

Generalized Fourier Series for Solutions of Linear Differential Equations

Alexandre Benoit¹
Joint work with Bruno Salvy²

¹CNRS, INRIA, UPMC

²INRIA

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I Introduction

Generalized Fourier Series

$$f(x) = \sum a_n \psi_n(x)$$

Some Examples

$$\sin(x) = 2 \sum_{n=0}^{\infty} (-1)^n J_{2n}(x)$$

$$\arccos(x) = \frac{1}{2\pi} T_0(x) - \sum_{n=0}^{\infty} \frac{4}{(2n+1)^2 \pi} T_{2n+1}(x)$$

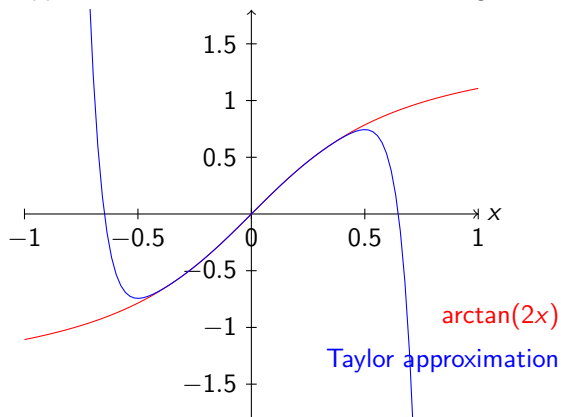
$$\operatorname{erf}(x) = 2 \sum_{n=0}^{\infty} \left(-\frac{1}{4}\right)^n \frac{1}{\sqrt{\pi} (2n+1) n!} {}_1F_1\left(\begin{matrix} n + \frac{1}{2} \\ 2n + 2 \end{matrix} \middle| -x\right)$$

More generally $(\psi_n(x))_{n \in \mathbb{N}}$ can be an orthogonal basis of a Hilbert space.

Applications: Good approximation properties.

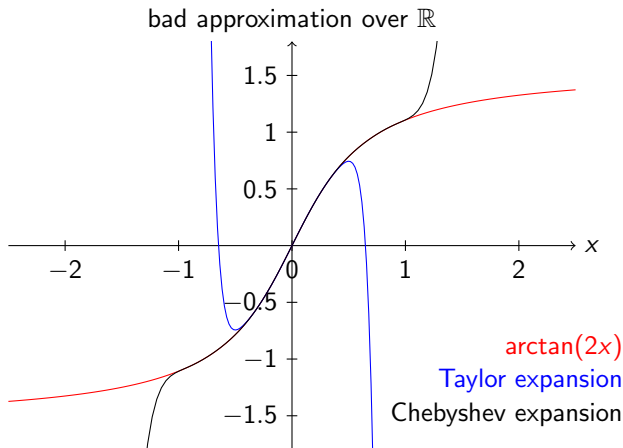
Applications: Good approximation properties.

Bad approximation outside its circle of convergence



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Applications: Good approximation properties.

Our framework

Families of functions $\psi_n(x)$ with two special properties

Mult by x (\mathcal{P}_x)

$$\mathcal{R}ec_{x^2}(x\psi_n(x)) = \mathcal{R}ec_{x^1}(\psi_n(x))$$

Examples

- Monomial polynomials
($M_n = x^n$)
- All orthogonal polynomials
- Bessel functions
- Legendre functions
- Parabolic cylinder functions

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$$xM_n = M_{n+1}$$

$$2xT_n(x) = T_{n+1}(x) + T_{n-1}(x)$$

$$\frac{1}{n}(xJ_{n+1} - xJ_{n-1}) = 2J_n$$

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Differentiation (\mathcal{P}_∂)

$$\mathcal{R}ec_{\partial 2}(\psi'_n(x)) = \mathcal{R}ec_{\partial 1}(\psi_n(x))$$

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$$M'_n = nM_{n-1}$$

$$\frac{1}{n+1} T'_{n+1}(x) - \frac{1}{n-1} T'_{n-1}(x) = 2T_n(x)$$

$$2J'_n(x) = J_{n-1}(x) - J_{n+1}(x)$$

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$$\mathcal{R}ec_{x2}(x\psi_n(x)) = \mathcal{R}ec_{x1}(\psi_n(x))$$

Differentiation (\mathcal{P}_∂)

$$\mathcal{R}ec_{\partial2}(\psi_n'(x)) = \mathcal{R}ec_{\partial1}(\psi_n(x))$$

This is our data-structure for $\psi_n(x)$

Main Idea

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If $\psi_n(x)$ satisfies (\mathcal{P}_x) and (\mathcal{P}_∂) , for any $f(x) = \sum a_n \psi_n(x)$ solution of a **linear differential** equation with polynomial coefficients, the coefficients a_n are cancelled by a **linear recurrence relation** with polynomial coefficients.

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Applications:

- Efficient numerical computation of the coefficients.
- Computation of closed-form for the coefficients (when it's possible).

Previous work

- Clenshaw (1957): **numerical scheme** to compute the coefficients when $\psi_n(x) = T_n(x)$ (**Chebyshev** series).
- Lewanowicz (1976-2004): **algorithms to compute a recurrence relation** when ψ_n is an **orthogonal or semi-orthogonal polynomial family**.
- Rebillard and Zakrajšek (2006): **General algorithm** computing a recurrence relation when ψ_n is a family of hypergeometric polynomials
- Benoit and Salvy (2009) : **Simple unified presentation** and complexity analysis of the previous algorithms using **Fractions of recurrence relations** when $\psi_n = T_n$. New and fast algorithm to compute the Chebyshev recurrence.

New Results (2012)

- Simple unified presentation of the previous algorithms using [Pairs of recurrence relations](#).
- **New general algorithm** computing the recurrence relation of the coefficients for a Generalized Fourier Series when $\psi_n(x)$ satisfies (\mathcal{P}_x) and (\mathcal{P}_∂) .

II Pairs of Recurrence Relations

Examples: Chebyshev case ($f(x) = \sum u_n T_n(x)$)

Basic rules:

$$\begin{aligned}
 xf &= \sum a_n T_n && \underbrace{(\mathcal{P}_x)} && a_n &= \frac{u_{n-1} + u_{n+1}}{2} \\
 f' &= \sum b_n T_n && \underbrace{(\mathcal{P}_\partial)} && b_{n-1} - b_{n+1} &= 2nu_n.
 \end{aligned}$$

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$$f' = \sum b_n T_n \quad (\mathcal{P}_\partial) \qquad b_{n-1} - b_{n+1} = 2n u_n.$$

Combine:

$$f' + 2xf = \sum c_n T_n \quad (\mathcal{P}_\partial + 2\mathcal{P}_x) \qquad c_{n-1} - c_{n+1} = \text{Rec}_1(u_n).$$

Application: Chebyshev series for $\exp(-x^2)$.

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$$(f' + 2xf)' = \sum d_n T_n \quad \underbrace{(\mathcal{P}_\partial)} \qquad d_{n-1} - d_{n+1} = 2nc_n,$$

\rightarrow

$$\text{Rec}_2(d_n) = \text{Rec}_3(u_n),$$

$$(f' + 2xf)' - 2f = \sum e_n T_n \quad \rightarrow$$

$$\text{Rec}_4(e_n) = \text{Rec}_5(u_n).$$

Application: Chebyshev series for $\text{erf}(x)$.

Rings of Pairs of Recurrence Relations

Theorem (Least Common Left Multiple (Ore 33))

Given Rec_1 and Rec_2 , there exists a recurrence relation Rec and a pair $(\widetilde{\text{Rec}}_1, \widetilde{\text{Rec}}_2)$ such that for all sequences $(u_n)_{n \in \mathbb{N}}$:

$$\text{Rec}(u_n) = \widetilde{\text{Rec}}_1 \circ \text{Rec}_1(u_n) = \widetilde{\text{Rec}}_2 \circ \text{Rec}_2(u_n)$$

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- The LCLM is the recurrence relation Rec with **minimal order**.
- Computation : Euclidean algorithm.

Operations of addition and composition

$$\text{Rec} = \text{lcm}(\text{Rec}_1, \text{Rec}_2) = \widetilde{\text{Rec}}_1 \circ \text{Rec}_1 = \widetilde{\text{Rec}}_2 \circ \text{Rec}_2$$

Operation 1: Addition

$$\text{Rec}_1(a_n) = \widetilde{\text{Rec}}_3(u_n), \text{Rec}_2(b_n) = \widetilde{\text{Rec}}_4(u_n)$$

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$$\rightarrow \text{Rec}(a_n + b_n) = \left(\widetilde{\text{Rec}}_1 \circ \text{Rec}_3 + \widetilde{\text{Rec}}_2 \circ \text{Rec}_4 \right) (u_n).$$

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Operation 2: Composition

$$\text{Rec}_1(u_n) = \text{Rec}_3(a_n), \text{Rec}_2(u_n) = \text{Rec}_4(b_n)$$

$$\text{Rec}(u_n) = \widetilde{\text{Rec}}_1 \circ \text{Rec}_1(u_n) = \widetilde{\text{Rec}}_2 \circ \text{Rec}_2(u_n)$$

$$\rightarrow \text{Rec}_1 \circ \text{Rec}_3(a_n) = \text{Rec}_2 \circ \text{Rec}_4(b_n).$$

Main Result

Main Result : Morphism

There exists a morphism φ such that if $f = \sum u_n \psi_n(x)$ and $g = \sum v_n \psi_n(x)$ are related by $L(f) = g$ (L a linear differential operator), then:

$$\varphi(L) = (\text{Rec}_1, \text{Rec}_2) \quad \text{with} \quad \text{Rec}_1(u_n) = \text{Rec}_2(v_n)$$

In particular if $L(f) = 0$, then $\text{Rec}_1(u_n) = 0$.

Definition of the Morphism φ

$$f = \sum u_n \psi_n(x) \quad g = \sum v_n \psi_n(x)$$

$$\mathcal{R}ec_{x_2}(x\psi_n(x)) = \mathcal{R}ec_{x_1}(\psi_n(x))$$

 $\varphi(x)$

$$\text{if } xf = g, \text{ then} \\ \mathcal{R}ec_{x_2}(u_n) = \mathcal{R}ec_{x_1}(v_n)$$

$$\mathcal{R}ec_{\partial_2}(\psi'_n(x)) = \mathcal{R}ec_{\partial_1}(\psi_n(x))$$

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$$\text{if } f' = g, \text{ then} \\ \mathcal{R}ec_{\partial_1}(u_n) = \mathcal{R}ec_{\partial_2}(v_n)$$

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Example for Chebyshev series:

$$2xT_n(x) = T_{n+1}(x) + T_{n-1}(x)$$

$$\frac{T'_{n+1}(x)}{n+1} - \frac{T'_{n-1}(x)}{n-1} = 2T_n(x)$$

 φ

$$u_{n+1} + u_{n-1} = 2v_n$$

$$2u_n = \frac{1}{n} (v_{n-1} - v_{n+1})$$

Example for Bessel series

$$\frac{1}{n} (xJ_{n+1} - xJ_{n-1}) = 2J_n$$

$$2J'_n(x) = J_{n-1}(x) - J_{n+1}(x)$$

 φ

$$2u_n = \frac{v_{n+1}}{n+1} + \frac{v_{n-1}}{n-1}$$

$$u_{n+1} - u_{n-1} = 2v_n$$

General Algorithm

Recall

- Definition of $\varphi(x)$ and $\varphi(\partial)$
- Algorithms to compute addition and composition between two pairs

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General Algorithm

We deduce from this morphism a general Horner-like **algorithm to compute the recurrence relation** satisfied by the coefficients of a generalized Fourier series solution of a linear differential equation.

III Recurrences of Smaller Order

Greatest Common **Left** Divisor and Reduction of Order

GCLD

Given a pair $(\text{Rec}_1, \text{Rec}_2)$, the Euclidean algorithm computes the greatest recurrence relation Rec (GCLD) such that there exists a pair $(\widetilde{\text{Rec}}_1, \widetilde{\text{Rec}}_2)$ with the following relations for all sequences $(u_n)_{n \in \mathbb{N}}$ and $(v_n)_{n \in \mathbb{N}}$:

$$\text{Rec} \circ \widetilde{\text{Rec}}_1 (u_n) = \text{Rec}_1 (u_n)$$

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The **orders** of the recurrence relations $\widetilde{\text{Rec}}_i$ are **at most** those of Rec_i .

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Remark

In a general case, we don't have :

$$\text{Rec}_1(u_n) = \text{Rec}_2(v_n) \Rightarrow \widetilde{\text{Rec}}_1(u_n) = \widetilde{\text{Rec}}_2(v_n),$$

GLCD for reduction of order

Theorem

Given L a linear differential operator, $f = \sum u_n \psi_n(x)$, $g = \sum v_n \psi_n(x)$ such that $L(f) = g$ and a pair $(\text{Rec}_1, \text{Rec}_2) = \varphi(L)$. We have

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Application: Adaptation of the previous algorithm

At the end of the previous algorithm, add a final step:
Remove the GLCD of the two recurrence relations of the pair.

Example of reduction for Chebyshev series

$$\sqrt{1-x^2} = \sum_{n \in \mathbb{N}} \frac{4}{\pi(2n+1)} T_{2n}(x) = \sum_{n \in \mathbb{N}} c_n T_n(x)$$

$\sqrt{1-x^2}$ is the solution of the differential equation:

$$xy(x) + (1-x^2)y'(x) = 0$$

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With the general algorithm we obtain the pair of recurrence relations :

$$\text{Rec}_1(u_n) = (n+3)u_{n+2} - 2nu_n + (n-3)u_{n-2} \text{ and } \text{Rec}_2(v_n) = 2(-v_{n+1} + v_{n-1}).$$

We deduce : $(n+3)c_{n+2} - 2nc_n + (n-3)c_{n-2} = 0.$

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We deduce : $(n+2)c_{n+1} - (n-2)c_{n-1} = 0$.

Counterexample of the minimality by Lewanowicz

$x \exp(x)$ is solution of

$$deq := xy(x)' - (x + 1)y = 0.$$

By the morphism we obtain the recurrence

$$-u_n + 2nu_{n+1} + (2n + 8)u_{n+3} + u_{n+4}.$$

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This function is also solution of

$$deq2 = (-1 + x^2)y''' - (x^2 - 3x - 1)y'' - (4x - 1)y' - 3y(x) = 0.$$

By the morphism, we obtain the recurrence relation

$$(-n^2 - 3n - 3)u_n + (2n^3 + 6n^2 + 6n + 2)u_{n+1} + (n^2 + n + 1)u_{n+2}.$$

IV Conclusion

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- New general algorithm.
- Use of the GCLD to reduce order of the recurrence.

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Perspectives:

- Computation of the recurrence of **minimal order**.
- Numerical computation of the coefficients.
- Closed form for the coefficients.

Example

$$\operatorname{erf}(x) = \sum_{n=0}^{\infty} 2 \frac{4^{-n} (-1)^n {}_1F_1(n + 1/2; 2n + 2; -1)}{\sqrt{\pi} (2n + 1) n!} T_{2n+1}(x).$$

- Integration in the Dynamic Dictionary of Mathematical Functions.