Telescopers for Rational and Algebraic Functions via Residues

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Outline

- Motivation: enumerating 3D Walks.

- **Integrability** problems:

  Given $f \in K(y, z)$, decide whether

  $$f = D_y(g) + D_z(h)$$

  for some $g, h \in K(y, z)$.

- **Telescoping** problems:

  Given $f \in k(x, y, z)$, find $L \in k(x)\langle D_x \rangle$ such that

  $$L(x, D_x)(f) = D_y(g) + D_z(h)$$

  for some $g, h \in k(x, y, z)$. 

Shaoshi Chen  |  Telescopers and Residues
Enumerating 3D Walks

The Rook moves in a straight line as below in first quadrant of the 3D space.

\( R_n \): The number of different Rook walks from \((0, 0, 0)\) to \((n, n, n)\).
2D-diagonals

\( f(m, n) \): the number of different Rook walks from \((0, 0)\) to \((m, n)\).

\[
F(x, y) = \sum_{m,n \geq 0} f(m, n) x^m y^n = \frac{1}{1 - \frac{x}{1-x} - \frac{y}{1-y}}.
\]

The diagonal of \( F(x, y) \) is

\[
\text{diag}(F) := \sum_{n \geq 0} f(n, n) x^n.
\]

Notation: \( \mathbb{F} \) an algebraically closed field of char zero (\( = \overline{\mathbb{Q}}, \mathbb{C}, \ldots \)).

Lemma: Let \( G := y^{-1} \cdot F(y, x/y) \) and \( L(x, D_x) \) be a linear differential operator with coefficients in \( \mathbb{F}(x) \). Then

\[
L(x, D_x)(G) = D_y(H) \quad \text{with} \quad H \in \mathbb{F}(x, y) \quad \Rightarrow \quad L(\text{diag}(F)) = 0
\]
Telescopers for Rational Functions: The Bivariate Case

Let $\mathbb{F}(x)\langle D_x \rangle$ be the ring of linear differential operators in $x$ with coefficients in $\mathbb{F}(x)$.

**Problem.** For $f \in \mathbb{F}(x, y)$, find $L \in \mathbb{F}(x)\langle D_x \rangle$ such that

$$L(x, D_x)(f) = D_y(g) \quad \text{for some } g \in \mathbb{F}(x, y).$$

**Telescopers**

**Simpler Problem.** For $h \in \mathbb{F}(x, y)$, decide whether

$$h = D_y(g) \quad \text{for some } g \in \mathbb{F}(x, y)$$

**Answer.** $h = D_y(g)$ iff $\text{res}_y(h, \beta) = 0$ for any root $\beta$ of the $\text{den}(h)$.

**Idea.** To find $L \in \mathbb{F}(x)\langle D_x \rangle$ such that $h = L(f)$ has only zero residues.
Telescoping via Residues: The Bivariate Rational Case

Hermite Reduction.

\[ f = D_y(g_1) + \frac{A}{B}, \quad \text{where } \deg_y(A) < \deg_y(B) \text{ and } B \text{ squarefree.} \]

Rothstein-Trager Resultant. \( R(x, z) := \text{resultant}_y(B, A - zD_y(B)) \).

\[ R(x, \text{res}_y(A/B, \beta)) = 0 \quad \text{for any root } \beta \text{ of } B \text{ in } \overline{F(x)}. \]

Theorem (Abel 1827). There exists \( L \in \overline{F(x)}\langle D_x \rangle \) s.t. \( L(\gamma) = 0 \) for any root \( \gamma \in \overline{F(x)} \) of \( R(x, z) \).

\[ L(\text{res}_y(f, \beta)) = \text{res}_y(L(f), \beta) = 0 \quad (\forall \beta) \quad \Rightarrow \quad L(f) = D_y(g). \]
Telescopers for 2D Rook Walks

For the 2D Rook walks, the rational function is

\[
f := \frac{(-1 + y)(-y + x)}{y(y - 2x - 2y^2 + 3xy)}
\]

**Resultant:** The Rothstein-Trager Resultant is

\[
R(x, z) := (-x + 2zx)(40z^2x^2 + x - 2x^2 + x^3 - 4z^2x - 36z^2x^3)
\]

So the residues of \( f \) w.r.t. \( y \) are respectively

\[
r_1 = \frac{1}{2}, \quad r_2 = \frac{\sqrt{(9x - 1)(x - 1)}}{18x - 2}, \quad r_3 = -\frac{\sqrt{(9x - 1)(x - 1)}}{18x - 2}
\]

**Annihilators for residues:** \( L_1 = D_x \) and

\[
L_2 = L_3 = (9x^2 - 10x + 1)D_x + (18x - 14)
\]

Finally, the telescoper for \( f \) is

\[
L := (9x^2 - 10x + 1)D_x^2 + (18x - 14)D_x.
\]
Recurrences

**$R(n)$**: the number of different Rook walks from $(0, 0)$ to $(n, n)$.

Let $S_n$ be the shift operator defined by $S_n(R(n)) = R(n + 1)$.

\[
L(x, D_x) \left( \sum_{n \geq 0} R(n)x^n \right) = 0 \quad \Rightarrow \quad P(n, S_n)(R(n)) = 0.
\]

For the 2D Rook walks, we get the linear recurrence:

\[
R(n + 2) = \frac{(-10n - 14)R(n + 1) + 9nR(n)}{n + 2} \quad (R(1) = 2, \ R(2) = 14).
\]

Running the recurrence, $R(n)$ is as follows.

2, 14, 106, 838, 6802, 56190, 470010, 3968310, ... \quad \text{OEIS:A051708}
Enumerating 3D Walks

The Rook moves in 3-dimensional space.

**Question:** How many different Rook walks from \((0, 0, 0)\) to \((n, n, n)\)?
3D-diagonals

\( f(m, n, k) \): the number of different Rook walks from \((0, 0, 0)\) to \((m, n, k)\).

\[
F(x, y, z) = \sum_{m,n \geq 0} f(m, n, k) x^m y^n z^k = \frac{1}{1 - x - y - z}.
\]

The diagonal of \(F(x, y, z)\) is

\[
\text{diag}(F) := \sum_{n \geq 0} f(n, n, n) x^n.
\]

Lemma: Let \(\tilde{F} := (yz)^{-1} \cdot F(y, z/y, x/z)\) and \(L(x, D_x) \in \mathbb{F}(x)\langle D_x \rangle\). Then

\[
L(x, D_x)(\tilde{F}) = D_y(G) + D_z(H) \quad \text{with} \quad G, H \in \mathbb{F}(x, y, z) \Rightarrow L(\text{diag}(F)) = 0.
\]

Telescopers and Residues
Telescoping Problems

Telescopers for trivariate rational functions:
Given $f \in \mathbb{F}(x, y, z)$, find $L \in \mathbb{F}(x) \langle D_x \rangle$ such that

$$L(x, D_x)(f) = D_y(g) + D_z(h) \quad \text{for some } g, h \in \mathbb{F}(x, y, z).$$

Telescopers for bivariate algebraic functions:
Given $\alpha(x, y)$ algebraic over $\mathbb{F}(x, y)$, find $L \in \mathbb{F}(x) \langle D_x \rangle$ such that

$$L(x, D_x)(\alpha) = D_y(\beta) \quad \text{for some algebraic } \beta(x, y) \text{ over } \mathbb{F}(x, y).$$

Goal: The two telescoping problems above are equivalent!
Integrability Problems

Rational Integrability:

Given $f(y, z) \in \mathbb{E}(y, z)$, decide

$$f = D_y(g) + D_z(h) \text{ for some } g, h \in \mathbb{E}(y, z).$$

If such $g, h$ exist, we say that $f$ is rational Integrable w.r.t. $y$ and $z$.

Algebraic Integrability:

Given $\alpha(y)$ algebraic over $\mathbb{E}(y)$, decide

$$\alpha = D_y(\beta) \text{ for some algebraic } \beta \text{ over } \mathbb{E}(y).$$

If such $\beta$ exists, we say that $\alpha$ is algebraic Integrable w.r.t. $y$.

Goal: The two Integrable problems above are equivalent!
Residues

**Definition.** Let $f \in \mathbb{F}(x, y)(z)$. The *residue* of $f$ at $\beta_i$ w.r.t. $z$, denoted by $\text{res}_z(f, \beta_i)$, is the coefficient $\alpha_{i,1}$ in

$$f = \sum_{i=1}^{n} \sum_{j=1}^{m_i} \frac{\alpha_{i,j}}{(z - \beta_i)^j}, \quad \text{where } \alpha_{i,j}, \beta_i \in \overline{\mathbb{F}(x, y)}.$$

**Lemma.** Let $f \in \mathbb{F}(x, y)(z)$ and $\beta \in \overline{\mathbb{F}(x, y)}$.

- $\partial(\text{res}_z(f, \beta)) = \text{res}_z(\partial(f), \beta)$ with $\partial \in \{D_x, D_y\}$.
- $f = D_z(g) \iff$ All residues of $f$ w.r.t. $z$ are zero.

**Remark.** The second assertion is not true for algebraic functions!!!
Equivalence between Two Integrability Problems

**Theorem (Integrability).** Let \( f = A/B \in \mathbb{F}(x)(y, z) \). Then

\[
f = D_y(g) + D_z(h) \iff \text{res}_z(f, \beta) = D_y(\gamma \beta) \quad \text{for all} \ \beta \text{ s.t. } B(\beta) = 0.
\]

**Example 1.** Let \( f = (x + y + z)^{-1} \). Since \( \text{res}_z(f, -x - y) = 1 = D_y(y) \), \( f \) is rational Integrable w.r.t. \( y \) and \( z \). In fact,

\[
f = D_y \left( \frac{x + y}{x + y + z} \right) + D_z \left( -\frac{x + y}{x + y + z} \right).
\]

**Example 2.** Let \( f = (xyz)^{-1} \). Since \( \text{res}_z(f, 0) = (xy)^{-1} \) is not algebraic integrable, \( f \) is not rational Integrable w.r.t. \( y \) and \( z \).
Theorem (Telescoping). Let $f \in \mathbb{F}(x, y, z)$ and $L \in \mathbb{F}(x) \langle D_x \rangle$. Then

$L(x, D_x)$ is a telescoper for $f$ w.r.t. $y$ and $z$

$\uparrow$

$L(x, D_x)$ is a telescoper for every residue of $f$ w.r.t. $z$

Remark.

$L_i(x, D_x)(\alpha_i) = D_y(\beta_i), \ 1 \leq i \leq n$

$\Downarrow$

$L = \text{LCLM}(L_1, L_2, \ldots, L_n)$ is a telescoper for all $\alpha_i$. 
Differentials and Residues

Let $K = \mathbb{F}(x, y)(\alpha)$ where $\alpha$ is an algebraic function over $\mathbb{F}(x, y)$. Think of $\alpha(x, y)$ as a parameterized family of algebraic functions of $y$ (with parameter $x$).

**Differentials.**

$$\Omega_{K/\mathbb{F}(x)} := \{ \beta \, dy \mid \beta \in K \}.$$

- $df = 0$ for all $f \in \mathbb{F}(x)$ and $D_x(\beta dy) = D_x(\beta)dy$.

**Residues.** Let $\mathcal{P}$ be a place of $K$ (with no ramification). Then any $\beta \in K$ has a $\mathcal{P}$-adic expansion

$$\beta = \sum_{i \geq \rho} a_i t^i, \quad \text{where } \rho \in \mathbb{Z}, \, a_i \in \overline{\mathbb{F}(x)} \text{ and } t \in K.$$

The residues of $\beta$ at $\mathcal{P}$ is $a_{-1}$, denoted by $\text{res } \mathcal{P}(\beta)$.

- $\text{res } \mathcal{P}(D_x(\beta)) = D_x(\text{res } \mathcal{P}(\beta))$.  

Differential Equations for Residues

Let \( K = \mathbb{F}(x, y)(\alpha) \) and \( \beta = A/B \) with \( A \in \mathbb{F}(x)[y, \alpha] \) and \( B \in \mathbb{F}(x)[y] \). Let \( B^* \) be the squarefree part of \( B \) w.r.t. \( y \).

**Theorem.** There exists \( L \in \mathbb{F}(x)\langle D_x \rangle \) such that all residues of \( L(\alpha) \) are zero and

\[
\deg_{D_x}(L) \leq [K : \mathbb{F}(x, y)] \cdot \deg_y(B^*).
\]

**Definition.** A differential \( \omega \in \Omega_{K/\mathbb{F}(x)} \) is of **second kind** if all residues of \( \omega \) are zero.

**Lemma.**

- If \( \omega \) is exact i.e. \( \omega = d(\beta) \), then \( \omega \) is of second kind.
- Let \( \Phi_{K/\mathbb{F}(x)} := \{ \text{differentials of second kind} \} / \{ \text{exact differentials} \} \). Then
  \[
  \dim_{\mathbb{F}(x)}(\Phi_{K/\mathbb{F}(x)}) = 2 \cdot \text{genus}(K).
  \]
Telescopers for Bivariate Algebraic Functions

Algorithm. Given $\alpha(x, y)$ algebraic over $\mathbb{F}(x, y)$, do

1. Compute $L_1 \in \mathbb{F}(x)\langle D_x \rangle$ such that $\omega = L_1(\alpha) dy$ is of second kind.

2. Find $a_0, \ldots, a_{2g} \in \mathbb{F}(x)$ with $g := \text{genus}(K)$ with $K = \mathbb{F}(x, y)(\alpha)$, not all zero, such that

$$a_{2g} D_x^{2g}(\omega) + \cdots + a_0 \omega = d(\beta) \quad \text{for some } \beta \in K.$$

Remark.

- If $\alpha \in \mathbb{F}(x, y)$, Step 2 is not needed since $g = 0$.
- If $\omega$ is of second kind, so is $D_x^i(\omega)$ for all $i \in \mathbb{N}$. 
Telescopers for 3D Rook Walks

Transformation. \( F = P/Q := (yz)^{-1}f(y, z/y, x/z) \).

\[
\begin{align*}
\frac{P}{Q} &= \frac{(-1 + y)(y - z)(-z + x)}{zy (zy - 2 yx - 2 z^2 + 3 xz - 2 y^2 z + 3 y^2 x + 3 z^2 y - 4 zyx)}
\end{align*}
\]

Residues. Roots of \( R(x, y, u) := \text{Resultant}_z(Q, P - u \cdot D_z(Q)) \) are

\[
\begin{align*}
r_1 &= \frac{y - 1}{y(3y - 2)}, \quad r_2 = -r_3 = \frac{(y - 1)^2}{y(3y - 2)\sqrt{-4y^3 + 16xy^2 + 4y^2 - y - 24xy + 9x}}.
\end{align*}
\]

Telescopers. \( L_1 = D_x \) and \( L_2 = L_3 \) with

\[
\begin{align*}
L_2 &= D_x^3 + \frac{(4608 x^4 - 6372 x^3 + 813 x^2 + 514 x - 4) D_x^2}{x (-2 + 121 x + 475 x^2 - 1746 x^3 + 1152 x^4)} \\
&\quad + \frac{4 (576 x^3 - 801 x^2 - 108 x + 74) D_x}{x (-2 + 121 x + 475 x^2 - 1746 x^3 + 1152 x^4)}.
\end{align*}
\]
Recurrences for 3D Rook Walks

\[ L = \text{LCLM}(L_1, L_2, L_3) \] is a telescopers for \( F(x, y, z) \).

\[ \downarrow \]

\[ L(x, D_x) \left( \sum_n f(n, n, n) x^n \right) = 0 \]

**Recurrence.** Let \( r(n) := f(n, n, n) \). From \( L(x, D_x) \) via gfun, we get

\[
(1152n^2 + 1152n^3)r(n) + (-7830n - 3204 - 6372n^2 - 1746n^3)r(n + 1) + (2957n + 762 + 2238n^2 + 475n^3)r(n + 2) + (4197n + 4698 + 1240n^2 + 121n^3)r(n + 3) + (-22n^2 - 80n - 96 - 2n^3)r(n + 4) = 0.
\]

With initial values \( r(0) = 1, r(1) = 6, r(2) = 222, r(3) = 9918 \), we get

\[ 1, 6, 222, 9918, 486924, 25267236, 1359631776, 75059524392, \ldots \]
### Implementation and Experiments

**Timings.** We compare different algorithms for examples in combinatorics.

<table>
<thead>
<tr>
<th></th>
<th>Chyzak</th>
<th>Koutschan</th>
<th>Residue</th>
</tr>
</thead>
<tbody>
<tr>
<td>3D Rook 1</td>
<td>3.48</td>
<td>24.5</td>
<td>0.59</td>
</tr>
<tr>
<td>3D Rook 2</td>
<td>31</td>
<td>182</td>
<td>2.3</td>
</tr>
<tr>
<td>3D Queen 1</td>
<td>11805</td>
<td>&gt; 30h</td>
<td>1203</td>
</tr>
<tr>
<td>3D Queen 2</td>
<td>12109</td>
<td>&gt; 30h</td>
<td>1186</td>
</tr>
<tr>
<td>Random example</td>
<td>221</td>
<td>1232</td>
<td>26</td>
</tr>
</tbody>
</table>

*Figure:* Timings are in seconds.

For more examples, please visit

http://www.risc.jku.at/people/mkauers/residues/
Summary

Equivalence.

\[ L(x, D_x)(f) = D_y(g) + D_z(h), \quad f, g, h \in \mathbb{F}(x, y, z) \]

\[ \uparrow \]

\[ L(x, D_x)(\alpha) = D_y(\beta) \quad \text{for any residue } \alpha \text{ of } f \text{ w.r.t. } z. \]

Note. One can also reduce rational \( m \) vars to algebraic \( m - 1 \) vars.

Order Bound. Let \( K = \mathbb{F}(x, y)(\alpha) \) and \( n \) be the number of poles of \( \alpha \).

\[ L(x, D_x)(\alpha) = D_y(\beta) \quad \Rightarrow \quad \text{ord}(L) \leq [K : \mathbb{F}(x, y)] \cdot n + 2 \cdot \text{genus}(K). \]

Future Work. Walks in higher dimension (4D, 5D, ...).