Generalized Fourier Series for Solutions of Linear Differential Equations

Alexandre Benoit\textsuperscript{1}
Joint work with Bruno Salvy\textsuperscript{2}

\textsuperscript{1}CNRS, INRIA, UPMC
\textsuperscript{2}INRIA

I Introduction
Generalized Fourier Series

\[ f(x) = \sum a_n \psi_n(x) \]

Some Examples

\[ \sin(x) = 2 \sum_{n=0}^{\infty} (-1)^n J_{2n}(x) \]

\[ \arccos(x) = \frac{1}{2\pi} \sum_{n=0}^{\infty} \frac{4}{(2n+1)^2} T_{2n+1}(x) \]

\[ \text{erf}(x) = 2 \sum_{n=0}^{\infty} \left(-\frac{1}{4}\right)^n \frac{1}{\sqrt{\pi}(2n+1)n!} {}_{1}F_{1}\left(n+\frac{1}{2},2n+2,-x\right) \]

More generally \( (\psi_n(x))_{n \in \mathbb{N}} \) can be an orthogonal basis of a Hilbert space.
Applications: Good approximation properties.

Approximation of $\arctan(2x)$ by Taylor expansion of degree 1

Taylor approximation
Applications: Good approximation properties.

Bad approximation outside its circle of convergence

\[ \text{arctan}(2x) \]

Taylor approximation
Applications: Good approximation properties.

approximation of $\arctan(2x)$ by Chebyshev expansion of degree 1
Applications: Good approximation properties.

bad approximation over $\mathbb{R}$

arctan(2x)

Taylor expansion

Chebyshev expansion
Applications: Good approximation properties.

approximation of $\arctan(2x)$ by Hermite expansion of degree 1

$\arctan(2x)$
Taylor expansion
Chebyshev expansion
Hermite expansion
Our framework

Families of functions $\psi_n(x)$ with two special properties

### Mult by $x$ ($\mathcal{P}_x$)

$$\text{Rec}_{x_2} (x\psi_n(x)) = \text{Rec}_{x_1} (\psi_n(x))$$

### Examples

- Monomial polynomials
  $$(M_n = x^n)$$
- All orthogonal polynomials
- Bessel functions
- Legendre functions
- Parabolic cylinder functions
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Mult by $x$ ($P_x$)

$$\text{Rec}_{x_2}(x\psi_n(x)) = \text{Rec}_{x_1}(\psi_n(x))$$

Examples

- Monomial polynomials
  $(M_n = x^n)$
  $xM_n = M_{n+1}$
- All orthogonal polynomials
  $2xT_n(x) = T_{n+1}(x) + T_{n-1}(x)$
- Bessel functions
  $\frac{1}{n}(xJ_{n+1} - xJ_{n-1}) = 2J_n$
Our framework

Families of functions $\psi_n(x)$ with two special properties

Mult by $x$ ($P_x$)

$$Rec_{x2}(x\psi_n(x)) = Rec_{x1}(\psi_n(x))$$

Differentiation ($P_\partial$)

$$Rec_{\partial 2}(\psi_n'(x)) = Rec_{\partial 1}(\psi_n(x))$$

Examples

- Monomial polynomials
- Classical orthogonal polynomials
- Bessel functions
- Legendre functions
- Parabolic cylinder functions
Our framework

Families of functions $\psi_n(x)$ with two special properties

**Mult by $x$ ($P_x$)**

\[
\text{Rec}_{x2}(x\psi_n(x)) = \text{Rec}_{x1}(\psi_n(x))
\]

**Differentiation ($P_\partial$)**

\[
\text{Rec}_{\partial 2}(\psi'_n(x)) = \text{Rec}_{\partial 1}(\psi_n(x))
\]

**Examples**

- Monomial polynomials
- Classical orthogonal polynomials
- Bessel functions
- Legendre functions
- Parabolic cylinder functions

\[
M'_n = nM_{n-1}
\]

\[
\frac{1}{n+1} T'_{n+1}(x) - \frac{1}{n-1} T'_{n-1}(x) = 2T_n(x)
\]

\[
2J'_n(x) = J_{n-1}(x) - J_{n+1}(x)
\]
Families of functions $\psi_n(x)$ with two special properties

**Mult by $x$ ($\mathcal{P}_x$)**

$$\mathcal{R}ec_{x2}(x\psi_n(x)) = \mathcal{R}ec_{x1}(\psi_n(x))$$

**Differentiation ($\mathcal{P}_\partial$)**

$$\mathcal{R}ec_{\partial2}(\psi'_n(x)) = \mathcal{R}ec_{\partial1}(\psi_n(x))$$

This is our data-structure for $\psi_n(x)$
Main Idea

If $\psi_n(x)$ satisfies $(P_x)$ and $(P_\partial)$, for any $f(x) = \sum a_n \psi_n(x)$ solution of a linear differential equation with polynomial coefficients, the coefficients $a_n$ are cancelled by a linear recurrence relation with polynomial coefficients.
Main Idea

If $\psi_n(x)$ satisfies $(P_x)$ and $(P_\partial)$, for any $f(x) = \sum a_n \psi_n(x)$ solution of a linear differential equation with polynomial coefficients, the coefficients $a_n$ are cancelled by a linear recurrence relation with polynomial coefficients.

Applications:

- Efficient numerical computation of the coefficients.
- Computation of closed-form for the coefficients (when it's possible).
Previous work

- Clenshaw (1957): numerical scheme to compute the coefficients when $\psi_n(x) = T_n(x)$ (Chebyshev series).
- Lewanowicz (1976-2004): algorithms to compute a recurrence relation when $\psi_n$ is an orthogonal or semi-orthogonal polynomial family.
- Rebillard and Zakravšek (2006): General algorithm computing a recurrence relation when $\psi_n$ is a family of hypergeometric polynomials.
- Benoit and Salvy (2009): Simple unified presentation and complexity analysis of the previous algorithms using Fractions of recurrence relations when $\psi_n = T_n$. New and fast algorithm to compute the Chebyshev recurrence.
New Results (2012)

- Simple unified presentation of the previous algorithms using Pairs of recurrence relations.
- New general algorithm computing the recurrence relation of the coefficients for a Generalized Fourier Series when $\psi_n(x)$ satisfies $(\mathcal{P}_x)$ and $(\mathcal{P}_\partial)$. 
II Pairs of Recurrence Relations
Examples: Chebyshev case \( (f(x) = \sum u_n T_n(x)) \)

Basic rules:

\[
x f = \sum a_n T_n \quad (\mathcal{P}_x)
\]
\[
f' = \sum b_n T_n \quad (\mathcal{P}_\partial)
\]

\[
a_n = \frac{u_{n-1} + u_{n+1}}{2}
\]

\[
b_{n-1} - b_{n+1} = 2nu_n.
\]
Examples: Chebyshev case \( f(x) = \sum u_n T_n(x) \)

Basic rules:

\[
xf = \sum a_n T_n \quad (\mathcal{P}_x)
\]
\[
f' = \sum b_n T_n \quad (\mathcal{P}_\partial)
\]

Combine:

\[
f' + 2xf = \sum c_n T_n \quad (\mathcal{P}_\partial + 2\mathcal{P}_x)
\]

Application: Chebyshev series for \( \exp(-x^2) \).

\[
a_n = \frac{u_{n-1} + u_{n+1}}{2}
\]
\[
b_{n-1} - b_{n+1} = 2nu_n.
\]
\[
c_{n-1} - c_{n+1} = \text{Rec}_1(u_n).
\]
Examples: Chebyshev case \( (f(x) = \sum u_n T_n(x)) \)

Basic rules:

\[
\begin{align*}
xf &= \sum a_n T_n \quad (P_x) \\
\frac{df}{dx} &= \sum b_n T_n \quad (P_\partial)
\end{align*}
\]

\[
a_n = \frac{u_{n-1} + u_{n+1}}{2}
\]

\[
b_{n-1} - b_{n+1} = 2nu_n.
\]

Combine:

\[
f' + 2xf = \sum c_n T_n \quad (P_\partial + 2P_x)
\]

\[
c_{n-1} - c_{n+1} = \text{Rec}_1(u_n).
\]

Application: Chebyshev series for \( \exp(-x^2) \).

\[
(f' + 2xf)' = \sum d_n T_n \quad (P_\partial)
\]

\[
d_{n-1} - d_{n+1} = 2nc_n,
\]

\[
\text{Rec}_2(d_n) = \text{Rec}_3(u_n),
\]

\[
\text{Rec}_4(e_n) = \text{Rec}_5(u_n).
\]

Application: Chebyshev series for \( \text{erf}(x) \).
Rings of Pairs of Recurrence Relations

Theorem (Least Common Left Multiple (Ore 33))

Given $\text{Rec}_1$ and $\text{Rec}_2$, there exists a recurrence relation $\text{Rec}$ and a pair $(\tilde{\text{Rec}}_1, \tilde{\text{Rec}}_2)$ such that for all sequences $(u_n)_{n \in \mathbb{N}}$:

$$\text{Rec} \ (u_n) = \tilde{\text{Rec}}_1 \circ \text{Rec}_1 \ (u_n) = \tilde{\text{Rec}}_2 \circ \text{Rec}_2 \ (u_n)$$
Theorem (Least Common Left Multiple (Ore 33))

Given $\text{Rec}_1$ and $\text{Rec}_2$, there exists a recurrence relation $\text{Rec}$ and a pair $(\tilde{\text{Rec}}_1, \tilde{\text{Rec}}_2)$ such that for all sequences $(u_n)_{n \in \mathbb{N}}$:

$$\text{Rec} (u_n) = \tilde{\text{Rec}}_1 \circ \text{Rec}_1 (u_n) = \tilde{\text{Rec}}_2 \circ \text{Rec}_2 (u_n)$$

- The LCLM is the recurrence relation $\text{Rec}$ with minimal order.
Rings of Pairs of Recurrence Relations

Theorem (Least Common Left Multiple (Ore 33))

Given $\text{Rec}_1$ and $\text{Rec}_2$, there exists a recurrence relation $\text{Rec}$ and a pair $(\tilde{\text{Rec}}_1, \tilde{\text{Rec}}_2)$ such that for all sequences $(u_n)_{n \in \mathbb{N}}$:

$$\text{Rec} (u_n) = \tilde{\text{Rec}}_1 \circ \text{Rec}_1 (u_n) = \tilde{\text{Rec}}_2 \circ \text{Rec}_2 (u_n)$$

- The LCLM is the recurrence relation $\text{Rec}$ with minimal order.
- Computation: Euclidean algorithm.
Operations of addition and composition

\[
\text{Rec} = \text{lclm}(\text{Rec}_1, \text{Rec}_2) = \tilde{\text{Rec}}_1 \circ \text{Rec}_1 = \tilde{\text{Rec}}_2 \circ \text{Rec}_2
\]

**Operation 1: Addition**

\[
\text{Rec}_1(a_n) = \text{Rec}_3(u_n), \quad \text{Rec}_2(b_n) = \text{Rec}_4(u_n)
\]

\[
\text{Rec}(a_n) = \tilde{\text{Rec}}_1 \circ \text{Rec}_3(u_n), \quad \text{Rec}(b_n) = \tilde{\text{Rec}}_2 \circ \text{Rec}_4(u_n)
\]

\[
\rightarrow \text{Rec}(a_n + b_n) = \left(\tilde{\text{Rec}}_1 \circ \text{Rec}_3 + \tilde{\text{Rec}}_2 \circ \text{Rec}_4\right)(u_n).
\]
Operations of addition and composition

Rec = \text{lclm}(Rec_1, Rec_2) = \bar{Rec}_1 \circ Rec_1 = \bar{Rec}_2 \circ Rec_2

**Operation 1: Addition**

Rec_1(a_n) = \bar{Rec}_3(u_n), \quad Rec_2(b_n) = \bar{Rec}_4(u_n)

Rec(a_n) = \bar{Rec}_1 \circ Rec_3(u_n), \quad Rec(b_n) = \bar{Rec}_2 \circ Rec_4(u_n)

\rightarrow Rec(a_n + b_n) = \left(\bar{Rec}_1 \circ Rec_3 + \bar{Rec}_2 \circ Rec_4\right)(u_n).

**Operation 2: Composition**

Rec_1(u_n) = \bar{Rec}_3(a_n), \quad Rec_2(u_n) = \bar{Rec}_4(b_n)

Rec(u_n) = \bar{Rec}_1 \circ Rec_1(u_n) = \bar{Rec}_2 \circ Rec_2(u_n)

\rightarrow Rec_1 \circ Rec_3(a_n) = Rec_2 \circ Rec_4(b_n).
Main Result

Main Result: Morphism

There exists a morphism \( \varphi \) such that if \( f = \sum u_n \psi_n(x) \) and \( g = \sum v_n \psi_n(x) \) are related by \( L(f) = g \) (\( L \) a linear differential operator), then:

\[
\varphi(L) = (\text{Rec}_1, \text{Rec}_2) \quad \text{with} \quad \text{Rec}_1(u_n) = \text{Rec}_2(v_n)
\]

In particular if \( L(f) = 0 \), then \( \text{Rec}_1(u_n) = 0 \).
Definition of the Morphism $\phi$

\[
f = \sum u_n \psi_n(x) \quad g = \sum v_n \psi_n(x)
\]

$\mathcal{R}ec_{x2}(x\psi_n(x)) = \mathcal{R}ec_{x1}(\psi_n(x))$

if $xf = g$, then \(\mathcal{R}ec_{x2}(u_n) = \mathcal{R}ec_{x1}(v_n)\)

$\mathcal{R}ec_{\partial2}(\psi'_n(x)) = \mathcal{R}ec_{\partial1}(\psi_n(x))$

if $f' = g$, then \(\mathcal{R}ec_{\partial1}(u_n) = \mathcal{R}ec_{\partial2}(v_n)\)
Definition of the Morphism $\varphi$

$$f = \sum u_n \psi_n(x) \quad g = \sum v_n \psi_n(x)$$

$\mathcal{Rec}_{x2}(x \psi_n(x)) = \mathcal{Rec}_{x1}(\psi_n(x))$

$\mathcal{Rec}_{\partial 2}(\psi'_n(x)) = \mathcal{Rec}_{\partial 1}(\psi_n(x))$

Example for Chebyshev series:

$$2x T_n(x) = T_{n+1}(x) + T_{n-1}(x)$$

$$\frac{T_{n+1}'(x)}{n+1} - \frac{T_{n-1}'(x)}{n-1} = 2 T_n(x)$$

$\varphi$:

$$u_{n+1} + u_{n-1} = 2v_n$$

$$2u_n = \frac{1}{n} (v_{n-1} - v_{n+1})$$

Example for Bessel series

$$\frac{1}{n} (x J_{n+1} - x J_{n-1}) = 2J_n$$

$$2J_n'(x) = J_{n-1}(x) - J_{n+1}(x)$$

$\varphi$:

$$2u_n = \frac{v_{n+1}}{n+1} + \frac{v_{n-1}}{n-1}$$

$$u_{n+1} - u_{n-1} = 2v_n$$
General Algorithm

Recall

- Definition of $\varphi(x)$ and $\varphi(\partial)$
- Algorithms to compute addition and composition between two pairs
Recall

- Definition of $\varphi(x)$ and $\varphi(\partial)$
- Algorithms to compute addition and composition between two pairs

General Algorithm

We deduce from this morphism a general Horner-like algorithm to compute the recurrence relation satisfied by the coefficients of a generalized Fourier series solution of a linear differential equation.
III Recurrences of Smaller Order
Given a pair \((\text{Rec}_1, \text{Rec}_2)\), the Euclidean algorithm computes the greatest recurrence relation \(\text{Rec}\) (GCLD) such that there exists a pair \(\left(\tilde{\text{Rec}}_1, \tilde{\text{Rec}}_2\right)\) with the following relations for all sequences \((u_n)_{n\in\mathbb{N}}\) and \((v_n)_{n\in\mathbb{N}}\):

\[
\text{Rec} \circ \tilde{\text{Rec}}_1 (u_n) = \text{Rec}_1 (u_n)
\]
\[
\text{Rec} \circ \tilde{\text{Rec}}_2 (v_n) = \text{Rec}_2 (v_n)
\]
GCLD

Given a pair \((\text{Rec}_1, \text{Rec}_2)\), the Euclidean algorithm computes the greatest recurrence relation \(\text{Rec}\) (GCLD) such that there exists a pair \(\left(\tilde{\text{Rec}}_1, \tilde{\text{Rec}}_2\right)\) with the following relations for all sequences \((u_n)_{n \in \mathbb{N}}\) and \((v_n)_{n \in \mathbb{N}}\):

\[
\text{Rec} \circ \tilde{\text{Rec}}_1 (u_n) = \text{Rec}_1 (u_n)
\]
\[
\text{Rec} \circ \tilde{\text{Rec}}_2 (v_n) = \text{Rec}_2 (v_n)
\]

The orders of the recurrence relations \(\tilde{\text{Rec}}_i\) are at most those of \(\text{Rec}_i\).
Greatest Common **Left** Divisor and Reduction of Order

**GCLD**

Given a pair \((\text{Rec}_1, \text{Rec}_2)\), the Euclidean algorithm computes the greatest recurrence relation \(\text{Rec} \) (GCLD) such that there exists a pair \((\overline{\text{Rec}}_1, \overline{\text{Rec}}_2)\) with the following relations for all sequences \((u_n)_{n \in \mathbb{N}}\) and \((v_n)_{n \in \mathbb{N}}\):

\[
\begin{align*}
\text{Rec} \circ \overline{\text{Rec}}_1 (u_n) &= \text{Rec}_1 (u_n) \\
\text{Rec} \circ \overline{\text{Rec}}_2 (v_n) &= \text{Rec}_2 (v_n)
\end{align*}
\]

The orders of the recurrence relations \(\overline{\text{Rec}}_i\) are at most those of \(\text{Rec}_i\).

**Remark**

In a general case, we don’t have:

\[
\text{Rec}_1(u_n) = \text{Rec}_2(v_n) \Rightarrow \overline{\text{Rec}}_1(u_n) = \overline{\text{Rec}}_2(v_n),
\]
Theorem

Given $L$ a linear differential operator, $f = \sum u_n \psi_n(x)$, $g = \sum v_n \psi_n(x)$ such that $L(f) = g$ and a pair $(\text{Rec}_1, \text{Rec}_2) = \varphi(L)$. We have

$$\hat{\text{Rec}}_1(u_n) = \hat{\text{Rec}}_2(v_n)$$
GLCD for reduction of order

Theorem

Given $L$ a linear differential operator, $f = \sum u_n \psi_n(x)$, $g = \sum v_n \psi_n(x)$ such that $L(f) = g$ and a pair $(\text{Rec}_1, \text{Rec}_2) = \varphi(L)$. We have

$$\tilde{\text{Rec}}_1(u_n) = \tilde{\text{Rec}}_2(v_n)$$

Application: Adaptation of the previous algorithm

At the end of the previous algorithm, add a final step: Remove the GC LD of the two recurrence relations of the pair.
Example of reduction for Chebyshev series

\[ \sqrt{1 - x^2} = \sum_{n \in \mathbb{N}} \frac{4}{\pi(2n + 1)} T_{2n}(x) = \sum_{n \in \mathbb{N}} c_n T_n(x) \]

\( \sqrt{1 - x^2} \) is the solution of the differential equation:

\[ xy(x) + (1 - x^2)y'(x) = 0 \]
Example of reduction for Chebyshev series

$$\sqrt{1 - x^2} = \sum_{n \in \mathbb{N}} \frac{4}{\pi(2n + 1)} T_{2n}(x) = \sum_{n \in \mathbb{N}} c_n T_n(x)$$

\(\sqrt{1 - x^2}\) is the solution of the differential equation:

$$xy(x) + (1 - x^2)y'(x) = 0$$

With the general algorithm we obtain the pair of recurrence relations:

\(\text{Rec}_1(u_n) = (n+3)u_{n+2} - 2nu_n + (n-3)u_{n-2}\) and \(\text{Rec}_2(v_n) = 2(-v_{n+1} + v_{n-1})\).

We deduce: 
\((n + 3)c_{n+2} - 2nc_n + (n - 3)c_{n-2} = 0.\)
Example of reduction for Chebyshev series

\[ \sqrt{1-x^2} = \sum_{n \in \mathbb{N}} \frac{4}{\pi(2n+1)} T_{2n}(x) = \sum_{n \in \mathbb{N}} c_n T_n(x) \]

\( \sqrt{1-x^2} \) is the solution of the differential equation:

\[ xy(x) + (1-x^2)y'(x) = 0 \]

With the general algorithm we obtain the pair of recurrence relations:

Rec\( _1(u_n) = (n+3)u_{n+2} - 2nu_n + (n-3)u_{n-2} \) and Rec\( _2(v_n) = 2(-v_{n+1} + v_{n-1}) \).

We deduce:

\[ (n+3)c_{n+2} - 2nc_n + (n-3)c_{n-2} = 0. \]

\[ \tilde{\text{Rec}}_1(u_n) = (n+2)u_{n+1} - (n-2)u_{n-1} \) and \( \tilde{\text{Rec}}_2(v_n) = 2v_n. \)

We deduce:

\[ (n+2)c_{n+1} - (n-2)c_{n-1} = 0. \]
Counterexample of the minimality by Lewanowicz

$x \exp(x)$ is solution of

$$deq := xy(x)' - (x + 1)y = 0.$$ 

By the morphism we obtain the recurrence

$$-u_n + 2nu_{n+1} + (2n + 8)u_{n+3} + u_{n+4}.$$
Counterexample of the minimality by Lewanowicz

$x \exp(x)$ is solution of

$$deq := xy(x)' - (x + 1)y = 0.$$  

By the morphism we obtain the recurrence

$$-u_n + 2nu_{n+1} + (2n + 8)u_{n+3} + u_{n+4}.$$  

This function is also solution of

$$deq2 = (-1 + x^2)y''' - (x^2 - 3x - 1)y'' - (4x - 1)y' - 3y(x) = 0.$$  

By the morphism, we obtain the recurrence relation

$$(-n^2 - 3n - 3)u_n + (2n^3 + 6n^2 + 6n + 2)u_{n+1} + (n^2 + n + 1)u_{n+2}.$$
IV Conclusion
Conclusion

Contributions:

- Use of Pairs of recurrence relations.
- New general algorithm.
- Use of the GCLD to reduce order of the recurrence.

\[
\text{erf}(x) = \sum_{n=0}^{\infty} \frac{2^{-n}}{(2n+1)!} F_{1}(n+\frac{1}{2}; 2n+2; \frac{1}{2}) \sqrt{(2n+1)}
\]
Conclusion

Contributions:

- Use of Pairs of recurrence relations.
- New general algorithm.
- Use of the GCLD to reduce order of the recurrence.

Perspectives:

- Computation of the recurrence of minimal order.
- Numerical computation of the coefficients.
- Closed form for the coefficients.

Example

\[
\text{erf} (x) = \sum_{n=0}^{\infty} 2 \frac{4^{-n} (-1)^n \, _1F_1(n + 1/2; 2n + 2; -1)}{\sqrt{\pi} (2n + 1) n!} \, T_{2n+1} (x).
\]

- Integration in the Dynamic Dictionary of Mathematical Functions.