Mean asymptotic behaviour of radix-rational sequences and
dilation equations

Philippe Dumas,
Algorithms, INRIA Paris-Rocquencourt

September 5th, 2011
What is a radix-rational sequence (aka a $k$-regular sequence)?

- **A very simple example**

  $$ u_n = \begin{cases} 
  1 & \text{if } n \text{ is a power of 2} \\
  0 & \text{otherwise} 
  \end{cases} $$

  - $u_{2n} = u_n$
  - $u_{2n+1} = u_{2n+1}$
  - $u_{4n+1} = u_{4n+3} = 0$

$$ A_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad A_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} $$

$$ L = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad C = \begin{pmatrix} 1 \\ 0 \end{pmatrix} $$
Formal definition A (complex) sequence is rational with respect to radix $B$ if there is a finite dimensional vector space which contains the sequence and is left stable by the $B$-section operators. (Allouche and Shallit, 1992)

Linear representation A sequence is $B$-rational if and only if it admits a linear representation $A_0, A_1, \ldots, A_{B-1}, L, C.$
• **Domains**
  
  ▶ binary coding of integers (sum of digits, Thue-Morse sequence, Rudin-Shapiro sequence...)
  
  ▶ theory of numbers (Pascal triangle reduced modulo a power of 2, sum of three squares)
  
  ▶ divide-and-conquer algorithms (binary powering, Euclidean matching...)

• **A more natural example** Cost of mergesort in the worst case

\[
 u_n = u_{\lceil n/2 \rceil} + u_{\lfloor n/2 \rfloor} + n - 1, \quad u_0 = 0, \quad u_1 = 0
\]

\[
 B = 2 \quad A_0 = \begin{pmatrix}
 2 & 1 & 0 & 0 & 0 \\
 0 & 1 & 0 & 0 & 0 \\
 0 & 1 & 2 & 1 & 1 \\
 1 & 0 & 0 & 1 & 0 \\
 0 & 0 & 0 & 0 & 1
\end{pmatrix} \quad A_1 = \begin{pmatrix}
 1 & 0 & 0 & 0 & 0 \\
 1 & 2 & 0 & 0 & 0 \\
 1 & 0 & 0 & 1 & -1 \\
 0 & 0 & 0 & 0 & 0 \\
 0 & 1 & 2 & 1 & 3
\end{pmatrix}
\]

\[
 L = \begin{pmatrix}
 0 & 0 & 0 & 0 & 1
\end{pmatrix} \quad C = \begin{pmatrix}
 1 & 0 & 0 & 0 & 0
\end{pmatrix}^\text{tr}
\]

\[
 13 = (1101)_2 \quad u_{13} = LA_1A_1A_0A_1C
\]

• **Classical case** \( B = 1 \), \( u_n = LA_0^nC \)
Asymptotic behaviour of radix-rational sequences

**Theorem** Each $B$-rational sequence admits an asymptotic expansion of the form

$$u_n = \sum_{\alpha > \alpha^*, \ell \geq 0} n^{\alpha} \log_B^\ell(n) \sum_\omega \omega^{[\log_B n]} \Psi_{\alpha, \ell, \omega}(\log_B n) + O(n^{\alpha^*})$$

$\omega$ modulus 1 complex number, $\Psi$ 1-periodic function

**Example** Worst mergesort: $u_n = n \log_2 n + n \Psi(\log_2 n) + 1$, with $\Psi(t) = 1 - \{t\} - 2^{1-\{t\}}$ (Flajolet and Golin, 1994)

**Average or not?** Study of $\sum_{0 \leq n \leq N} u_n$

**Tools**
- rational formal power series
- dilation equation
- joint spectral radius
- Jordan reduction
- numeration system
Radix-rational sequence and rational formal power series

- Every radix-rational sequence hides a rational formal power series.
  alphabet $\mathcal{B} = \{0, 1, \ldots, B - 1\}$
  formal power series $S = \sum_{w \in \mathcal{B}^*} (S, w)w$
  here $(S, w) = LA_{w_1}A_{w_2} \cdots A_{w_K}C = LA_wC$ if $w = w_1w_2 \cdots w_K$
- Every rational formal power series defines a radix-rational sequence.
  $n = (w)_B$, $u_n = (S, w)$
- The rational formal power series is the essential object.
Running sum

\[ S_K(x) = \sum_{|w|=K} A_w C \] 
\[ (0.w)_B \leq x \]

\[ S_K(x) = \sum_{r_1<x_1} A_{r_1} Q^{K-1} C + \sum_{r_2<x_2} A_{x_1} A_{r_2} Q^{K-2} C + \sum_{r_3<x_3} A_{x_1} A_{x_2} A_{r_3} Q^{K-3} C \]
\[ + \cdots + \sum_{r_K \leq x_K} A_{x_1} A_{x_2} \cdots A_{r_K} C \]

\[ Q = A_0 + A_1 + \cdots + A_{B-1} \]

\[ K = (0.x_1)_3 \]
\[ (0.x_1 x_2)_3 \]
\[ (0.x_1 x_2 x_3)_3 \]
\[ B = 3 \]
\[ x = (0.121 \ldots)_3 \]
Lemma

With \( Q = A_0 + A_1 + \cdots + A_{B-1} \), the sequence of running sums \((S_K)\) satisfies the recursion

\[
S_{K+1}(x) = \sum_{r_1 < x_1} A_{r_1} Q^K C + A_{x_1} S_K(Bx - x_1),
\]

where \( x_1 \) is the first digit in the radix-\( B \) expansion of \( x \) in \([0, 1)\), with \( S_0(x) = C \).
Basic case

Hypothesis: \( Q^K C \xrightarrow{K \to +\infty} R(K) \left( V + O \left( \frac{1}{K} \right) \right) \)
for some nonzero vector \( V \) with \( R(K + 1)/R(K) = \rho \omega (1 + O(1/K)) \) and \( \rho > 0, |\omega| = 1. \)

\[
F_K(x) = \frac{1}{R(K)} S_K(x), \quad F_{K+1} = \mathcal{L}_K F_K
\]

\[
\mathcal{L}_K \Phi(x) = \frac{1}{R(K + 1)} \sum_{r_1 < x_1} A_{r_1} Q^K C + \frac{R(K)}{R(K + 1)} A_{x_1} \Phi(Bx - x_1)
\]

Basic dilation equation:

\( \Phi(0) = 0, \Phi(1) = V, \)
\( \Phi(x) = \frac{1}{\rho \omega} \sum_{r_1 < r} A_{r_1} V + \frac{1}{\rho \omega} A_r \Phi(Bx - r). \)
Wavelets

Scaling function \( \varphi \) (or father wavelet), data \( (c_k) \) with hypotheses

Daubechies, 1988: There exists a unique function \( \varphi \in L^2(\mathbb{R}) \) such that

1. \( \varphi(x) = \sum_{k=0}^{K-1} c_k \varphi(2x-k) \)
2. \( \int_{\mathbb{R}} \varphi(x) \, dx = 1 \)
3. \( \text{supp} \varphi \subset [0, K - 1] \)

Mother wavelet \( \psi(x) = \sum_k (-1)^k c_{g-1-k} \varphi(2x-k) \)

Wavelets \( \varphi_k(x) = \varphi(x-k) \), \( \psi_{j,k}(x) = 2^{-j/2} \psi(2^{-j}x-k) \)

Expansion \( f = \sum_k \langle f, \varphi_k \rangle \varphi_k + \sum_j \sum_k \langle f, \psi_{j,k} \rangle \psi_{j,k} \) for \( f \in L^2(\mathbb{R}) \)

Example: iteration from the box function (contracting operator)
hat  
\[ c_0 = \frac{1}{2}, \quad c_1 = 1, \quad c_2 = \frac{1}{2} \]

cubic B-spline
\[ c_0 = \frac{1}{8}, \quad c_1 = \frac{4}{8}, \quad c_2 = \frac{6}{8}, \quad c_3 = \frac{4}{8}, \quad c_4 = \frac{1}{8} \]

Daubechies  
\[ c_0 = \frac{1}{4} + \frac{\alpha}{4}, \quad c_1 = \frac{3}{4} + \frac{\alpha}{4}, \quad c_2 = \frac{3}{4} - \frac{\alpha}{4}, \quad c_3 = \frac{1}{4} - \frac{\alpha}{4}, \quad \alpha = \sqrt{3} \]
Refinement schemes (Deslauriers and Dubuc, 1986)

- Interpolation scheme (gliding Lagrange interpolation)
  data: \((v_k)_{k \in \mathbb{Z}}\) and \(L > 0\),
  output: \(f\) function such that \(f(k) = v_k\)
  if \(f\) defined on \(\frac{1}{2^j} \mathbb{Z}\) and \(x_{j,k} = \frac{k}{2^j} + \frac{1}{2^{j+1}}\) then \(f(x_{j,k}) = \pi_{j,k}(x_{j,k})\) where
    \(\pi_{j,k}\) is the Lagrange interpolation polynomial at \(p/2^j\) with \(k - L \leq p \leq k + L + 1\)
- Correction If \((v_k)\) bounded, \(f\) extends to \(\mathbb{R}\) as a continuous function
- Scaling function \(\varphi\)
  \(v_0 = 1, v_k = 0\) for \(k \neq 0\)
  \(f(x) = \sum_{k \in \mathbb{Z}} v_k \varphi(x - k)\)
  \(\varphi(x) = \sum_k c_k \varphi(2x - k)\)
- Example with \(L = 2\) (cascade algorithm)
Basic result

Theorem

Let $L$, $(A_r)_{0 < r < B}$, $C$ be a linear representation of dimension $d$ for the radix $B$. It is assumed that

1. $Q^K C \xrightarrow{K \to +\infty} R(K) \left( V + O \left( \frac{1}{K} \right) \right)$ for some nonzero vector $V$ with $R(K + 1)/R(K) = \rho \omega (1 + O(1/K))$ and $\rho > 0$, $|\omega| = 1$
2. there exists and induced norm $\| \|$ and a constant $\lambda$, with $0 < \lambda < \rho$ such that all matrices $A_r$, $0 \leq r < B$, satisfy $\|A_r\| \leq \lambda$.

Then

1. the basic dilation equation has a unique solution $F$, which is continuous from $[0, 1]$ into $\mathbb{C}^d$,
2. the sequence $(F_K)$ converges uniformly towards $F$, with speed essentially $O((\lambda/\rho)^K)$.

Concretely $S_K(x) \xrightarrow{K \to +\infty} R(K)F(x) + O(\lambda^K)$
Contribution of dilation equations

- Contracting operator
- Cascade algorithm
- Regularity $F$ is Hölder with exponent $\log_B (\rho/\lambda)$
- Form of the dilation equation ($B = 2$)
  - Piecewise equation, non homogeneous
    \[
    F(x) = \begin{cases} 
    T_0 F(2x) & \text{if } 0 \leq x \leq 1/2 \\
    T_0 V + T_1 F(2x - 1) & \text{if } 1/2 \leq x \leq 1 
    \end{cases}
    \]
  - Global equation, homogeneous
    \[
    F(x) = T_0 F(2x) + T_1 F(2x - 1) \quad \text{for } x \text{ real}
    \]
    with $F$ constant on the left of 0 and on the right of 1
- Example Billingsley’s distribution functions (Billingsley, 1995)
  \[
  X = \sum_{n \geq 0} \frac{X_n}{2^n}, \quad X_n = \text{Bernoulli}(p), \quad 0 < p < 1
  \]
\( L = (1), \ A_0 = (1 - p), \ A_1 = (p), \ C = (1) \)
\( Q = (1), \ \rho = 1, \ \omega = 1 \ V = (1), \ R(K) = 1, \)
\( F \) distribution function, \( F(x) = (1 - p)F(2x) + pF(2x - 1), \ F(0) = 0, \ F(1) = 1 \)

\[ \lambda = \max(p, 1 - p) \]
Hölder exponent \( \alpha = \log_2(1/\max(p, 1 - p)) \simeq 0.62 \) (here \( p = 13/20 \))
if \( 1/2 < p < 1 \)
essentially best Hölder exponent on the right \( \alpha_+ = \log_2(1/(1 - p)) \simeq 1.51 > 1 \)
essentially best Hölder exponent on the left \( \alpha_- = \alpha \)
Joint spectral radius

Controlling products $A_w$ Rota and Strang, 1960: $\lambda_T = \max_{|w| = T} \|A_w\|^{1/T}$

Joint spectral radius $\lambda_* = \lim_{T \to +\infty} \lambda_T$

Example with worst mergesort:
1 = ⋄
$\infty$ = ⋄
2 = ⋄
Changing the radix

- From $B$ to $B^T$ If a sequence is $B$-rational, it is $B^T$-rational for all $T \in \mathbb{N}_{>0}$.
  Linear representation $L, A_r, C$ with $0 \leq r < B$ for $B$
  becomes $L, A_w, C$ with $|w| = T$ for $B^T$.
- Eigenvalues $Q = \sum_{0 \leq r < B} A_r$ becomes $Q(T) = \sum_{|w| = T} A_w = Q^T$
  $\rho$ becomes $\rho^T$ (and $S_K$ becomes $S_{KT}$)
- Dichotomy (desired)

\[ \rho \]

\[ 0 \quad \lambda_* \quad \lambda_T \quad \lambda_1 \]

error term ? expansion
• **Jordan reduction**
  - **Idea**
    Jordan reduction of matrix \( Q \) and processing of each generalized eigenspace
    Vector valued functions become matrix valued functions, but same arguments
  - **Qualitative result**

**Theorem**

Let \( L, (A_r)_{0 \leq r < B}, C \) be a linear representation of a formal power series \( S \). The sequence of running sums

\[
S_K(x) = L S_K(x) = \sum_{|w| = K \atop (0,w) \leq x} L A_w C
\]

admits an asymptotic expansion with error term \( O(\lambda^K) \) for every \( \lambda > \lambda_* \), where \( \lambda_* \) is the joint spectral radius of the family \((A_r)_{0 \leq r < B}\). The used asymptotic scale is the family of sequences \( \rho^K(\ell) \), \( \rho > 0, \ell \in \mathbb{N}_{\geq 0} \). The coefficients are related to solutions of dilation equations. The error term is uniform with respect to \( x \in [0, 1] \).
Numeration system

We return to $\sum_{n=0}^{N} u_n$.

**Idea**

$N = B^K + t$, $K = \lfloor \log_B N \rfloor$, $t = \{\log_B N\}$

sum up to $N =$

$[\text{sum of } u_n \text{ up to } B^K - 1] \text{ plus } [\text{sum of } u_n \text{ from } B^k \text{ to } N]=$

$[(\text{sum of } (S, w) \text{ for } |w| \leq K) \text{ minus (sum for those words beginning with 0)})$ plus

$[(\text{running sum of } (S, w) \text{ up to } N \text{ for words of length } K + 1) \text{ minus (sum for those words beginning with 0)})]
Technique

\[ \sum_{n \leq BK+t} u(n) = \sum_{0 \leq k \leq K} \left( \sum_{|w|=k} LA_w C - \sum_{|w'|=k-1} LA_0 A_{w'} C \right) \]
\[ + \left( \sum_{|w|=K+1} LA_w C - \sum_{|w'|=K} LA_0 A_{w'} C \right) \]

that is

\[ \sum_{n \leq BK+t} u(n) = L(I_d - A_0) \sum_{0 \leq k \leq K} Q^k C + \sum_{|w|=K+1} LA_w C \]
\[ \left( \sum_{|w|=K+1} LA_w C - \sum_{|w'|=K} LA_0 A_{w'} C \right) \]

or

\[ \sum_{n \leq BK+t} u(n) = L(I_d - A_0) \sum_{0 \leq k \leq K} Q^k C + LS_{K+1}(B^{t-1}) \]

and we are at home.
Qualitative result

Theorem

Let $L, (A_r)_{0 \leq r < B}, C$ be a linear representation of a radix rational sequence $(u_n)$. The running sum $\sum_{n=0}^{N} u_n$ admits an asymptotic expansion with error term $O(N^\log_B \lambda)$ for every $\lambda > \lambda_*$, where $\lambda_*$ is the joint spectral radius of the family $(A_r)_{0 \leq r < B}$. The used asymptotic scale is the family of sequences $N^\alpha \left(\left\lfloor \log_B N \right\rfloor \ell\right)$, $\alpha \in \mathbb{R}$, $\ell \in \mathbb{N}_{\geq 0}$. The coefficients write $\omega \log_B N \Phi(\log_B N)$ where $\omega$ is modulus 1 complex numer and $\Phi(t)$ is 1-periodic and related to some solution of a dilation equation by the change of variable $x = B^{\{t\}-1}$.

$$\rho^K \omega^K \left(\frac{K}{\ell}\right) F(x) \rightarrow N^\log_B \rho \left(\left\lfloor \log_B N \right\rfloor \ell\right) \times \Phi(\log_B N)$$

$$\Phi(t) = \omega^{\lfloor t \rfloor} \rho^{1-\{t\}} F(B^{\{t\}-1})$$
Example Discrepancy of the van der Corput sequence (Béjian and Faure, 1977)

Van der Corput sequence: \( n = (n_{\ell-1} \ldots n_1 n_0)_{2} \) \( u_n = (0.n_0 n_1 \ldots n_{\ell-1})_{2} \)

Discrepancy:

\[
D(n) = \sup_{0 \leq \alpha < \beta \leq 1} \left| \frac{\nu(n, \alpha, \beta)}{n} - (\beta - \alpha) \right|
\]

Béjian and Faure sequence: \( E(n) = nD(n) \)

\[
E(1) = 1, \quad E(2n) = E(n), \quad E(2n + 1) = \frac{1}{2}(E(n) + E(n + 1) + 1)
\]

basis \((E(n), E(n + 1), 1)\), linear representation

\[
L = \begin{pmatrix} 0 & 1 & 1 \end{pmatrix}, \quad A_0 = \begin{pmatrix} 1 & 1/2 & 0 \\ 0 & 1/2 & 0 \\ 0 & 1/2 & 1 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 1/2 & 0 & 0 \\ 1/2 & 1 & 0 \\ 1/2 & 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 \end{pmatrix}.
\]

\[
\lambda_* = 1, \quad Q = \begin{pmatrix} 3/2 & 1/2 & 0 \\ 1/2 & 3/2 & 0 \\ 1/2 & 1/2 & 2 \end{pmatrix}
\]
Jordan reduction, with basis \((V_1, V_2^0, V_2^1)\)

\[V_1 = \begin{pmatrix} 1/2 & -1/2 & 0 \end{pmatrix}^\text{tr}, \quad V_2^0 = \begin{pmatrix} 0 & 0 & 1/2 \end{pmatrix}^\text{tr}, \quad V_2^1 = \begin{pmatrix} 1/2 & 1/2 & 0 \end{pmatrix}^\text{tr},\]

\[J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}, \quad C = V_1 + V_2^1\]

\[S_K(x) = \frac{1}{2} 2^K K F^0(x) + 2^K F^1(x) + O(K)\]

\[F^0(x) = \frac{1}{2} A_0 F^0(2x), \quad \text{for } 0 \leq x < 1/2,\]

\[F^0(x) = \frac{1}{2} A_0 V_2^0 + \frac{1}{2} F^0(2x - 1), \quad \text{for } 1/2 \leq x < 1;\]

\[F^1(x) = -\frac{1}{2} F^0(x) + \frac{1}{2} A_0 F^0(2x), \quad \text{for } 0 \leq x < 1/2,\]

\[F^1(x) = -\frac{1}{2} F^0(x) + \frac{1}{2} A_0 V_2^1 + \frac{1}{2} F^1(2x - 1), \quad \text{for } 1/2 \leq x < 1\]

\[F^0(0) = 0, \quad F^0(1) = V_2^0, \quad F^1(0) = 0, \quad F^1(1) = V_2^1. \quad F^0(x) = x V_2^0, \quad F^1 \text{ is not explicit.}\]

\[\frac{1}{N} \sum_{n=1}^{N} E(n) = \lim_{N \to +\infty} \frac{1}{4} \log_2 N + \frac{1}{4} \left(1 - \{t\} + 2^{3-t} \left(F_2^1(2^\{t\} - 1) + F_3^1(2^\{t\} - 1)\right)\right) + O\left(\frac{\log N}{N}\right)\]
comparison between the (red) empirical and (blue) theoretical periodic functions
Example Newman-Coquet sequence (Newman, 1969; Coquet, 1983)

\[ u(n) = (-1)^{s_2(3n)} \] 4-rational sequence (changing the radix! \( \pm \sqrt{3}, 0 \rightarrow 3, 0 \))

\[ \sum_{n \leq N} (-1)^{s_2(3n)} = N^{\log_4 3} 3^{1-\{t\}} F(4^{\{t\}}-1) + O(1), \]

\[ F = F_1 + F_2 + F_3 \]

\[
\begin{align*}
F_1(x) &= \frac{1}{3} F_1(4x) + \frac{1}{3} F_2(4x) + \frac{1}{3} F_3(4x) + \frac{1}{3} F_1(4x - 1), \\
F_2(x) &= \frac{1}{3} F_2(4x - 1) - \frac{1}{3} F_3(4x - 1) + \frac{1}{3} F_1(4x - 2) + \frac{1}{3} F_2(4x - 2), \\
F_3(x) &= \frac{1}{3} F_3(4x - 2) + \frac{1}{3} F_1(4x - 3) - \frac{1}{3} F_2(4x - 3) + \frac{1}{3} F_3(4x - 3),
\end{align*}
\]

\[ F_1(0) = F_2(0) = F_3(0) = 0, \ F_1(1) = 2/3, \ F_2(1) = F_3(1) = 1/3. \]
Example Rudin-Shapiro sequence (Shapiro, 1951; Rudin, 1959; Brillhart and Carlitz, 1970)
\[ u(n) = (-1)^{e_2;11(n)} \]
\[
\sum_{n \leq N} u_n = \sqrt{N} \Phi(\log_4 N) + O(1)
\]
Periodicity versus pseudo-periodicity

\[
\Phi(t) = \omega^{\lfloor t \rfloor} \rho^{1 - \{t\}} F(B^{t-1})
\]

Example Rosettes

\[
A_0 = \begin{pmatrix} \cos \vartheta & 0 \\ 0 & \cos \vartheta \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & -\sin \vartheta \\ \sin \vartheta & 0 \end{pmatrix},
\]
Context

- **Exact expansion**
  - Delange, 1975
  - sum of digits
  - Allouche and Shallit, 2003
    - extension

- **Asymptotic expansion**
    - automata and substitutions
    - divide-and-conquer recurrences, Dirichlet series

\[
U(s)(B^s I_N - Q) = B^s \sum_{r=1}^{B-1} \frac{U_r}{r^s} + \sum_{r=1}^{B-1} \sum_{k=1}^{+\infty} (-1)^k \binom{s+k-1}{k} \left(\frac{r}{B}\right)^k U(s+k)A_r
\]