Generalized Fourier Series for Solutions of Linear Differential Equations

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I Introduction

Alexandre Benoit

GFS for Solutions of Linear Differential Equations.
Generalized Fourier Series

\[ f(x) = \sum a_n \psi_n(x) \]

### Some Examples

\[ \sin(x) = 2 \sum_{n=0}^{\infty} (-1)^n J_{2n}(x) \]

\[ \text{arccos}(x) = \frac{1}{2\pi} T_0(x) - \sum_{n=0}^{\infty} \frac{4}{(2n+1)^2} \pi T_{2n+1}(x) \]

\[ \text{erf}(x) = 2 \sum_{n=0}^{\infty} \left( -\frac{1}{4} \right)^n \frac{1}{\sqrt{\pi} (2n+1) n!} \mathbf{1}_{1} \left( \begin{array}{c} n + \frac{1}{2} \vline \cr 2n + 2 \end{array} \right) \left( -x \right) \]

More generally \((\psi_n(x))_{n \in \mathbb{N}}\) can be an orthogonal basis of a Hilbert space.
Applications: Good approximation properties.

Approximation of $\arctan(2x)$ by Taylor expansion of degree 1.
Applications: Good approximation properties.

Bad approximation outside its circle of convergence

\[ \text{arctan}(2x) \]

Taylor approximation
Applications: Good approximation properties.

approximation of $\arctan(2x)$ by Chebyshev expansion of degree 1
Applications: Good approximation properties.

![Graph showing bad approximation over $\mathbb{R}$](image)

- **arctan(2x)**
- Taylor expansion
- Chebyshev expansion
Applications: Good approximation properties.

approximation of $\arctan(2x)$ by Hermite expansion of degree 1
Our framework

Families of functions $\psi_n(x)$ with two special properties

Mult by $x$ ($\mathcal{P}_x$)

$$\text{Rec}_{x2}(x\psi_n(x)) = \text{Rec}_{x1}(\psi_n(x))$$

Examples

- Monomial polynomials
  \((M_n = x^n)\)
- All orthogonal polynomials
- Bessel functions
- Legendre functions
- Parabolic cylinder functions
Our framework

Families of functions $\psi_n(x)$ with two special properties

Mult by $x$ ($P_x$)

$$\text{Rec}_{x2}(x\psi_n(x)) = \text{Rec}_{x1}(\psi_n(x))$$

Examples

- Monomial polynomials ($M_n = x^n$) \hspace{2cm} $xM_n = M_{n+1}$
- All orthogonal polynomials \hspace{2cm} $2xT_n(x) = T_{n+1}(x) + T_{n-1}(x)$
- Bessel functions \hspace{2cm} $\frac{1}{n}(xJ_{n+1} - xJ_{n-1}) = 2J_n$
Our framework

Families of functions $\psi_n(x)$ with two special properties

Mult by $x$ ($P_x$)

$$\text{Rec}_{x2} (x\psi_n(x)) = \text{Rec}_{x1} (\psi_n(x))$$

Differentiation ($P_\partial$)

$$\text{Rec}_{\partial 2} (\psi'_n(x)) = \text{Rec}_{\partial 1} (\psi_n(x))$$

Examples

- Monomial polynomials
- Classical orthogonal polynomials
- Bessel functions
- Legendre functions
- Parabolic cylinder functions
### Our framework

Families of functions $\psi_n(x)$ with two special properties

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### Examples

- Monomial polynomials
- Classical orthogonal polynomials
- Bessel functions
- Legendre functions
- Parabolic cylinder functions

\[ M'_n = nM_{n-1} \]

\[ \frac{1}{n+1} T'_{n+1}(x) - \frac{1}{n-1} T'_{n-1}(x) = 2T_n(x) \]

\[ 2J'_n(x) = J_{n-1}(x) - J_{n+1}(x) \]
Families of functions $\psi_n(x)$ with two special properties

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This is our data-structure for $\psi_n(x)$
Main Idea

If $\psi_n(x)$ satisfies $(\mathcal{P}_x)$ and $(\mathcal{P}_\partial)$, for any $f(x) = \sum a_n \psi_n(x)$ solution of a linear differential equation with polynomial coefficients, the coefficients $a_n$ are cancelled by a linear recurrence relation with polynomial coefficients.
Main Idea

If \( \psi_n(x) \) satisfies \((\mathcal{P}_x)\) and \((\mathcal{P}_\partial)\), for any \( f(x) = \sum a_n \psi_n(x) \) solution of a linear differential equation with polynomial coefficients, the coefficients \( a_n \) are cancelled by a linear recurrence relation with polynomial coefficients.

Applications:
- Efficient numerical computation of the coefficients.
- Computation of closed-form for the coefficients (when it's possible).
Previous work

- Clenshaw (1957): numerical scheme to compute the coefficients when \( \psi_n(x) = T_n(x) \) (Chebyshev series).
- Lewanowicz (1976-2004): algorithms to compute a recurrence relation when \( \psi_n \) is an orthogonal or semi-orthogonal polynomial family.
- Rebillard and Zakrajšek (2006): General algorithm computing a recurrence relation when \( \psi_n \) is a family of hypergeometric polynomials.
- Benoit and Salvy (2009): Simple unified presentation and complexity analysis of the previous algorithms using Fractions of recurrence relations when \( \psi_n = T_n \). New and fast algorithm to compute the Chebyshev recurrence.
New Results (2012)

- Simple unified presentation of the previous algorithms using Pairs of recurrence relations.
- New general algorithm computing the recurrence relation of the coefficients for a Generalized Fourier Series when $\psi_n(x)$ satisfies $(\mathcal{P}_x)$ and $(\mathcal{P}_\partial)$. 
II Pairs of Recurrence Relations
Examples: Chebyshev case \((f(x) = \sum u_n T_n(x))\)

Basic rules:

\[
xf = \sum a_n T_n \quad (\mathcal{P}_x) \\
\frac{df}{dx} = \sum b_n T_n \quad (\mathcal{D})
\]

\[
a_n = \frac{u_{n-1} + u_{n+1}}{2} \\
b_{n-1} - b_{n+1} = 2nu_n.
\]
Examples: Chebyshev case \((f(x) = \sum u_n T_n(x))\)

Basic rules:

\[
xf = \sum a_n T_n \quad (\mathcal{P}_x)
\]

\[
f' = \sum b_n T_n \quad (\mathcal{P}_\partial)
\]

\[
a_n = \frac{u_{n-1} + u_{n+1}}{2}
\]

\[
b_{n-1} - b_{n+1} = 2nu_n.
\]

Combine:

\[
f' + 2xf = \sum c_n T_n \quad (\mathcal{P}_\partial + 2\mathcal{P}_x)
\]

\[
c_{n-1} - c_{n+1} = \text{Rec}_1(u_n).
\]

Application: Chebyshev series for \(\exp(-x^2)\).
Examples: Chebyshev case \((f(x) = \sum u_n T_n(x))\)

Basic rules:

\[
xf = \sum a_n T_n \quad (P_x) \quad a_n = \frac{u_{n-1} + u_{n+1}}{2}
\]

\[
f' = \sum b_n T_n \quad (P_{\partial}) \quad b_{n-1} - b_{n+1} = 2nu_n.
\]

Combine:

\[
f' + 2xf = \sum c_n T_n \quad (P_{\partial} + 2P_x) \quad c_{n-1} - c_{n+1} = \text{Rec}_1(u_n).
\]

Application: Chebyshev series for \(\exp(-x^2)\).

\[
(f' + 2xf)' = \sum d_n T_n \quad (P_{\partial}) \quad d_{n-1} - d_{n+1} = 2nc_n,
\]

\[
(f' + 2xf)' - 2f = \sum e_n T_n \quad \rightarrow \quad \text{Rec}_2(d_n) = \text{Rec}_3(u_n), \quad \text{Rec}_4(e_n) = \text{Rec}_5(u_n).
\]

Application: Chebyshev series for \(\text{erf}(x)\).
Theorem (Least Common Left Multiple (Ore 33))

Given $\text{Rec}_1$ and $\text{Rec}_2$, there exists a recurrence relation $\text{Rec}$ and a pair $(\tilde{\text{Rec}}_1, \tilde{\text{Rec}}_2)$ such that for all sequences $(u_n)_{n \in \mathbb{N}}$:

$$\text{Rec}(u_n) = \tilde{\text{Rec}}_1 \circ \text{Rec}_1(u_n) = \tilde{\text{Rec}}_2 \circ \text{Rec}_2(u_n)$$
**Theorem (Least Common Left Multiple (Ore 33))**

*Given* $\text{Rec}_1$ *and* $\text{Rec}_2$, *there exists a recurrence relation* $\text{Rec}$ *and a pair* $(\tilde{\text{Rec}}_1, \tilde{\text{Rec}}_2)$ *such that for all sequences* $(u_n)_{n \in \mathbb{N}}$ *:

$$\text{Rec} (u_n) = \tilde{\text{Rec}}_1 \circ \text{Rec}_1 (u_n) = \tilde{\text{Rec}}_2 \circ \text{Rec}_2 (u_n)$$

- The LCLM is the recurrence relation $\text{Rec}$ with *minimal order*. 
Rings of Pairs of Recurrence Relations

Theorem (Least Common Left Multiple (Ore 33))

Given $\text{Rec}_1$ and $\text{Rec}_2$, there exists a recurrence relation $\text{Rec}$ and a pair $\left(\widetilde{\text{Rec}}_1, \widetilde{\text{Rec}}_2\right)$ such that for all sequences $(u_n)_{n \in \mathbb{N}}$:

$$\text{Rec} (u_n) = \widetilde{\text{Rec}}_1 \circ \text{Rec}_1 (u_n) = \widetilde{\text{Rec}}_2 \circ \text{Rec}_2 (u_n)$$

- The LCLM is the recurrence relation $\text{Rec}$ with minimal order.
- Computation : Euclidean algorithm.
Operations of addition and composition

Rec = lclm(Rec₁, Rec₂) = Rec₁ ◦ Rec₁ = Rec₂ ◦ Rec₂

Operation 1: Addition

Rec₁(aₙ) = Rec₃(uₙ), Rec₂(bₙ) = Rec₄(uₙ)
Rec(aₙ) = Rec₁ ◦ Rec₃(uₙ), Rec(bₙ) = Rec₂ ◦ Rec₄(uₙ)
→ Rec(aₙ + bₙ) = (Rec₁ ◦ Rec₃ + Rec₂ ◦ Rec₄)(uₙ).
Operations of addition and composition

\[ \text{Rec} = \text{lcm}(\text{Rec}_1, \text{Rec}_2) = \tilde{\text{Rec}}_1 \circ \text{Rec}_1 = \tilde{\text{Rec}}_2 \circ \text{Rec}_2 \]

**Operation 1: Addition**

\[
\begin{align*}
\text{Rec}_1(a_n) &= \tilde{\text{Rec}}_3(u_n), \quad \text{Rec}_2(b_n) = \tilde{\text{Rec}}_4(u_n) \\
\text{Rec}(a_n) &= \tilde{\text{Rec}}_1 \circ \text{Rec}_3(u_n), \quad \text{Rec}(b_n) = \tilde{\text{Rec}}_2 \circ \text{Rec}_4(u_n) \\
\rightarrow \text{Rec}(a_n + b_n) &= \left( \tilde{\text{Rec}}_1 \circ \text{Rec}_3 + \tilde{\text{Rec}}_2 \circ \text{Rec}_4 \right)(u_n).
\end{align*}
\]

**Operation 2: Composition**

\[
\begin{align*}
\text{Rec}_1(u_n) &= \tilde{\text{Rec}}_3(a_n), \quad \text{Rec}_2(u_n) = \tilde{\text{Rec}}_4(b_n) \\
\text{Rec}(u_n) &= \tilde{\text{Rec}}_1 \circ \text{Rec}_1(u_n) = \tilde{\text{Rec}}_2 \circ \text{Rec}_2(u_n) \\
\rightarrow \text{Rec}_1 \circ \text{Rec}_3(a_n) &= \text{Rec}_2 \circ \text{Rec}_4(b_n).
\end{align*}
\]
Main Result: Morphism

There exists a morphism $\varphi$ such that if $f = \sum u_n \psi_n(x)$ and $g = \sum v_n \psi_n(x)$ are related by $L(f) = g$ ($L$ a linear differential operator), then:

$$\varphi(L) = (\text{Rec}_1, \text{Rec}_2) \quad \text{with} \quad \text{Rec}_1(u_n) = \text{Rec}_2(v_n)$$

In particular if $L(f) = 0$, then $\text{Rec}_1(u_n) = 0$. 
Definition of the Morphism $\varphi$

\[ f = \sum u_n \psi_n(x) \quad g = \sum v_n \psi_n(x) \]

- $\text{Rec}_x^2 (x \psi_n(x)) = \text{Rec}_x^1 (\psi_n(x))$
- $\text{Rec}_\partial^2 (\psi'_n(x)) = \text{Rec}_\partial^1 (\psi_n(x))$

**if $x f = g$, then**
- $\varphi(x)$
  - $\text{Rec}_x^2 (u_n) = \text{Rec}_x^1 (v_n)$

**if $f' = g$, then**
- $\varphi(\partial)$
  - $\text{Rec}_\partial^1 (u_n) = \text{Rec}_\partial^2 (v_n)$
Definition of the Morphism $\varphi$

\[ f = \sum u_n \psi_n(x) \quad g = \sum v_n \psi_n(x) \]

\[ \text{Rec}_{x2} (x \psi_n(x)) = \text{Rec}_{x1} (\psi_n(x)) \quad \varphi(x) \]

\[ \text{if } xf = g, \text{ then} \quad \text{Rec}_{x2} (u_n) = \text{Rec}_{x1} (v_n) \]

\[ \text{Rec}_{\partial 2} (\psi_n(x)) = \text{Rec}_{\partial 1} (\psi_n(x)) \quad \varphi(\partial) \]

\[ \text{if } f' = g, \text{ then} \quad \text{Rec}_{\partial 1} (u_n) = \text{Rec}_{\partial 2} (v_n) \]

Example for Chebyshev series:

\[ 2x T_n(x) = T_{n+1}(x) + T_{n-1}(x) \]

\[ \frac{T_{n+1}'(x)}{n+1} - \frac{T_{n-1}'(x)}{n-1} = 2T_n(x) \]

\[ \varphi \]

\[ u_{n+1} + u_{n-1} = 2v_n \]

\[ 2u_n = \frac{1}{n} (v_{n-1} - v_{n+1}) \]

Example for Bessel series

\[ \frac{1}{n} \left( xJ_{n+1} - xJ_{n-1} \right) = 2J_n \]

\[ 2J_n'(x) = J_{n-1}(x) - J_{n+1}(x) \]

\[ \varphi \]

\[ 2u_n = \frac{v_{n+1}}{n+1} + \frac{v_{n-1}}{n-1} \]

\[ u_{n+1} - u_{n-1} = 2v_n \]
Recall

- Definition of $\varphi(x)$ and $\varphi(\partial)$
- Algorithms to compute addition and composition between two pairs
Recall

- Definition of $\varphi(x)$ and $\varphi(\partial)$
- Algorithms to compute addition and composition between two pairs

General Algorithm

We deduce from this morphism a general Horner-like algorithm to compute the recurrence relation satisfied by the coefficients of a generalized Fourier series solution of a linear differential equation.
III Recurrences of Smaller Order
Given a pair \((\text{Rec}_1, \text{Rec}_2)\), the Euclidean algorithm computes the greatest recurrence relation \(\text{Rec} \ (\text{GCLD})\) such that there exists a pair \((\tilde{\text{Rec}}_1, \tilde{\text{Rec}}_2)\) with the following relations for all sequences \((u_n)_{n \in \mathbb{N}}\) and \((v_n)_{n \in \mathbb{N}}\):

\[
\text{Rec} \circ \tilde{\text{Rec}}_1 (u_n) = \text{Rec}_1 (u_n) \\
\text{Rec} \circ \tilde{\text{Rec}}_2 (v_n) = \text{Rec}_2 (v_n)
\]
Greatest Common Left Divisor and Reduction of Order

GCLD

Given a pair \((\text{Rec}_1, \text{Rec}_2)\), the Euclidean algorithm computes the greatest recurrence relation \(\text{Rec} (\text{GCLD})\) such that there exists a pair \((\tilde{\text{Rec}}_1, \tilde{\text{Rec}}_2)\) with the following relations for all sequences \((u_n)_{n \in \mathbb{N}}\) and \((v_n)_{n \in \mathbb{N}}\):

\[
\begin{align*}
\text{Rec} \circ \tilde{\text{Rec}}_1 (u_n) &= \text{Rec}_1 (u_n) \\
\text{Rec} \circ \tilde{\text{Rec}}_2 (v_n) &= \text{Rec}_2 (v_n)
\end{align*}
\]

The orders of the recurrence relations \(\tilde{\text{Rec}}_i\) are at most those of \(\text{Rec}_i\).
Greatest Common Left Divisor and Reduction of Order

**GCLD**

Given a pair \((\text{Rec}_1, \text{Rec}_2)\), the Euclidean algorithm computes the greatest recurrence relation \(\text{Rec} \) (GCLD) such that there exists a pair \((\tilde{\text{Rec}}_1, \tilde{\text{Rec}}_2)\) with the following relations for all sequences \((u_n)_{n \in \mathbb{N}}\) and \((v_n)_{n \in \mathbb{N}}\):

\[
\text{Rec} \circ \tilde{\text{Rec}}_1 (u_n) = \text{Rec}_1 (u_n)
\]
\[
\text{Rec} \circ \tilde{\text{Rec}}_2 (v_n) = \text{Rec}_2 (v_n)
\]

The orders of the recurrence relations \(\tilde{\text{Rec}}_i\) are at most those of \(\text{Rec}_i\).

**Remark**

In a general case, we don’t have:

\[
\text{Rec}_1(u_n) = \text{Rec}_2(v_n) \Rightarrow \tilde{\text{Rec}}_1(u_n) = \tilde{\text{Rec}}_2(v_n),
\]
Theorem

*Given* $L$ a linear differential operator, $f = \sum u_n \psi_n(x)$, $g = \sum v_n \psi_n(x)$ *such that* $L(f) = g$ *and a pair* $(\text{Rec}_1, \text{Rec}_2) = \varphi(L)$. *We have*

$$\tilde{\text{Rec}}_1(u_n) = \tilde{\text{Rec}}_2(v_n)$$
Theorem

Given $L$ a linear differential operator, $f = \sum u_n \psi_n(x)$, $g = \sum v_n \psi_n(x)$ such that $L(f) = g$ and a pair $(\text{Rec}_1, \text{Rec}_2) = \varphi(L)$. We have

$$\tilde{\text{Rec}}_1(u_n) = \tilde{\text{Rec}}_2(v_n)$$

Application: Adaptation of the previous algorithm

At the end of the previous algorithm, add a final step:
Remove the GC LD of the two recurrence relations of the pair.
Example of reduction for Chebyshev series

\[
\sqrt{1 - x^2} = \sum_{n \in \mathbb{N}} \frac{4}{\pi(2n + 1)} T_{2n}(x) = \sum_{n \in \mathbb{N}} c_n T_n(x)
\]

\(\sqrt{1 - x^2}\) is the solution of the differential equation:

\[xy(x) + (1 - x^2)y'(x) = 0\]
Example of reduction for Chebyshev series

\[\sqrt{1 - x^2} = \sum_{n \in \mathbb{N}} \frac{4}{\pi(2n + 1)} T_{2n}(x) = \sum_{n \in \mathbb{N}} c_n T_n(x)\]

\[\sqrt{1 - x^2}\] is the solution of the differential equation:

\[xy(x) + (1 - x^2)y'(x) = 0\]

With the general algorithm we obtain the pair of recurrence relations:

Rec\(_1\) (\(u_n\)) = \((n + 3)u_{n+2} - 2nu_n + (n - 3)u_{n-2}\) and Rec\(_2\) (\(v_n\)) = \(2 (-v_{n+1} + v_{n-1})\).

We deduce:

\[(n + 3)c_{n+2} - 2nc_n + (n - 3)c_{n-2} = 0.\]
Example of reduction for Chebyshev series

\[ \sqrt{1-x^2} = \sum_{n \in \mathbb{N}} \frac{4}{\pi(2n+1)} T_{2n}(x) = \sum_{n \in \mathbb{N}} c_n T_n(x) \]

\( \sqrt{1-x^2} \) is the solution of the differential equation:

\[ xy(x) + (1-x^2)y'(x) = 0 \]

With the general algorithm we obtain the pair of recurrence relations:

\[ \text{Rec}_1(u_n) = (n+3)u_{n+2} - 2nu_n + (n-3)u_{n-2} \quad \text{and} \quad \text{Rec}_2(v_n) = 2(-v_{n+1} + v_{n-1}). \]

We deduce:

\[ (n+3)c_{n+2} - 2nc_n + (n-3)c_{n-2} = 0. \]

\[ \tilde{\text{Rec}}_1(u_n) = (n+2)u_{n+1} - (n-2)u_{n-1} \quad \text{and} \quad \tilde{\text{Rec}}_2(v_n) = 2v_n. \]

We deduce:

\[ (n+2)c_{n+1} - (n-2)c_{n-1} = 0. \]
Counterexample of the minimality by Lewanowicz

\[ x \exp(x) \text{ is solution of} \]
\[
\text{deq} := xy(x)' - (x + 1)y = 0.
\]

By the morphism we obtain the recurrence
\[
-u_n + 2nu_{n+1} + (2n + 8)u_{n+3} + u_{n+4}.
\]
Counterexample of the minimality by Lewanowicz

$x \exp(x)$ is solution of

$$deq := xy(x)' - (x + 1)y = 0.$$  

By the morphism we obtain the recurrence

$$-u_n + 2nu_{n+1} + (2n + 8)u_{n+3} + u_{n+4}.$$  

This function is also solution of

$$deq2 = (-1 + x^2)y''' - (x^2 - 3x - 1)y'' - (4x - 1)y' - 3y(x) = 0.$$  

By the morphism, we obtain the recurrence relation

$$(-n^2 - 3n - 3)u_n + (2n^3 + 6n^2 + 6n + 2)u_{n+1} + (n^2 + n + 1)u_{n+2}.$$
IV Conclusion
Conclusion

Contributions:

- Use of Pairs of recurrence relations.
- New general algorithm.
- Use of the GCLD to reduce order of the recurrence.

Example:

\[
\text{erf}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1) n! T_{2n+1}} \left( \frac{1}{\sqrt{\pi}} \right) \frac{1}{(2n+1)} F_{1} \left( \frac{n+1}{2}; 2n+2; -1 \right) \]

Integration in the Dynamic Dictionary of Mathematical Functions.
Conclusion

Contributions:

- Use of Pairs of recurrence relations.
- New general algorithm.
- Use of the GCLD to reduce order of the recurrence.

Perspectives:

- Computation of the recurrence of minimal order.
- Numerical computation of the coefficients.
- Closed form for the coefficients.

Example

\[ \text{erf}(x) = \sum_{n=0}^{\infty} 2 \frac{4^{-n} (-1)^n \, _1F_1(n + 1/2; 2n + 2; -1)}{\sqrt{\pi} (2n + 1) n!} T_{2n+1}(x). \]

- Integration in the Dynamic Dictionary of Mathematical Functions.