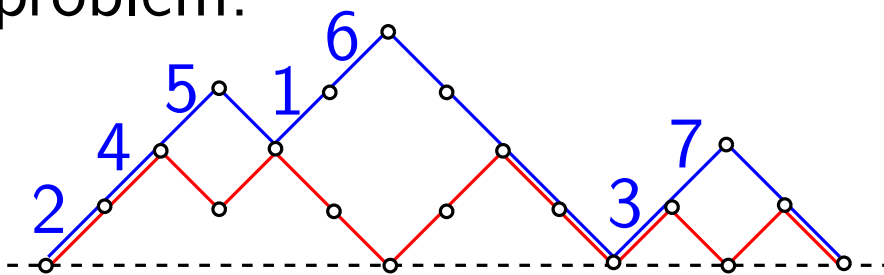
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Labelled problem:



Conjecture [Bergeron'10]:

$$a_n^{(m)} = (m + 1)^n (mn + 1)^{n-2}$$

now proved in [Bousquet-Mélou, Chapuy, Préville-Ratelle'11]

(also **guessing/checking**, but on **non-algebraic expressions**)