Computing Closed-Form Solutions of Integrable Connections

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[illegible]

Algorithms Project's Seminar

[illegible]

Introducing example - G. Letac, W. Bryc (1)

◇ [illegible] μ
[illegible] n [illegible] X [illegible] Y [illegible]
[illegible] μ, [illegible] $X + Y = S$ [illegible]
$S^{-1}X^2S^{-1} = Z$ [illegible]

◇ [illegible]
(*Bryc-Letac'12*):

[illegible] $y(x_1, \dots, x_n)$ [illegible]

$$\forall j \in \{1, \dots, n\}, \quad \frac{\beta}{2}(j-n)\frac{\partial y}{\partial x_{j+1}} + \mathrm{Tr}(P_j\,\mathrm{Hess}(y)) = 0 \quad ,$$

[illegible] β is *the Peirce constant* ($\beta \in \{1, 2, 4, 8, -2\}$) Hess [illegible]
[illegible] P_j's [illegible]

Introducing example - G. Letac, W. Bryc (2)

- Case $n = 2$

$$
\begin{cases}
-\frac{\beta}{2}\,\frac{\partial y}{\partial x_2} + \frac{\partial^2 y}{\partial x_1^2} - x_2\,\frac{\partial^2 y}{\partial x_2^2} & = 0 \\
2\,\frac{\partial^2 y}{\partial x_1\,\partial x_2} + x_1\,\frac{\partial^2 y}{\partial x_2^2} & = 0
\end{cases}
$$

- Case $n = 3$

$$
\begin{cases}
-\beta\,\frac{\partial y}{\partial x_2} + \frac{\partial^2 y}{\partial x_1^2} - x_2\,\frac{\partial^2 y}{\partial x_2^2} - 2\,x_3\,\frac{\partial^2 y}{\partial x_2\,\partial x_3} & = 0 \\
-\frac{\beta}{2}\,\frac{\partial y}{\partial x_3} + 2\,\frac{\partial^2 y}{\partial x_1\,\partial x_2} + x_1\,\frac{\partial^2 y}{\partial x_2^2} - x_3\,\frac{\partial^2 y}{\partial x_3^2} & = 0 \\
\frac{\partial^2 y}{\partial x_2^2} + 2\,\frac{\partial^2 y}{\partial x_1\,\partial x_3} + 2\,x_1\,\frac{\partial^2 y}{\partial x_2\,\partial x_3} + x_2\,\frac{\partial^2 y}{\partial x_3^2} & = 0
\end{cases}
$$

- [illegible]

Contributions

◇ **Remark**: [illegible] *D-finite* (*Chyzak-Salvy'98*)

◇ [illegible]

- [illegible]
- [illegible]

[illegible] *D*-fi[illegible]

◇ [illegible]

http://www.ensil.unilim.fr/~cluzeau/PDS.html

◇ [illegible]

Outline of the talk

1. D-fi[illegible]
2. [illegible]
3. [illegible]
4. [illegible]
5. [illegible]

I

D-finite linear systems of PDEs

Notations and a definition

- ◇ C [illegible] $\overline{C}$ [illegible]
- ◇ $k = C(x_1, \ldots, x_m)$ [illegible] $K = \overline{C}(x_1, \ldots, x_m)$ $\partial_i = \partial/\partial x_i$

D[illegible]

$\mathcal{U}$ [illegible] k [illegible]
[illegible] k [illegible] *Kolchin'73*)

A[illegible] D-fi[illegible] [illegible]
in $\mathcal{U}$ [illegible] C.

- ◇ [illegible] D-fi[illegible]
(*Chyzak-Salvy*[illegible]) [illegible]
[illegible] OreModules (*Chyzak-Quadrat-Robertz*)

Integrable connections

k n in m

$$\left\{\begin{array}{lll} \Delta_1\, Y = 0 & \text{with} & \Delta_1 := \partial_1\, I_n - A_1 \\ \quad\vdots & & \\ \Delta_m\, Y = 0 & \text{with} & \Delta_m := \partial_m\, I_n - A_m \end{array}\right.$$

$A_i' s \in \mathbb{M}_n(k)$

$$\partial_i(A_j) - A_i\, A_j = \partial_j(A_i) - A_j\, A_i, \quad \forall\, i, j \in \{1, \ldots, m\}$$

◇ *D-finite*

(*Chyzak-Salvy*)

OreModules (*Chyzak-Quadrat-Robertz*)

Example: Bryc-Letac system for $n = 2$

$$\begin{cases} -\frac{\beta}{2}\,\partial_2 y + \partial_1^2 y - x_2\,\partial_2^2 y & = 0 \\ 2\,\partial_1\,\partial_2 y + x_1\,\partial_2^2 y & = 0 \end{cases}$$

◇ [illegible] $\mathbb{Q}(\beta)$ [illegible]

$$\partial_i Y - A_i Y = 0\ , \quad i = 1\ , 2,$$

$$A_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\frac{1}{2}\,x_1 \\ 0 & \frac{1}{2}\,\beta & 0 & x_2 \\ 0 & 0 & 0 & \frac{(-3-\beta)x_1}{x_1{}^2-4\,x_2} \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -\frac{1}{2}\,x_1 \\ 0 & 0 & 0 & \frac{6+2\,\beta}{x_1{}^2-4\,x_2} \end{pmatrix}$$

◇ $Y = (\ y \quad \partial_2 y \quad \partial_1 y \quad \partial_2^2 y)^T$

Existing works

- [illegible] D-finite [illegible]
  - *Chyzak'00, Oaku-Takayama-Tsai'01*: [illegible]
  - *Li-Schwarz-Tsarev'03*: [illegible]
  - *Barkatou-Cluzeau-Weil'05*: [illegible] p
  - *Wu'05, Li-Singer-Wu-Zheng'06*: [illegible]
- [illegible]
  - [illegible]
  - [illegible]

II

Rational solutions

Rational solutions of OD systems (1)

◇ C [illegible] $\overline{C}$ [illegible]
$k = C(x)$ [illegible] $K = \overline{C}(x)$

$$Y' = A\,Y, \quad A \in \mathbb{M}_n(k), \quad \mathrm{denom}(A) = \prod_{i=1}^{s} q_i(x)^{r_i+1}$$

◇ [illegible] ([illegible] *Barkatou'99*)

- [illegible] $Q = \prod_{i=1}^{s} q_i(x)^{m_i}$
- [illegible] $Z' = (A - (Q'/Q)\, I_n)\, Z$

Complexity estimate

$$Y' = A\,Y, \quad A = (a_{i,j})_{i,j} \in \mathbb{M}_n(k), \quad \mathrm{denom}(A) = \prod_{i=1}^{s} q_i(x)^{r_i+1}$$

$$d := \sum_{i=1}^{s} (r_i + 1)\deg q_i)$$

$$r_\infty := \max\left(\max_{i,j} (\deg \mathrm{num}(a_{i,j}) - \deg \mathrm{den}(a_{i,j})), 0 \right)$$

◇ [illegible] C) [illegible] (*BCEW'12*)

- [illegible] q_i, [illegible] *indicial polynomial*: $\mathcal{O}(n^5 \max_i(r_i)\, d)$
- [illegible] ∞) [illegible] $\mathcal{O}(n^5\, r_\infty^2 + n^3\, N^2)$

⇝ [illegible] $Y' = A\,Y$: $\mathcal{O}(n^5 (\max_i(r_i)\, d + r_\infty^2) + n^3\, N^2)$

◇ [illegible] (*El Bacha's PhD'11*)

Rational solutions of integrable connections (1)

◇ $k = \mathbb{C}(x_1, \ldots, x_m)$ $K = \overline{\mathbb{C}}(x_1, \ldots, x_m)$

$$\begin{cases} \Delta_1\, Y = 0 & \text{with} \quad \Delta_1 \coloneqq \partial_1\, I_n - A_1, \\ \quad \vdots & \\ \Delta_m\, Y = 0 & \text{with} \quad \Delta_m \coloneqq \partial_m\, I_n - A_m, \end{cases} \qquad A_i \in \mathbb{M}_n(k)$$

◇ **Notation**: $[\, A_1, \ldots, A_m]$

[illegible]

[illegible] $Y \in K^n$ [illegible] $\Delta_i(Y) = 0$, $\forall\, i$.

◇ [illegible]

- [illegible] $\mathcal{V} \coloneqq \{ Y \in K^n \,;\, \Delta_1(Y) = 0 \}$
- [illegible](m [illegible] n) [illegible]

Rational solutions of integrable connections (2)

- $K_1 \coloneqq \overline{C}(x_2, \dots, x_m)$ $K = K_1(x_1)$ $\mathcal{V} \coloneqq \{Y \in K^n ; \Delta_1(Y) = 0\}$
- $\mathcal{V}$ is a K_1 [illegible] Δ [illegible] i
- [illegible] $x_2, \dots, x_m$ [illegible]

[illegible]

One can compute a non-singular matrix $P \in \mathbb{M}_n(K)$ *such that,* $\forall\, i$*:*

$$B_i \coloneqq P^{-1}(A_i P - \partial_i(P)) = \begin{pmatrix} B_i^{11} & B_i^{12} \\ 0 & B_i^{22} \end{pmatrix}, \quad B_i^{11} \in \mathbb{M}_s(K).$$

Moreover, $B_1^{11} = 0$ *and* $\forall\, i = 2, \dots, m$, $B_i^{11} \in \mathbb{M}_s(K_1)$.

Rational solutions of integrable connections (3)

◇ $v_1, \dots, v_s$ K_1 [illegible] $\mathcal{V}$, $V = (v_1 \; \dots \; v_s) \in \mathbb{M}_{n \times s}(K)$

[illegible] (BCEW'12)

$Y = V\,\Gamma \in K^n$ *rat. sol. of* $[A_1, \dots, A_m]$ *iff* $\Gamma \in K_1^s$ *rat. sol. of*

$$\begin{cases} \tilde{\Delta}_2\,\Gamma = 0 & \text{with} \quad \tilde{\Delta}_2 := \partial_2\, I_s - B_2^{11}, \\ \quad \vdots & \\ \tilde{\Delta}_m\,\Gamma = 0 & \text{with} \quad \tilde{\Delta}_m := \partial_m\, I_s - B_m^{11}, \end{cases}$$

No more x_1!

⇝ [illegible] B_i^{11}[illegible]

◇ [illegible] k) ⇝ [illegible]

◇ [illegible] q [illegible]

[illegible] $\partial_{i_0}(q) \neq 0 \;\Rightarrow\; q \,|\, \mathrm{denom}(A_{i_0})$ (BCEW'12)

III

Hyperexponential solutions

Exponential solutions of ordinary differential systems (1)

◇ C [illegible] $\overline{C}$ [illegible]
$k = C(x)$ [illegible] $K = \overline{C}(x)$

$$Y' = A\,Y, \quad A \in \mathbb{M}_n(k), \quad \mathrm{denom}(A) = \prod_{i=1}^{s} q_i(x)^{r_i+1}$$

[illegible]

[illegible] ([illegible] $\int f \,\mathrm{d}x) z$, [illegible] $f \in K$ [illegible] $z \in K^n$.

◇ [illegible] ([illegible] *Pfluegel'01*)

- [illegible]
- [illegible]

◇ [illegible]
[illegible] C [illegible]

Exponential parts and complexity estimate

$$Y' = A\,Y, \quad A = \frac{1}{x^{r+1}}\,(A_0 + A_1\,x + A_2\,x^2 + \cdots),\ r \in \mathbb{N},\ A_i \in \mathbb{M}_n(\overline{C})$$

[illegible]

[illegible] $x = 0$ [illegible] $\tilde{f}$ [illegible] $1/x$

$$\tilde{f} = \frac{\alpha_{p+1}}{x^{p+1}} + \frac{\alpha_p}{x^p} + \cdots + \frac{\alpha_1}{x},$$

[illegible] $0 \leq p \leq r$ [illegible] $\alpha_i's \in \overline{C}$ [illegible]

[illegible] $\int \tilde{f}\,\mathrm{d}x)\tilde{z}$, [illegible] $\tilde{z}$ [illegible] z [illegible]

[illegible] x.

◇ [illegible] (*BCEW'12*) $\mathcal{O}(n^5\,r^3$ [illegible] $n, r)$ [illegible]

[illegible] C [illegible] $\leq n$ [illegible] *Barkatou-Pfluegel'09*)

Complexity estimate

$$Y' = A\,Y, \quad A = (a_{i,j})_{i,j} \in \mathbb{M}_n(k), \quad \mathrm{denom}(A) = \textstyle\prod_{i=1}^{s} q_i(x)^{r_i+1}$$

$$d := \sum_{i=1}^{s} (r_i + 1) \deg q_i$$

$$r_\infty := \max\left(\max_{i,j} \left(\deg \mathrm{num}(a_{i,j}) - \deg \mathrm{den}(a_{i,j})\right), 0\right)$$

$\rightsquigarrow$ [illegible] $Y' = A\,Y$ (BCEW'12):

- $\mathcal{O}(n^5 (\max_i (r_i)^2\, d \sum_i (\ldots n, r_i) + r_\infty^3 (\ldots n, r_\infty))$ [illegible] C [illegible] $\leq n$
- $\mathcal{O}(n^{\delta+3} N^2)$ [illegible] C [illegible] $\leq n^\delta\,\delta!$

(δ: [illegible] N: [illegible])

Hyperexponential solutions of integrable connections (1)

$$\left\{\begin{array}{lll} \Delta_1\, Y = 0 & \text{with} & \Delta_1 \coloneqq \partial_1 I_n - A_1, \\ \vdots & & \\ \Delta_m\, Y = 0 & \text{with} & \Delta_m \coloneqq \partial_m I_n - A_m, \end{array}\right. \qquad A_i \in \mathbb{M}_n(C(x_1,\ldots,x_m))$$

◇ $K = \overline{C}(x_1, \ldots, x_m)$

[illegible]

L [illegible] K [illegible]

⊙ $u \neq 0 \in L$ [illegible] K: $\forall\, i,\ f_i \coloneqq \partial_i(u)/u \in K$.

⊙ [illegible] $u\, z$ [illegible] u [illegible]

[illegible] K [illegible] $z \in K^n$.

◇ u [illegible] $K \Rightarrow \partial_j(f_i) = \partial_i(f_j)\ \ \forall\, i, j$

◇ $u\, z$ [illegible] $[A_1, \ldots, A_m]$

$\Rightarrow z$ [illegible] $[A_1 - f_1 I_n, \ldots, A_m - f_m I_n]$

Hyperexponential solutions of integrable connections (2)

◇ [illegible]

- [illegible] $Y' = A_1\,Y$ [illegible]
  [illegible] $u\,z$ [illegible]
- $f_i \coloneqq \partial_i(u)/u \in K$ [illegible] $\Delta_{i,u} \coloneqq \partial_i - (A_i - f_i\,I_n)$
- $w_1, \ldots, w_s$ [illegible] $\mathcal{W}_u = \{w \in K^n ; \Delta_{1,u}(w) = 0\}$, [illegible]
  [illegible] $K^n \rightsquigarrow$ [illegible] $P = (W_u \quad \tilde{W})$

[illegible] (BCEW'12)

$Y = u\,W_u\,\Gamma_u$ hyperexp. sol. of $[A_1, \ldots, A_m]$ iff Γ_u hyperexp. sol. of $[B_2^{11}, \ldots, B_m^{11}]$ where $B_i = P^{-1}(\,A_i - f_i\,I_n)\,P - \partial_i(P)$ and $B_i^{11} \in \mathbb{M}_s(K_1)$ denotes the first $s \times s$ submatrix of B_i.

◇ [illegible] $\rightsquigarrow$ [illegible]

◇ [illegible] x_j's $j \neq 1$

IV

Implementation

Maple package INTEGRABLECONNECTIONS

◇ [illegible] INTEGRABLECONNECTIONS

- [illegible]

  http://www.ensil.unilim.fr/~cluzeau/PDS.html

- [illegible] *RationalSolutions* (& *Eigenring*) *HyperexponentialSolutions*
- [illegible] ISOLDE [illegible] (*Barkatou-Pfluegel*)

[illegible]

V

Conclusions

Contributions and Perspectives

- [illegible]
  - [illegible]
    [illegible] *Grigoriev'90*)
  - [illegible]
    [illegible]
  - [illegible] (IntegrableConnections)
- [illegible]
  - [illegible]
    [illegible]
  - [illegible]