On Contraction Method in function spaces and the partial match problem

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joint work with N. Broutin & R. Neininger

Contraction Method - Example

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Example - Quickselect

Task: Given a list of n different numbers, find the element of rank k , for simplicity assume $k = 1$.

Algorithm:

- Choose one element x uniformly at random among all (*pivot*)
- Comparing all elements with x gives sublists $S_<$ and $S_>$
- If $I_n = |S_<| = 1$ return x otherwise search recursively in $S_<$

Quickselect - Analysis

Let X_n be the number of key comparisons and $I_n = |S_<|$. Then

$$X_n \stackrel{d}{=} X_{I_n} + n - 1$$

for $(X_j), I_n$ independent and I_n uniformly distributed on $\{0, \ldots, n-1\}$.

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Observe that Y (or rather $\mathcal{L}(Y)$) satisfies this if $\mathcal{L}(Y)$ is a fixed-point of the following map

$$\begin{aligned} F &: \mathcal{M}(\mathbb{R}) \to \mathcal{M}(\mathbb{R}) \\ F(\mu) &= \mathcal{L}(UY + 1), \end{aligned}$$

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Idea: Use Banach fixed point theorem.

Quickselect - Contraction

For $\mu, \nu \in \mathcal{M}(\mathbb{R})$ let

$$\ell_1(\mu, \nu) = \inf_{X,Y:\mathcal{L}(X)=\mu,\mathcal{L}(Y)=\nu} \mathbb{E}[|X - Y|].$$

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ℓ_1 is a complete metric on the subset $\mathcal{M}'(\mathbb{R})$ of $\mathcal{M}(\mathbb{R})$ consisting of probability measures with finite first moment and

$$\ell_1(\mu_n, \mu) \to 0 \Rightarrow \mu \xrightarrow{w} \mu$$

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. Show: F is a contraction according to ℓ_1 in $\mathcal{M}'(\mathbb{R})$.

Quickselect - Contraction

Proof: Let X, Y s.t. $\mathcal{L}(X) = \mu$ and $\mathcal{L}(Y) = \nu$ and

$$\mathbb{E}[\|X - Y\|] \leq \ell_1(\mu, \nu) + \varepsilon.$$

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Then

$$\begin{aligned} \ell_1(F(\mu), F(\nu)) &\leq \mathbb{E}[\|UX + 1 - (UY + 1)\|] = \mathbb{E}U\mathbb{E}[\|X - Y\|] \\ &\leq \mathbb{E}U\ell_1(\mu, \nu) + \varepsilon \end{aligned}$$

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which gives

$$\ell_1(F(\mu), F(\nu)) \leq \mathbb{E}U\ell_1(\mu, \nu).$$

□

Quickselect - Contraction

The stochastic fixed-point equation

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Typically the number of subproblems is larger than one. For example, if X_n denotes the number of key comparisons performed by Quicksort sorting a list of n elements, then

$$X_n \stackrel{d}{=} X'_{I_n} + X''_{n-1-I_n} + n - 1$$

with independent copies $(X'_j), (X''_j)$ independent of I_n.

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Usual situation of an affine recursion after scaling:

$$X_n \stackrel{d}{=} \sum_{r=1}^{K} A_r^{(n)} \circ X^r_{I_r^{(n)}} + b^{(n)}$$

with $r.v.$ $(X_n), b^{(n)}$ taking values in some space S, $A_r^{(n)}$ random operators from S to S, and independent copies $(X_n^1), \dots, (X_n^K)$ of (X_n).

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If $A_r^{(n)} \to A_r$ and $b^{(n)} \to b$ for some S valued processes A_r, b, this suggests

$$X_n \to X,$$

where X solves $X \stackrel{d}{=} \sum_{r=1}^{K} A_r \circ X^{(r)} + b$ (uniquely).

Applications - The $\mathbb{R}^d$ case

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- $K = 1$: Quickselect
- $K = 2$: BST, RRT: Pathlength, Profile,. . . , Size of random Tries
- $K = m$: m-ary search trees
- $K = K(n)$ random: Galton-Watson trees
- $d = 2$: Wiener Index

Example in $\mathcal{C}([0,1])$ - Donsker's Theorem

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Let $X_1, X_2, \dots$ be iid random variables with $\mathbb{E}X_1 = 0, \mathbb{E}X_1^2 = 1$.
The process

$$S_t^n = \frac{1}{\sqrt{n}} \left(\sum_{k=1}^{\lfloor nt \rfloor} X_k + (nt - \lfloor t \rfloor) X_{\lfloor nt \rfloor + 1} \right), \quad t \in [0,1]$$

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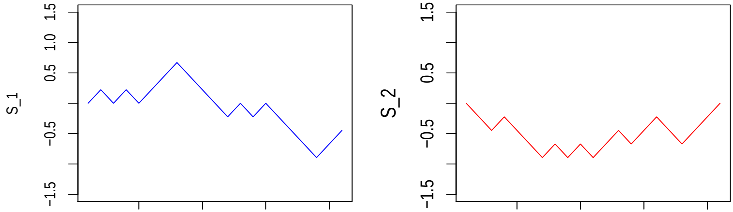
converges in distribution to a standard Brownian Motion in $\mathcal{C}([0,1])$ endowed with the uniform topology.

Example in $\mathcal{C}([0,1])$ - Donsker's Theorem



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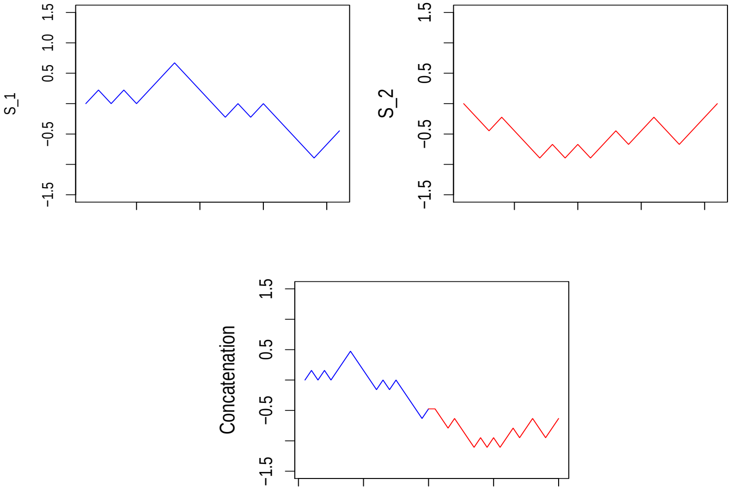

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It holds:

$$(S_t^n)_{t\in[0,1]} \stackrel{d}{=} \left(\sqrt{\frac{1}{2}}\left(\mathbf{1}_{\{t\leq 1/2\}} S_{2t}^{n/2} + \mathbf{1}_{\{t>1/2\}}\left(S_1^{n/2} + \widehat{S}_{2t-1}^{n/2}\right)\right)\right)_{t\in[0,1]}$$

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&= \left(A_1^{(n)} \circ S_t^{n/2} + A_2^{(n)} \circ \widehat{S}_t^{n/2}\right)_{t\in[0,1]}
\end{aligned}
$$

Observe that $A_r^{(n)}$ and $I_r^{(n)}$ are deterministic and $A_r^{(n)}$ is linear and bounded for $r = 1, 2$.

Example in $\mathcal{D}([0,1])$ - Partial match query

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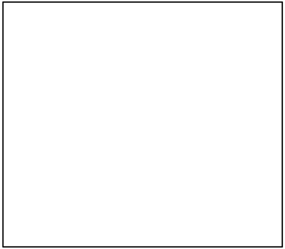
We consider a partial match query in a random two-dimensional quadtree under the uniform model.

Quadtrees, introduced by Finkel & Bentley in 1974, are generalizations of BSTs and used to store multidimensional data.

Partial match query in quadtrees



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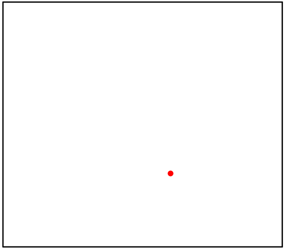

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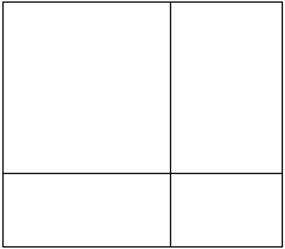

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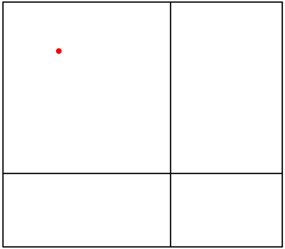

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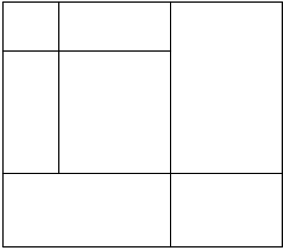

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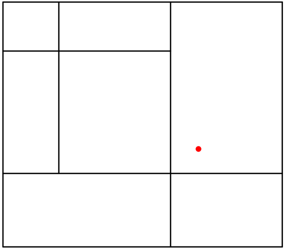

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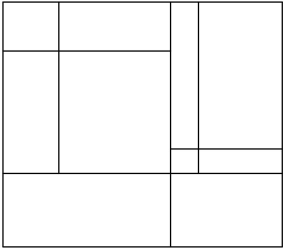

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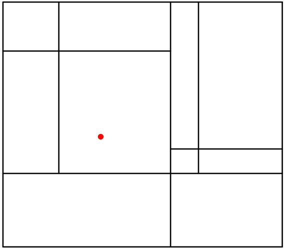

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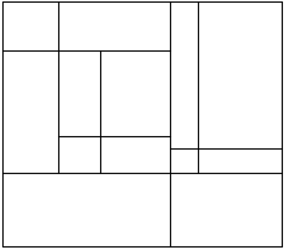

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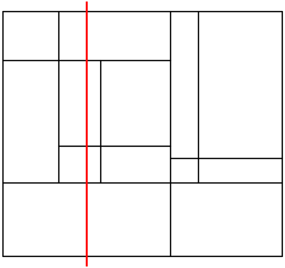

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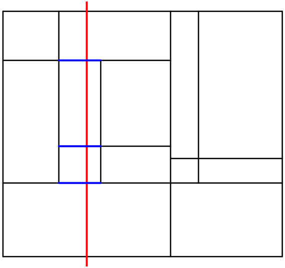

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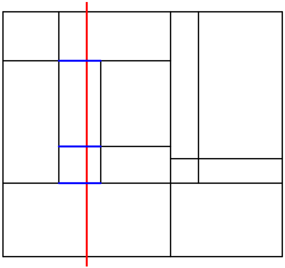

Input: $U_1, U_2, \ldots, U_n$ independent and uniformly on $[0, 1]$ distributed.

$C_n(t)$ number of horizontal lines intersecting the vertical line $x = t$.

Partial match query - Recursion

Let $I_1^{(n)}, I_2^{(n)}, I_3^{(n)}, I_4^{(n)}$ be the number of points in the four quadrants and (U_1, V_1) be the coordinates of the first point.

Partial match query - Recursion

Let $I_1^{(n)}, I_2^{(n)}, I_3^{(n)}, I_4^{(n)}$ be the number of points in the four quadrants and (U_1, V_1) be the coordinates of the first point. Then

$$\begin{aligned}(C_n(t))_{t\in[0,1]} \stackrel{d}{=} & \Bigg(\mathbf{1}_{\{t<U_1\}}\left(C^{(1)}_{I_1^{(n)}}\left(\frac{t}{U_1}\right)+C^{(2)}_{I_2^{(n)}}\left(\frac{t}{U_1}\right)\right)\\ &+\mathbf{1}_{\{t\geq U_1\}}\left(C^{(3)}_{I_3^{(n)}}\left(\frac{t-U_1}{1-U_1}\right)+C^{(4)}_{I_4^{(n)}}\left(\frac{t-U_1}{1-U_1}\right)\right)+1\Bigg)_{t\in[0,1]}\end{aligned}$$

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considered as process in $\mathcal{D}([0,1])$. Given (U_1, V_1) it holds

$$\mathcal{L}\left(I_1^{(n)}, I_2^{(n)}, I_3^{(n)}, I_4^{(n)}\right) = M(n-1; U_1 V_1, U_1(1-V_1), (1-U_1)V_1, (1-U_1)(1-V_1)).$$

The Zolotarev metric on a Banach space

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$$\zeta_s(\mu, \nu) = \sup_{f \in \mathcal{F}_s} |\mathbb{E}[f(X) - f(Y)]|,$$

with $\mathcal{L}(X) = \mu, \mathcal{L}(Y) = \nu$ and for $s = m + \alpha$ with $0 < \alpha \leq 1$ and $m := \lceil s \rceil - 1 \in \mathbb{N}_0$

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Set $\zeta_s(X, Y) = \zeta_s(\mathcal{L}(X), \mathcal{L}(Y))$.

Properties of the Zolotarev distance ζ_s

It holds $\zeta_s(X, Y) < \infty$, if

$$\mathbb{E}||X||^s, \mathbb{E}||Y||^s < \infty, \quad \mathbb{E}[g(X, \dots, X)] = \mathbb{E}[g(Y, \dots, Y)]$$

for all $k \leq m$ and multilinear, bounded functions $g : B^k \to \mathbb{R}$. In the following assume finiteness of the considered ζ_s-distances.

Properties of the Zolotarev distance ζ_s

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Lemma

ζ_s is $(s,+)$-ideal, i.e.

$$\zeta_s(\varphi(X),\varphi(Y)) \leq ||\varphi||^s \zeta_s(X,Y)$$

for any continuous and linear function $\varphi : \mathcal{B} \to \mathcal{B}$ with

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Furthermore

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for (X_1, Y_1) and (X_2, Y_2) independent.

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Proof: Zolotarev ['77]

Properties of the Zolotarev distance ζ_s

Theorem

Let $\mathcal{B} = \mathcal{C}([0,1])$*. Then* $\zeta_s(X_n, X) \to 0$ *implies*

- $X_n \xrightarrow{fdd} X$ *and* $\mathbb{E}[|X_n(t)|^s] \to \mathbb{E}|X(t)|^s]$ *for all* t
- $X_(U) \xrightarrow{d} X(U)$ *and* $\mathbb{E}[|X_n(U)|^s] \to \mathbb{E}|X(U)|^s]$ *for any random variable* U *on* $[0,1]$ *independent of* (X_n) *and* X*.*

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The question whether ζ_s convergence implies weak convergence is strongly related to the regularity of the norm function $||x||$.

Properties of ζ_s in $\mathcal{C}([0,1])$

Theorem

Let $B = \mathcal{C}([0,1])$ and $0 < s \leq 3$, i.e, $m \in \{0,1,2\}$. Let $(X_n)_{n\geq 1}$, X be random variables in $\mathcal{C}([0,1])$ where, for each $n \geq 1$, X_n is piecewise linear on intervals of length at least r_n. If

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Proof: The proof is based upon the approximation $||x||_p \to ||x||$ for $p \to \infty$. These ideas go back to a paper of Barbour in the context of Steins method.

General Theorem for $1 < s \leq 2$

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Assume X_n *is recursively given by*

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- $||X_n||_s < \infty, ||A_r^{(n)} - A_r||_s \to 0, ||b^{(n)} - b||_s \to 0,$
- $\sum_{r=1}^{K} \mathbb{E}[||A_r||^s] < 1,$
- *X solves the fixed-point equation with $\mathbb{E}[X(t)] = f(t)$ and $||X||_s < \infty$.*

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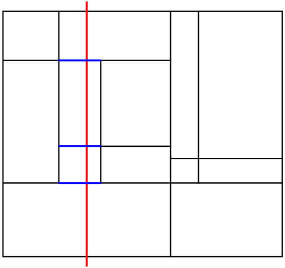
- X *solves the fixed-point equation with* $\mathbb{E}[X(t)] = f(t)$ *and* $||X||_s < \infty$ *.*

Then $\zeta_s(X_n, X) \to 0$ *. Appropriate rates for the convergence of the coefficients imply* $X_n \to X$ *in distribution in* $((\mathcal{C}([0,1]), || \cdot ||_{sup})$*.*

Application - Partial match query



Input: $U_1, U_2, \ldots, U_n$ independent and uniformly on $[0, 1]$ distributed.

$C_n(t)$ number of horizontal lines intersecting the vertical line $x = t$.

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Theorem (Curien, Joseph '10+)

$\mathbb{E}[C_n(t)] = C_2(t(1-t))^{\beta/2} n^\beta + o(n^\beta).$

Partial match - Recursion and scaling

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Remember the recursion

$$
\begin{aligned}
(C_n(t))_{t\in[0,1]} \quad &\stackrel{d}{=} \quad \Bigg(\mathbf{1}_{\{t<U_1\}} \left(C^{(1)}_{I_1^{(n)}}\left(\frac{t}{U_1}\right) + C^{(2)}_{I_2^{(n)}}\left(\frac{t}{U_1}\right)\right) \\
&+ \quad \mathbf{1}_{\{t\geq U_1\}} \left(C^{(3)}_{I_3^{(n)}}\left(\frac{t-U_1}{1-U_1}\right) + C^{(4)}_{I_4^{(n)}}\left(\frac{t-U_1}{1-U_1}\right)\right) + 1 \Bigg)_{t\in[0,1]}
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$$+\ \mathbf{1}_{\{t\geq U_1\}}\left(C^{(3)}_{I_3^{(n)}}\left(\frac{t-U_1}{1-U_1}\right)+C^{(4)}_{I_4^{(n)}}\left(\frac{t-U_1}{1-U_1}\right)\right)+1\Bigg)_{t\in[0,1]}$$

Let $\mathbb{E}C_n(t)] = a_n(t)C_2(t(1-t))^{\beta/2}n^\beta$ and

$$Y_n(t) = \frac{C_n(t)}{a_n(t)C_2 n^\beta}.$$

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The recursion in terms of Y_n is

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\begin{aligned}
(Y_n(t))_{t\in[0,1]} \quad \stackrel{d}{=} \quad & \Bigg(\mathbf{1}_{\{t<U_1\}} \left(\frac{I_1^{(n)}}{n}\right)^{\beta} \frac{a_{I_1^{(n)}}(t)}{a_n(t)} Y^{(1)}_{I_1^{(n)}}\left(\frac{t}{U_1}\right) \\
+ \quad & \mathbf{1}_{\{t<U_1\}} \left(\frac{I_2^{(n)}}{n}\right)^{\beta} \frac{a_{I_2^{(n)}}(t)}{a_n(t)} Y^{(2)}_{I_2^{(n)}}\left(\frac{t}{U_1}\right) \\
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Partial match - Fixed point equation

This gives rise to following fixed-point equation

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Existence of a continuous solution Y with $\mathbb{E}[Y(t)] = (t(1-t))^{\beta/2}$ and $\mathbb{E}[||Y||^2] < \infty$ is proved.

Partial match - Convergence

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Uniform convergence of the coefficients, i.e. $\mathbb{E}[||A_r^{(n)} - A_r||^2] \to 0$, follows from uniformity of the convergence of

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Lemma (Monotonicity)

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Now, the general Theorem implies

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Let Y be the solution of the appearing fixed-point equation. Then $\zeta_s(Y_n, Y) \to 0$.

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- $\frac{C_n(t)}{C_2 n^\beta} \xrightarrow{d,L^2} Y(t)$
- $\frac{C_n(U)}{C_2 n^\beta} \xrightarrow{d,L^2} Y(U)$
- $\mathrm{Var}\, C_n(U) = \gamma n^{2\beta} + o(n^{2\beta})$

for U uniform on $[0,1]$, independent of (C_n) and C.

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For $\mu_2(u) = \mathbb{E}[Y(u)^2]$ *it holds*

$$\begin{aligned}\mu_2(t) \quad &= \quad \int_0^t \frac{2(1-u)^{2\beta}}{2\beta+1}\mu_2\left(\frac{t-u}{1-u}\right) + 2B(\alpha)(1-u)^{2\beta}\left(\frac{t-u}{1-u}\frac{1-t}{1-u}\right)^\beta du \\ &+ \quad \int_t^1 \frac{2u^{2\beta}}{2\beta+1}\mu_2\left(\frac{t}{u}\right) + 2B(\alpha)u^{2\beta}\left(\frac{t}{u}\frac{u-t}{u}\right)^\beta du\end{aligned}$$

Future work

- weak convergence of Y_n in $(\mathcal{D}([0,1]), ||\cdot||_{\text{sup}})$
- results for $\sup_{t\in[0,1]} C_n(t)$
- higher dimensions
- related tree models (K-d trees, ...)