Computing Closed-Form Solutions of Integrable Connections

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Problem in probability theory: find all probability distributions \( \mu \) on real symmetric matrices of order \( n \) such that if \( X \) and \( Y \) are independent with the same distribution \( \mu \), then \( X + Y = S \) and \( S^{-1} X^2 S^{-1} = Z \) are independent.

Under some restrictions, the problem can be reduced to (Bryc-Letac’12):

Find \( y(x_1, \ldots, x_n) \) such that

\[
\forall j \in \{1, \ldots, n\}, \quad \frac{\beta}{2} (j - n) \frac{\partial y}{\partial x_{j+1}} + \text{Tr}(P_j \text{Hess}(y)) = 0,
\]

where \( \beta \) is the Peirce constant (\( \beta \in \{1, 2, 4, 8, -2\} \)), \( \text{Hess} \) the Hessian matrix and the \( P_j \)'s are given symmetric matrices.
Introducing example - G. Letac, W. Bryc (2)

◊ Case $n = 2$:

$$-\frac{\beta}{2} \frac{\partial y}{\partial x_2} + \frac{\partial^2 y}{\partial x_1^2} - x_2 \frac{\partial^2 y}{\partial x_2^2} = 0$$

$$2 \frac{\partial^2 y}{\partial x_1 \partial x_2} + x_1 \frac{\partial^2 y}{\partial x_2^2} = 0$$

◊ Case $n = 3$:

$$-\beta \frac{\partial y}{\partial x_2} + \frac{\partial^2 y}{\partial x_1^2} - x_2 \frac{\partial^2 y}{\partial x_2^2} - 2x_3 \frac{\partial^2 y}{\partial x_2 \partial x_3} = 0$$

$$-\frac{\beta}{2} \frac{\partial y}{\partial x_3} + 2 \frac{\partial^2 y}{\partial x_1 \partial x_2} + x_1 \frac{\partial^2 y}{\partial x_2^2} - x_3 \frac{\partial^2 y}{\partial x_2 \partial x_3} = 0$$

$$\frac{\partial^2 y}{\partial x_2^2} + 2 \frac{\partial^2 y}{\partial x_1 \partial x_3} + 2x_1 \frac{\partial^2 y}{\partial x_2 \partial x_3} + x_2 \frac{\partial^2 y}{\partial x_2 \partial x_3} = 0$$

◊ Problem: compute “solutions” of such linear systems of PDEs
-contributions

- **Remark:** the latter systems are *D*-finite (*Chyzak-Salvy’98*)

- In this talk, we provide **algorithms** for computing:
  - rational solutions
  - hyperexponential solutions

  of such *D*-finite linear systems of PDEs.

- Maple **implementation** available at
  
  [http://www.ensil.unilim.fr/~cluzeau/PDS.html](http://www.ensil.unilim.fr/~cluzeau/PDS.html)

- **Complexity analysis**
Outline of the talk

1. $D$-finite linear systems of PDEs
2. Rational solutions
3. Hyperexponential solutions
4. Implementation
5. Conclusions
$D$-finite linear systems of PDEs
Notations and a definition

- $C$ computable field of char. zero, $\overline{C}$ its algebraic closure
- $k = C(x_1, \ldots, x_m)$ and $K = \overline{C}(x_1, \ldots, x_m)$, $\partial_i = \partial/\partial x_i$

**Definition**

$\mathcal{U}$ universal differential extension of $k$ containing all solutions of linear systems of PDEs over $k$ (existence, e.g., Kolchin’73).

A linear system of PDEs is said to be **$D$-finite** if its solution space in $\mathcal{U}$ is of finite dimension over $C$.

- **Algorithms** to test if a given system is $D$-finite exist
  - (Chyzak-Salvy’98 - Gröbner or Janet basis computations)
- **Implementation**: OreModules (Chyzak-Quadrat-Robertz)
Integrable connections

Definition

Integrable connection over $k$ of size $n$ in $m$ variables:

\[
\begin{align*}
\Delta_1 Y &= 0 \quad \text{with} \quad \Delta_1 := \partial_1 I_n - A_1 \\
&\vdots \\
\Delta_m Y &= 0 \quad \text{with} \quad \Delta_m := \partial_m I_n - A_m
\end{align*}
\]

where $A_i's \in \mathbb{M}_n(k)$ and the integrability conditions are satisfied:

\[
\partial_i(A_j) - A_i A_j = \partial_j(A_i) - A_j A_i, \quad \forall i, j \in \{1, \ldots, m\}
\]

- Every $D$-finite linear system of PDEs can be written as an integrable connection (Chyzak-Salvy’98), implementation in OreModules (Chyzak-Quadrat-Robertz)
Example: Bryc-Letac system for $n = 2$

\[-\frac{\beta}{2} \partial_2 y + \partial_1^2 y - x_2 \partial_2^2 y = 0\]
\[2 \partial_1 \partial_2 y + x_1 \partial_2^2 y = 0\]

◊ Integrable connection over $Q(\beta)$ of size 4 in 2 variables:

\[\partial_i Y - A_i Y = 0, \quad i = 1, 2, \quad \text{with}\]

\[A_1 = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -\frac{1}{2} x_1 \\
0 & \frac{1}{2} \beta & 0 & x_2 \\
0 & 0 & 0 & \frac{(-3-\beta)x_1}{x_1^2-4x_2}
\end{pmatrix}, \quad A_2 = \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & -\frac{1}{2} x_1 \\
0 & 0 & 0 & \frac{6+2 \beta}{x_1^2-4x_2}
\end{pmatrix}\]

◊ $Y = (y \quad \partial_2 y \quad \partial_1 y \quad \partial_2^2 y)^T$
Existing works

- Algorithmic studies of $D$-finite linear systems of PDEs:
  - Chyzak’00, Oaku-Takayama-Tsai’01: rational solutions of holonomic systems
  - Li-Schwarz-Tsarev’03: factorization, hyperexp. solutions
  - Barkatou-Cluzeau-Weil’05: factorization in char. $p$
  - Wu’05, Li-Singer-Wu-Zheng’06: Picard-Vessiot extensions, factorization, hyperexp. solutions over Laurent-Ore algebras

- Strategy of our work:
  - Consider integrable connections
  - Proceed recursively: benefit from algorithms for ordinary differential (OD) systems
Rational solutions
Rational solutions of OD systems (1)

\( C \) computable field of char. zero, \( \overline{C} \) its algebraic closure, 
\( k = C(x) \) and \( K = \overline{C}(x) \)

\[ Y' = AY, \quad A \in \mathbb{M}_n(k), \quad \text{denom}(A) = \prod_{i=1}^{s} q_i(x)^{r_i+1} \]

Algorithm for computing rational solutions (for ex. Barkatou'99):

- Compute a universal denominator \( Q = \prod_{i=1}^{s} q_i(x)^{m_i} \)
- Compute polynomial solutions of \( Z' = (A + (Q'/Q) I_n) Z \)
Complexity estimate

\[ Y' = A \cdot Y, \quad A = (a_{i,j})_{i,j} \in M_n(k), \quad \text{denom}(A) = Q_{i=1}^s q_i(x)^{r_i+1} \]

\[ X^s \]

\[ d := (r_i + 1) \deg(q_i) \]

\[ r_\infty := \max_{i,j} \max_i \left( 1 + \deg(\text{num}(a_{i,j})) - \deg(\text{den}(a_{i,j})) \right) + 0 \]

◊ **Arithmetic (operations in \( C \)) complexity estimate (BCEW'12):**

- **Universal denominator:** simple form at \( q_i \), integer roots of the *indicial polynomial*: \( \mathcal{O}(n^5 \max_i(r_i) d) \)

- **Polynomial solutions:** degree bound (simple form at \( \infty \)),
  coefficients: \( \mathcal{O}(n^5 r_\infty^2 + n^3 N^2) \)

\[ \leadsto \text{rational solutions of } Y' = A \cdot Y: \mathcal{O}(n^5 (\max_i(r_i) d + r_\infty^2) + n^3 N^2) \]

◊ **Main tool:** simple form (arithm. compl. in El Bacha’s PhD’11)
Rational solutions of integrable connections (1)

\[ k = C(x_1, \ldots, x_m), \quad K = \overline{C}(x_1, \ldots, x_m) \]

\[ \Delta_1 Y = 0 \quad \text{with} \quad \Delta_1 := \partial_1 I_n - A_1, \]
\[ \vdots \]
\[ \Delta_m Y = 0 \quad \text{with} \quad \Delta_m := \partial_m I_n - A_m, \]

\[ A_i \in \mathbb{M}_n(k) \]

Notation: \([A_1, \ldots, A_m]\)

Definition

Rational solution: vector \( Y \in K^n \) such that \( \Delta_i(Y) = 0, \ \forall \ i. \)

Recursive process:

- Compute \( \mathcal{V} := \{ Y \in K^n ; \Delta_1(Y) = 0 \} \)
- Reduce the size (\( m \) and \( n \)) of the problem
Rational solutions of integrable connections (2)

◊ $K_1 := \overline{C}(x_2, \ldots, x_m), \ K = K_1(x_1), \ \mathcal{V} := \{ Y \in K^n ; \Delta_1(Y) = 0 \}$

◊ $\mathcal{V}$ is a $K_1$-vector space stable under the action of each $\Delta_i$.

◊ A basis can be computed using an algorithm for OD systems and viewing $x_2, \ldots, x_m$ as transcendental constants.

Lemma

One can compute a non-singular matrix $P \in \mathbb{M}_n(K)$ such that, $\forall i$:

$$B_i := P^{-1} (A_i P - \partial_i(P)) = \begin{pmatrix} B_{i11}^{11} & B_{i12}^{12} \\ 0 & B_{i22}^{22} \end{pmatrix}, \quad B_{i11}^{11} \in \mathbb{M}_s(K).$$

Moreover, $B_{11}^{11} = 0$ and $\forall i = 2, \ldots, m$, $B_{i11}^{11} \in \mathbb{M}_s(K_1)$. 
Rational solutions of integrable connections (3)

- $\nu_1, \ldots, \nu_s$ $K_1$-basis of $\mathcal{V}$, $\mathcal{V} = (\nu_1 \ldots \nu_s) \in \mathbb{M}_{n \times s}(K)$

**Theorem (BCEW'12)**

$Y = V \Gamma \in K^n$ rat. sol. of $[A_1, \ldots, A_m]$ iff $\Gamma \in K_1^s$ rat. sol. of

$$\tilde{\Delta}_2 \Gamma = 0 \quad \text{with} \quad \tilde{\Delta}_2 := \partial_2 l_s - B_{211}$$

$$\vdots$$

$$\tilde{\Delta}_m \Gamma = 0 \quad \text{with} \quad \tilde{\Delta}_m := \partial_m l_s - B_{m11},$$

No more $x_1$!

$\Rightarrow$ Recursive algorithm (with efficient method for computing $B_{i11}$'s)

- Complexity: worst case estimate (op. in $k$) $\Rightarrow$ to be improved!

- Denominators: $q$ irreducible factor of the denom. of a rat. sol. such that $\partial_{i_0}(q) \neq 0 \Rightarrow q | \text{denom}(A_{i_0})$ (BCEW’12)
III

Hyperexponential solutions
Exponential solutions of ordinary differential systems (1)

- $C$ computable field of char. zero, $\overline{C}$ its algebraic closure, $k = C(x)$ and $K = \overline{C}(x)$

$$Y' = A Y, \quad A \in M_n(k), \quad \text{denom}(A) = \prod_{i=1}^{s} q_i(x)^{r_i+1}$$

**Definition**

**Exponential solution:** $\exp(\int f \, dx) z$, where $f \in K$ and $z \in K^n$.

- **Algorithm for computing exponential solutions** (*Pfluegel'01*):
  - Compute the non-ramified local exponential parts at each sing.
  - For each combination, compute polynomial solutions

- **Bottlenecks:** large number of comb. & computations in algebraic extensions of $C$ of large degree
\( Y' = A Y, \quad A = \frac{1}{x^{r+1}} (A_0 + A_1 x + A_2 x^2 + \cdots), \ r \in \mathbb{N}, \ A_i \in \mathbb{M}_n(\overline{C}) \)

**Definition**

Non-ramified local exponential part at \( x = 0 \): polynomial \( \tilde{f} \) in \( 1/x \)

\[
\tilde{f} = \frac{\alpha_{p+1}}{x^{p+1}} + \frac{\alpha_p}{x^p} + \cdots + \frac{\alpha_1}{x},
\]

where \( 0 \leq p \leq r \) and \( \alpha_i's \in \overline{C} \) such that there exists a formal local solution of the system of the form \( \exp(\int \tilde{f} \, dx) \tilde{z} \), where \( \tilde{z} \) is a vector of formal power series in \( x \).

◊ **Arithmetic cost** \( (BCEW'12): O(n^5 \, r^3 \, \min(n, r)) \) op. in an alg. ext. of \( C \) of degree \( \leq n \) (super-reduction, Barkatou-Pfluegel’09)
Complexity estimate

\[ Y' = A Y, \quad A = (a_{i,j})_{i,j} \in \mathbb{M}_n(k), \quad \text{denom}(A) = \prod_{i=1}^{s} q_i(x)^{r_i+1} \]

\[ d := (r_i + 1) \deg(q_i) \quad (i=1) \]

\[ r_{\infty} := \max_{i,j} \max (1 + \deg(\text{num}(a_{i,j})) - \deg(\text{den}(a_{i,j}))) , 0 \]

\[ \Rightarrow \text{Exponential solutions of } Y' = A Y \ (BCEW'12): \]

- \[ O(n^5 (\max_i (r_i)^2 d \sum_i \min(n, r_i) + r_{\infty}^3 \min(n, r_{\infty})) ) \] op. in an alg. ext. of \( C \) of degree \( \leq n \)

- \[ O(n^{\delta+3} N^2) \] op. in an alg. ext. of \( C \) of degree \( \leq n^\delta \delta! \)

(\( \delta \): number of singularities, \( N \): degree bound for all the computed polynomial solutions)
Hyperexponential solutions of integrable connections (1)

\[ \Delta_1 Y = 0 \quad \text{with} \quad \Delta_1 := \partial_1 \ln - A_1, \]
\[ \vdots \]
\[ \Delta_m Y = 0 \quad \text{with} \quad \Delta_m := \partial_m \ln - A_m, \]

\[ K = \overline{\mathbb{C}}(x_1, \ldots, x_m) \]

**Definition**

- *L differential extension of K* having the same field of constants.
- \( u \neq 0 \in L \) hyperexponential over \( K \): \( \forall i, f_i := \partial_i(u)/u \in K \).
- Hyperexponential solution: solution \( u z \) with \( u \) hyperexponential over \( K \) and \( z \in K^n \).

\[ u \text{ hyperexponential over } K \Rightarrow \partial_j(f_i) = \partial_i(f_j), \forall i, j \]

\[ u z \text{ hyperexp. sol. of } [A_1, \ldots, A_m] \]
\[ \Rightarrow z \text{ rat. sol. of } [A_1 - f_1 l_n, \ldots, A_m - f_m l_n] \]
Hyperexponential solutions of integrable connections (2)

- **Recursive algorithm** as for rational solutions
  - Exp. sol. of $Y' = A_1 Y$ computed with algorithm for OD systems. Let $u z$ be such a solution
  - $f_i := \partial_i(u)/u \in K$ and $\Delta_{i,u} := \partial_i - (A_i - f_i I_n)$
  - $w_1, \ldots, w_s$ basis of $\mathcal{W}_u = \{w \in K^n; \Delta_{1,u}(w) = 0\}$, complete it into a basis of $K^n \rightsquigarrow$ matrix $P = (W_u \ W)$

**Theorem (BCEW'12)**

$$Y = u W_u \Gamma_u \text{ hyperexp. sol. of } [A_1, \ldots, A_m] \text{ iff } \Gamma_u \text{ hyperexp. sol. of } [B_{11}^{21}, \ldots, B_{11}^{m1}] \text{ where } B_i = P^{-1} \left( (A_i - f_i I_n) P - \partial_i(P) \right) \text{ and } B_i^{11} \in \mathbb{M}_s(K_1) \text{ denotes the first } s \times s \text{ submatrix of } B_i.$$}

- **Complexity**: worst case estimate $\rightsquigarrow$ to be improved
- **Discard local exp. parts** involving non-rat. functions of $x_j$'s, $j \neq 1$
IV

Implementation
Maple package **IntegrableConnections**

○ Algorithms are implemented in a Maple package called **IntegrableConnections**

- Available with some examples at http://www.ensil.unilim.fr/~cluzeau/PDS.html
- Main procedures: *RationalSolutions* (\& *Eigenring*), *HyperexponentialSolutions*
- Some adaptations of **ISOLDE** code (*Barkatou-Pfluegel*)

Demo.
Conclusions
Contributions and Perspectives

- **Summary of the contributions:**
  - **Complexity estimates** for computing rational and exponential solutions of ODE systems (in the literature of ODE systems, Grigoriev’90).
  - Algorithms for computing rational and hyperexponential solutions of integrable connections.
  - Implementation available (IntegrableConnections).

- **Perspectives:**
  - Precise complexity analysis of algorithms for integrable connections.
  - Algorithms for other types of solutions and factorization.