Telescopers for Rational and Algebraic Functions via Residues

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Outline

- Motivation: enumerating 3D Walks.

- Integrability problems:

  Given $f \in K(y,z)$, decide whether

  $$f = D_y(g) + D_z(h) \quad \text{for some } g, h \in K(y,z).$$

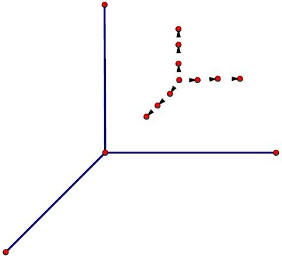
- Telescoping problems:

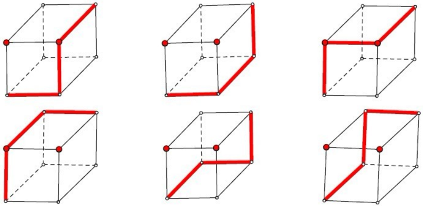
  Given $f \in k(x,y,z)$, find $L \in k(x)\langle D_x \rangle$ such that

  $$L(x, D_x)(f) = D_y(g) + D_z(h) \quad \text{for some } g, h \in k(x,y,z).$$

Enumerating 3D Walks

The Rook moves in a straight line as below in first quadrant of the 3D space.





R(1) = 6

R_n: The number of different Rook walks from $(0, 0, 0)$ to (n, n, n).

2D-diagonals

$f(m,n)$: the number of different Rook walks from $(0,0)$ to (m,n).

$$F(x,y) = \sum_{m,n\geq 0} f(m,n)\, x^m y^n = \frac{1}{1 - \frac{x}{1-x} - \frac{y}{1-y}}.$$

The diagonal of $F(x,y)$ is

$$\operatorname{diag}(F) := \sum_{n\geq 0} f(n,n)\, x^n.$$

Notation: $\mathbb{F}$ an algebraically closed field of char zero $(= \overline{\mathbb{Q}}, \mathbb{C}, \ldots)$.

Lemma: Let $G := y^{-1} \cdot F(y, x/y)$ and $L(x, D_x)$ be a linear differential operator with coefficients in $\mathbb{F}(x)$. Then

$$\underbrace{L(x, D_x)}_{\text{Telescoper}}(G) = D_y(H) \quad \text{with } H \in \mathbb{F}(x,y) \quad \Rightarrow \quad L(\operatorname{diag}(F)) = 0$$

Telescopers for Rational Functions: The Bivariate Case

Let $\mathbb{F}(x)\langle D_x \rangle$ be the ring of linear differential operators in x with coefficients in $\mathbb{F}(x)$.

Problem. For $f \in \mathbb{F}(x, y)$, find $L \in \mathbb{F}(x)\langle D_x \rangle$ such that

$$\underbrace{L(x, D_x)}_{\text{Telescoper}}(f) = D_y(g) \quad \text{for some } g \in \mathbb{F}(x, y).$$

Simpler Problem. For $h \in \mathbb{F}(x, y)$, decide whether

$$h = D_y(g) \quad \text{for some } g \in \mathbb{F}(x, y)$$

Answer. $h = D_y(g)$ iff $\mathrm{res}_y(h, \beta) = 0$ for any root β of the $\mathrm{den}(h)$.

Idea. To find $L \in \mathbb{F}(x)\langle D_x \rangle$ such that $h = L(f)$ has only zero residues.

Telescoping via Residues: The Bivariate Rational Case

Hermite Reduction.

$$f = D_y(g_1) + \frac{A}{B}, \quad \text{where } \deg_y(A) < \deg_y(B) \text{ and } B \text{ squarefree.}$$

Rothstein-Trager Resultant. $R(x, z) := \text{resultant}_y(B, A - zD_y(B))$.

$$R(x, \operatorname{res}_y(A/B, \beta)) = 0 \quad \text{for any root } \beta \text{ of } B \text{ in } \overline{\mathbb{F}(x)}.$$

Theorem (Abel 1827). There exists $L \in \mathbb{F}(x)\langle D_x \rangle$ s.t. $L(\gamma) = 0$ for any root $\gamma \in \overline{\mathbb{F}(x)}$ of $R(x, z)$.

$$L(\operatorname{res}_y(f, \beta)) = \operatorname{res}_y(L(f), \beta) = 0 \ (\forall \beta) \quad \Rightarrow \quad L(f) = D_y(g).$$

Telescopers for 2D Rook Walks

For the 2D Rook walks, the rational function is

$$f := \frac{(-1+y)(-y+x)}{y(y-2x-2y^2+3xy)}$$

Resultant: The Rothstein-Trager Resultant is

$$R(x,z) := (-x+2zx)(40z^2x^2+x-2x^2+x^3-4z^2x-36z^2x^3)$$

So the residues of f w.r.t. y are respectively

$$r_1 = \frac{1}{2}, \quad r_2 = \frac{\sqrt{(9x-1)(x-1)}}{18x-2}, \quad r_3 = -\frac{\sqrt{(9x-1)(x-1)}}{18x-2}$$

Annihilators for residues: $L_1 = D_x$ and

$$L_2 = L_3 = (9x^2-10x+1)D_x + (18x-14)$$

Finally, the telescoper for f is

$$L := (9x^2-10x+1)D_x^2 + (18x-14)D_x.$$

Recurrences

$R(n)$: the number of different Rook walks from $(0,0)$ to (n,n).

Let S_n be the shift operator defined by $S_n(R(n)) = R(n+1)$.

$$L(x, D_x)\left(\sum_{n\geq 0} R(n)x^n\right) = 0 \quad \Rightarrow \quad P(n, S_n)(R(n)) = 0.$$

For the 2D Rook walks, we get the linear recurrence:

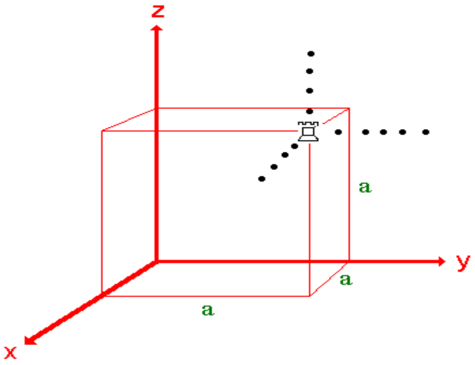
$$R(n+2) = \frac{(-10n-14)R(n+1) + 9nR(n)}{n+2} \quad (R(1) = 2, R(2) = 14).$$

Running the recurrence, $R(n)$ is as follows.

$$2, 14, 106, 838, 6802, 56190, 470010, 3968310, \ldots$$ OEIS:A051708

Enumerating 3D Walks

The Rook moves in 3-dimensional space.



Question: How many different Rook walks from $(0,0,0)$ to (n,n,n)?

3D-diagonals

$f(m,n,k)$: the number of different Rook walks from $(0,0,0)$ to (m,n,k).

$$F(x,y,z) = \sum_{m,n\geq 0} f(m,n,k)\, x^m y^n z^k = \frac{1}{1 - \frac{x}{1-x} - \frac{y}{1-y} - \frac{z}{1-z}}.$$

The diagonal of $F(x,y,z)$ is

$$\operatorname{diag}(F) := \sum_{n\geq 0} f(n,n,n)\, x^n.$$

Lemma: Let $\tilde{F} := (yz)^{-1} \cdot F(y, z/y, x/z)$ and $L(x, D_x) \in \mathbb{F}(x)\langle D_x \rangle$. Then

$$\underbrace{L(x, D_x)}_{\text{Telescoper}}(\tilde{F}) = D_y(G) + D_z(H) \quad \text{with } G, H \in \mathbb{F}(x,y,z) \;\Rightarrow\; L(\operatorname{diag}(F)) = 0.$$

Telescoping Problems

Telescopers for trivariate rational functions:

Given $f \in \mathbb{F}(x, y, z)$, find $L \in \mathbb{F}(x)\langle D_x \rangle$ such that

$$L(x, D_x)(f) = D_y(g) + D_z(h) \quad \text{for some } g, h \in \mathbb{F}(x, y, z).$$

Telescopers for bivariate algebraic functions:

Given $\alpha(x, y)$ algebraic over $\mathbb{F}(x, y)$, find $L \in \mathbb{F}(x)\langle D_x \rangle$ such that

$$L(x, D_x)(\alpha) = D_y(\beta) \quad \text{for some algebraic } \beta(x, y) \text{ over } \mathbb{F}(x, y).$$

Goal: The two telescoping problems above are equivalent!

Integrability Problems

Rational Integrability:

Given $f(y,z) \in \mathbb{E}(y,z)$, decide

$$f = D_y(g) + D_z(h) \quad \text{for some } g, h \in \mathbb{E}(y,z).$$

If such g, h exist, we say that f is rational Integrable w.r.t. y and z.

Algebraic Integrability:

Given $\alpha(y)$ algebraic over $\mathbb{E}(y)$, decide

$$\alpha = D_y(\beta) \quad \text{for some algebraic } \beta \text{ over } \mathbb{E}(y).$$

If such β exists, we say that α is algebraic Integrable w.r.t. y.

Goal: The two Integrable problems above are equivalent!

Residues

Definition. Let $f \in \mathbb{F}(x, y)(z)$. The residue of f at β_i w.r.t. z, denoted by $\mathrm{res}_z(f, \beta_i)$, is the coefficient $\alpha_{i,1}$ in

$$f = \sum_{i=1}^{n} \sum_{j=1}^{m_i} \frac{\alpha_{i,j}}{(z - \beta_i)^j}, \quad \text{where } \alpha_{i,j}, \beta_i \in \overline{\mathbb{F}(x, y)}.$$

Lemma. Let $f \in \mathbb{F}(x, y)(z)$ and $\beta \in \overline{\mathbb{F}(x, y)}$.

- $\partial(\mathrm{res}_z(f, \beta)) = \mathrm{res}_z(\partial(f), \beta)$ with $\partial \in \{D_x, D_y\}$.
- $f = D_z(g) \quad \Leftrightarrow \quad$ All residues of f w.r.t. z are zero.

Remark. The second assertion is not true for algebraic functions!!!

Equivalence between Two Integrability Problems

Theorem (Integrability). Let $f = A/B \in \mathbb{F}(x)(y,z)$. Then

$$f = D_y(g) + D_z(h) \;\Leftrightarrow\; \mathrm{res}_z(f,\beta) = D_y(\gamma_\beta) \text{ for all } \beta \text{ s.t. } B(\beta) = 0.$$

Example 1. Let $f = (x+y+z)^{-1}$. Since $\mathrm{res}_z(f, -x-y) = 1 = D_y(y)$, f is rational Integrable w.r.t. y and z. In fact,

$$f = D_y\left(\frac{x+y}{x+y+z}\right) + D_z\left(-\frac{x+y}{x+y+z}\right).$$

Example 2. Let $f = (xyz)^{-1}$. Since $\mathrm{res}_z(f, 0) = (xy)^{-1}$ is not algebraic integrable, f is not rational Integrable w.r.t. y and z.

Equivalence between Two Telescoping Problems

Theorem (Telescoping). Let $f \in \mathbb{F}(x, y, z)$ and $L \in \mathbb{F}(x)\langle D_x \rangle$. Then

$$L(x, D_x) \text{ is a telescoper for } f \text{ w.r.t. } y \text{ and } z$$

$$\Updownarrow$$

$$L(x, D_x) \text{ is a telescoper for every residue of } f \text{ w.r.t. } z$$

Remark.

$$L_i(x, D_x)(\alpha_i) = D_y(\beta_i), \ 1 \leq i \leq n$$

$$\Downarrow$$

$$L = \text{LCLM}(L_1, L_2, \ldots, L_n) \text{ is a telescoper for all } \alpha_i.$$

Differentials and Residues

Let $K = \mathbb{F}(x, y)(\alpha)$ where α is an algebraic function over $\mathbb{F}(x, y)$. Think of $\alpha(x, y)$ as a parameterized family of algebraic functions of y (with parameter x).

Differentials.

$$\Omega_{K/\mathbb{F}(x)} := \{\beta\, dy \mid \beta \in K\}.$$

- $df = 0$ for all $f \in \mathbb{F}(x)$ and $D_x(\beta dy) = D_x(\beta)dy$.

Residues. Let $\mathcal{P}$ be a place of K (with no ramification). Then any $\beta \in K$ has a $\mathcal{P}$-adic expansion

$$\beta = \sum_{i \geq \rho} a_i t^i, \quad \text{where } \rho \in \mathbb{Z},\ a_i \in \overline{\mathbb{F}(x)} \text{ and } t \in K.$$

The residues of β at $\mathcal{P}$ is a_{-1}, denoted by $\operatorname{res}_{\mathcal{P}}(\beta)$.

- $\operatorname{res}_{\mathcal{P}}(D_x(\beta)) = D_x(\operatorname{res}_{\mathcal{P}}(\beta))$.

Differential Equations for Residues

Let $K = \mathbb{F}(x, y)(\alpha)$ and $\beta = A/B$ with $A \in \mathbb{F}(x)[y, \alpha]$ and $B \in \mathbb{F}(x)[y]$. Let B^* be the squarefree part of B w.r.t. y.

Theorem. There exists $L \in \mathbb{F}(x)\langle D_x \rangle$ such that all residues of $L(\alpha)$ are zero and

$$\deg_{D_x}(L) \leq [K : \mathbb{F}(x, y)] \cdot \deg_y(B^*).$$

Definition. A differential $\omega \in \Omega_{K/\mathbb{F}(x)}$ is of second kind if all residues of ω are zero.

Lemma.

- If ω is exact i.e. $\omega = d(\beta)$, then ω is of second kind.
- Let $\Phi_{K/\mathbb{F}(x)} := \{\text{differentials of second kind}\}/\{\text{exact differentials}\}$. Then

$$\dim_{\mathbb{F}(x)}(\Phi_{K/\mathbb{F}(x)}) = 2 \cdot \text{genus}(K).$$

Telescopers for Bivariate Algebraic Functions

Algorithm. Given $\alpha(x, y)$ algebraic over $\mathbb{F}(x, y)$, do

1. Compute $L_1 \in \mathbb{F}(x)\langle D_x \rangle$ such that $\omega = L_1(\alpha)\, dy$ is of second kind.
2. Find $a_0, \ldots, a_{2g} \in \mathbb{F}(x)$ with $g := \text{genus}(K)$ with $K = \mathbb{F}(x, y)(\alpha)$, not all zero, such that

$$a_{2g} D_x^{2g}(\omega) + \cdots + a_0 \omega = d(\beta) \quad \text{for some } \beta \in K.$$

Remark.

- If $\alpha \in \mathbb{F}(x, y)$, Step 2 is not needed since $g = 0$.
- If ω is of second kind, so is $D_x^i(\omega)$ for all $i \in \mathbb{N}$.

Telescopers for 3D Rook Walks

Transformation. $F = P/Q := (yz)^{-1} f(y, z/y, x/z)$.

$$\frac{P}{Q} = \frac{(-1+y)(y-z)(-z+x)}{zy\left(zy - 2\,yx - 2\,z^2 + 3\,xz - 2\,y^2z + 3\,y^2x + 3\,z^2y - 4\,zyx\right)}$$

Residues. Roots of $R(x, y, u) := \text{Resultant}_z(Q, P - u \cdot D_z(Q))$ are

$$r_1 = \frac{y-1}{y(3y-2)}, \;\; r_2 = -r_3 = \frac{(y-1)^2}{y(3y-2)\sqrt{-4y^3 + 16xy^2 + 4y^2 - y - 24xy + 9x}}.$$

Telescopers. $L_1 = D_x$ and $L_2 = L_3$ with

$$L_2 = {D_x}^3 + \frac{\left(4608\,x^4 - 6372\,x^3 + 813\,x^2 + 514\,x - 4\right){D_x}^2}{x\left(-2 + 121\,x + 475\,x^2 - 1746\,x^3 + 1152\,x^4\right)} + \frac{4\left(576\,x^3 - 801\,x^2 - 108\,x + 74\right)D_x}{x\left(-2 + 121\,x + 475\,x^2 - 1746\,x^3 + 1152\,x^4\right)}$$

Recurrences for 3D Rook Walks

$L = \mathrm{LCLM}(L_1, L_2, L_3)$ is a telescopers for $F(x, y, z)$.

$$\Downarrow$$

$$L(x, D_x)\left(\sum_n f(n, n, n)x^n\right) = 0$$

Recurrence. Let $r(n) := f(n, n, n)$. From $L(x, D_x)$ via gfun, we get

$(1152n^2 + 1152n^3)r(n) + (-7830n - 3204 - 6372n^2 - 1746n^3)r(n+1) + (2957n + 762 + 2238n^2 + 475n^3)r(n+2) + (4197n + 4698 + 1240n^2 + 121n^3)r(n+3) + (-22n^2 - 80n - 96 - 2n^3)r(n+4) = 0.$

With initial values $r(0) = 1, r(1) = 6, r(2) = 222, r(3) = 9918$, we get

$$1, 6, 222, 9918, 486924, 25267236, 1359631776, 75059524392, \ldots$$

Implementation and Experiments

Timings. We compare different algorithms for examples in combinatorics.

	Chyzak	Koutschan	Residue
3D Rook 1	3.48	24.5	0.59
3D Rook 2	31	182	2.3
3D Queen 1	11805	$>$ 30h	1203
3D Queen 2	12109	$>$ 30h	1186
Random example	221	1232	26

Figure: Timings are in seconds.

For more examples, please visit

http://www.risc.jku.at/people/mkauers/residues/

Summary

Equivalence.

$$L(x, D_x)(f) = D_y(g) + D_z(h), \quad f, g, h \in \mathbb{F}(x, y, z)$$

$$\Updownarrow$$

$$L(x, D_x)(\alpha) = D_y(\beta) \quad \text{for any residue } \alpha \text{ of } f \text{ w.r.t. } z.$$

Note. One can also reduce rational m vars to algebraic $m - 1$ vars.

Order Bound. Let $K = \mathbb{F}(x, y)(\alpha)$ and n be the number of poles of α.

$$L(x, D_x)(\alpha) = D_y(\beta) \quad \Rightarrow \quad \mathrm{ord}(L) \leq [K : \mathbb{F}(x, y)] \cdot n + 2 \cdot \mathrm{genus}(K).$$

Future Work. Walks in higher dimension (4D, 5D, ...).