LLL-reducing in quasi-linear time

Damien Stehlé
Joint work with A. Novocin & G. Villard

LIP – CNRS/ENSL/INRIA/UCBL/U. Lyon

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Euclidean lattices

Lattice $\equiv$ discrete subgroup of $\mathbb{R}^n$

$$\equiv \{ \sum_{i \leq n} x_i b_i : x_i \in \mathbb{Z} \}$$

If the $b_i$'s are linearly independent, they are called a basis.

Bases are not unique, but they can be obtained from each other by integer transforms of determinant $\pm 1$:

$$\begin{bmatrix} -2 & 1 \\ 10 & 6 \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ 2 & 4 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}.$$
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Lattice reduction: a representation paradigm

**Lattice reduction:** Start from an arbitrary basis, and improve the norms/orthogonality of its vectors.

What for?

- Shorter vectors $\Rightarrow$ less space.
- Reduced bases provide intrinsic information about the lattice.
- Reduced bases are easier to compute with.

Lattice reduction as a matrix problem:

Given $B \in \mathbb{R}^{n \times n}$ full-rank, find $U \in GL_n(\mathbb{Z})$ s.t.

$$BU \text{ small and/or with a "nice" QR-factor } R.$$
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Why do we care about lattices?

- Computer algebra: factorisation of rational polynomials.
- Cryptography: cryptanalyses of variants of RSA.
- Communications theory: MIMO, GPS, error correcting codes.
- Combinatorial optimisation, algorithmic group theory, algorithmic number theory, computer arithmetic, etc.
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Lattices tend to pop out every time one wants to use linear algebra but is restricted to discrete transformations.
The LLL reduction [Lenstra-Lenstra-Lovász’82]

Let $\delta \in (1/4, 1)$. A basis $B = (b_i)_{i \leq n} \in \mathbb{R}^{n \times n}$ with QR-factorisation $B = QR$ is said LLL-reduced if:

- $\forall i, j : |r_{i,j}| \leq r_{i,i}/2$ [size-reduction]
- $\forall i : \delta \cdot r_{i,i}^2 \leq r_{i,i+1}^2 + r_{i+1,i+1}^2$ [Lovász’ condition].
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The \( r_{i,i}'s \) can’t drop too fast: \( r_{i+1,i+1} \geq \sqrt{\delta - \frac{1}{4} r_{i,i}} \).

\[\prod_i \|b_i\| \leq 2^{O(n^2)} \cdot \det(L)\].

\( \det(L) := \det(b_i)_i \) is a lattice invariant. \( \delta < 1 \) is crucial to get polynomial-time complexity.
The LLL reduction \cite{LLL82}

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Complexity bounds

Input: \( B \in \mathbb{Z}^{n \times n} \) of full rank, with \( \max \| b_i \| \leq 2^\beta \).

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Can we do better with respect to $\beta$?
Quasi-linear LLL-reduction

- Yap’92, Schönhage’91: $\beta^{1+\varepsilon}$ for $n = 2$.
- Eisenbrand-Rote’01: $\beta^{1+\varepsilon}$ for fixed any $n$.

**Our result**

We give an algorithm, called $\tilde{L}^1$, that computes “somewhat” LLL-reduced bases in time $O(n^{5+\varepsilon} \beta + n^{\omega+1+\varepsilon} \beta^{1+\varepsilon})$.

- $n^{\omega}$: cost of matrix mult. in dimension $n$.
- For fixed $n$: $O(M(\beta) \log \beta)$, where $M(\cdot)$ is for integer mult.
- Same total degree as before.
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Plan of the talk

1. Wishful thinking.
2. Reducing by deforming.
3. Reducing by truncating.
4. The $\tilde{L}^1$ algorithm.
A gcd analogy

Euclid’s algorithm for computing gcd\((r_0, r_1)\):

\[ i := 1. \text{ While } r_i \neq 0: \]
\[ \text{Compute } q_i := \lfloor r_{i-1}/r_i \rfloor, \quad r_{i+1} := r_{i-1} - q_i r_i. \]
Output \(r_{i-1}\).

Vectorial interpretation:

\[
\begin{pmatrix} r_i \\ r_{i+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & -q_i \end{pmatrix} \cdot \begin{pmatrix} r_{i-1} \\ r_i \end{pmatrix} = \prod_{j=i}^{1} \begin{pmatrix} 0 & 1 \\ 1 & -q_j \end{pmatrix} \cdot \begin{pmatrix} 0 \\ r_1 \end{pmatrix}
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LLL as a gcd: Given \(B_i\), find \(U_i\) s.t. \(B_i U_i\) is closer to reduced.
- \(L^3\): Compute \(r_{i-1}/r_i\) exactly before rounding it.
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Towards a quasi-linear time gcd algorithm

Euclid computes remainders $(r_i)_i$ and quotients $(q_i)_i$.

- Assume $r_0 \approx r_1 \approx 2^\beta$.
- Writing down all the $r_i$’s costs $O(\beta^2)$.

**Lehmer’38**

If $\frac{|r_0 - \bar{r}_0|}{r_0}, \frac{|r_1 - \bar{r}_1|}{r_1} \leq 2^{-2\ell}$, then $(q_i)_i$ and $(\bar{q}_i)_i$ share their first $\ell$ bits.

- Do not compute the $q_i$’s using and updating the lengthy $r_i$’s: Use the shorter $\bar{r}_i$’s instead!
- When the relevant bits of the $q_i$’s are known, apply them to $(r_0, r_1)$... and apply Lehmer again.
- Knuth’70, Schönhage’71: Do this recursively!
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The Knuth-Schönhage gcd algorithm

To compute the first $\ell$ quotient bits of $r_0, r_1$ of bit-sizes $2\ell$:

1. Take the first $\ell$ bits of $r_0$ and $r_1$.
2. Recursively get the first $\ell/2$ quotient bits.
3. Apply the quotients to $r_0, r_1$, to get $r'_0, r'_1$.
4. Take the first $\ell$ bits of $r'_0$ and $r'_1$.
5. Recursively get the first $\ell/2$ quotient bits.

- Applying the quotients: multiply a $O(\ell)$-bit $2 \times 2$ matrix to a $O(\ell)$-bit vector.
- Cost: $C_\ell = 2C_{\ell/2} + O(M(\ell)) = O(M(\ell) \log \ell)$.
- Can be used to compute gcdfs in time $O(M(\ell) \log \ell)$. 

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What about doing it for LLL?

To compute the “first” $\ell$ bits of $U$ reducing $B$:

1. Take the first $\ell$ bits of each $b_{ij}$.
2. Recursively get the first $\ell/2$ bits of $U$.
3. Apply them to $B$, to get a shorter $B'$.
4. Take the first $\ell$ bits of each $b'_{ij}$.
5. Recursively get the next $\ell/2$ bits of $U$.

- What is a “quotient” here?
- How to control the bit-size of a unimodular matrix?
- Can we truncate “remainders”, i.e., lattice bases?
- How to handle multidimensionality / unbalanced magnitudes?
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From reduced to reduced

- If $B$ is arbitrary, then a reducing $U$ can be huge (Cramer :-().
- If $B$ is reduced, any $U$ such that $BU$ is reduced is bounded.

Let $B$ be reduced with R-factor $R$, and $U$ s.t. $BU$ is reduced. Then:

$$\forall i, j : |u_{ij}| \leq 2^{O(n)} \cdot \frac{r_{jj}}{r_{ii}}.$$ 

- If $B$ is reduced, the $r_{ii}$’s can’t decrease fast.
- Assuming they don’t increase, we get $\max |u_{ij}| \leq 2^{O(n)}$. 
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Start from something reduced, deform it a bit, and reduce it!

The Belabas-van Hoeij-Novocin deformation:

\[ B \rightarrow \text{diag}(2^\ell, 1, \ldots, 1) \cdot B = \sigma_\ell B. \]

The \( r_{ii} \)'s cannot decrease.

Their product increases by a factor \( 2^\ell \).

Let \( \ell \geq 0 \), \( B \) be reduced with R-factor \( R \), and \( U \) s.t. \( \sigma_\ell BU \) is reduced. Then:

\[ \forall i, j : |u_{ij}| \leq 2^{\ell + O(n)} \cdot r_{jj} / r_{ii}. \]

\[ \rightarrow \] If \( B \) is “balanced”, each \( u_{ij} \) has at most \( \ell + O(n) \) bits.
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- Their product increases by a factor \( 2^\ell \).

Let \( \ell \geq 0 \), \( B \) be reduced with R-factor \( R \), and \( U \) s.t. \( \sigma_\ell BU \) is reduced. Then:

\[ \forall i, j : |u_{ij}| \leq 2^{\ell+\mathcal{O}(n)} \cdot r_{jj}/r_{ii}. \]

\( \rightarrow \) If \( B \) is “balanced”, each \( u_{ij} \) has at most \( \ell + \mathcal{O}(n) \) bits.
From reduced to deformed to reduced

- Start from something reduced, deform it a bit, and reduce it!
- The Belabas-van Hoeij-Novocin deformation:
  \[ B \mapsto \text{diag}(2^\ell, 1, \ldots, 1) \cdot B = \sigma_\ell B. \]
- The \( r_{ii} \)'s cannot decrease.
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Let \( \ell \geq 0 \), \( B \) be reduced with R-factor \( R \), and \( U \) s.t. \( \sigma_\ell BU \) is reduced. Then:

\[ \forall i, j : |u_{ij}| \leq 2^\ell + O(n) \cdot \frac{r_{jj}}{r_{ii}}. \]

\[ \rightarrow \text{If } B \text{ is “balanced”, each } u_{ij} \text{ has at most } \ell + O(n) \text{ bits.} \]
Lift-reducing suffices for reducing

Assume $B \in \mathbb{Z}^{n \times n}$ is upper triangular.

$$
\begin{bmatrix}
  b_{1,1} & \ldots & b_{1,n-2} & b_{1,n-1} & b_{1,n} \\
  \vdots & \ddots & \vdots & \vdots & \vdots \\
  0 & \ldots & b_{n-2,n-2} & b_{n-2,n-1} & b_{n-2,n} \\
  0 & \ldots & 0 & b_{n-1,n-1} & b_{n-1,n} \\
  0 & \ldots & 0 & 0 & b_{n,n}
\end{bmatrix}
$$
Lift-reducing suffices for reducing

Assume $B \in \mathbb{Z}^{n \times n}$ is upper triangular.

\[
\begin{bmatrix}
    b_{1,1} & \ldots & b_{1,n-2} & b_{1,n-1} & \vert & b_{1,n} \\
    \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
    0 & \ldots & b_{n-2,n-2} & b_{n-2,n-1} & b_{n-2,n} \\
    0 & \ldots & 0 & b_{n-1,n-1} & b_{n-1,n} \\
    0 & \ldots & 0 & 0 & b_{n,n}
\end{bmatrix}
\]

Bottom right $1 \times 1$ submatrix is reduced.
Lift-reducing suffices for reducing

Assume $B \in \mathbb{Z}^{n \times n}$ is upper triangular.

$$
\begin{bmatrix}
 b_{1,1} & \ldots & b_{1,n-2} \\
 \vdots & \ddots & \vdots \\
 0 & \ldots & b_{n-2,n-2} \\
 0 & \ldots & 0 \\
 0 & \ldots & 0
\end{bmatrix}
\begin{bmatrix}
 b_{1,n-1} & b_{1,n} \\
 \vdots & \vdots \\
 b_{n-2,n-1} & b_{n-2,n} \\
 \frac{b_{n-1,n-1}}{2^\ell} & \frac{b_{n-1,n}}{2^\ell} \\
 0 & b_{n,n}
\end{bmatrix}
$$

Scale down row $n-1$ so that bottom-right $2 \times 2$ submatrix is reduced: $\ell \approx \log b_{n-1,n-1}$. 
Lift-reducing suffices for reducing

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$$
\begin{bmatrix}
  b_{1,1} & \cdots & b_{1,n-2} \\
  \vdots & \ddots & \vdots \\
  0 & \cdots & b_{n-2,n-2} \\
  0 & \cdots & 0 \\
  0 & \cdots & 0
\end{bmatrix}
\begin{bmatrix}
  b_{1,n-1} & b_{1,n} \\
  \vdots & \vdots \\
  b_{n-2,n-1} & b_{n-2,n} \\
  b_{n-1,n-1} & b_{n-1,n} \\
  0 & b_{n,n}
\end{bmatrix}
$$

Lift row $n - 1$ by $\ell$ bits and reduce bottom-right $2 \times 2$ submatrix.
Lift-reducing suffices for reducing

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\[
\begin{bmatrix}
  b_{1,1} & \ldots & b_{1,n-2} & b_{1,n-1} & b_{1,n} \\
  \vdots & \ddots & \vdots & \vdots & \vdots \\
  0 & \ldots & b_{n-2,n-2} & b_{n-2,n-1} & b_{n-2,n} \\
  0 & \ldots & 0 & x & x \\
  0 & \ldots & 0 & x & x \\
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\begin{bmatrix}
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  0 & \ldots & b_{n-2,n-2} & x & x \\
  0 & \ldots & 0 & x & x \\
  0 & \ldots & 0 & x & x
\end{bmatrix}
\]

Propagate the transformations to the first $n - 2$ coordinates, and reduce wrt the diagonal coefficients.
Lift-reducing suffices for reducing

Assume $B \in \mathbb{Z}^{n \times n}$ is upper triangular.

\[
\begin{bmatrix}
  b_{1,1} & \cdots & b_{1,n-2} & x & x \\
  \vdots & \ddots & \vdots & \vdots & \vdots \\
  0 & \cdots & b_{n-2,n-2} & 2^\ell & 2^\ell \\
  0 & \cdots & 0 & x & x \\
  0 & \cdots & 0 & x & x \\
\end{bmatrix}
\]

Scale down row $n-2$ so that bottom-right $3 \times 3$ submatrix is reduced: $\ell \approx \log b_{n-2,n-2}$. 

Damien Stehlé
LLL-reducing in quasi-linear time 11/04/2011 16/36
Lift-reducing suffices for reducing

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    \vdots & \ddots & \vdots & \vdots & \vdots \\
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  0 & \ldots & x & x & x \\
  0 & \ldots & x & x & x \\
\end{bmatrix}
\]

Keep going.
Lift-reducing in quasi-linear time suffices

- McCurley-Hafner’91: 
  \[ H = HNF(B) \text{ can be computed in time } \mathcal{O}(n^{\omega+1+\varepsilon} \beta^{1+\varepsilon}). \]

- Cost of the lifts:
  \[
  \mathcal{P}oly(n) \cdot \left( \tilde{\mathcal{O}}(\log h_{n,n}) + \tilde{\mathcal{O}}(\log h_{n-1,n-1}) + \ldots \right)
  = \mathcal{P}oly(n) \cdot \tilde{\mathcal{O}}(\log \det H)
  = \mathcal{P}oly(n) \cdot \tilde{\mathcal{O}}(\log \det B).
  \]
  (in fact, we do a bit better than that)

- Cost of the propagations bounded using the smallness of the transforms:  
  \[ \mathcal{O}(n^{\omega+1+\varepsilon} (\beta^{1+\varepsilon} + n)). \]
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- McCurley-Hafner’91:
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  \]
  \[
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Where are we now?

• LLL-reduction $\longrightarrow$ sequence of Lift-reductions.
• We are to lift-reduce in quasi-linear time.
  More precisely: given $\ell$ and $B$ reduced, we will find $U$ unimodular such that $\sigma_{\ell}BU$ is reduced, in time $\tilde{O}(\ell)$.
• This is independent from the bit-size of $B$.

• The “LLL quotients” are the matrices $U$ that achieve some amount $\ell$ of lifting.
• The quotients have bounded magnitudes.
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Plan of the talk

1. Wishful thinking.
2. Reducing by deforming.
3. Reducing by truncating.
4. The $\widetilde{L}^1$ algorithm.
The LLL-reduction is inappropriate for truncations
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We can’t decide reducedness by looking at the (53) top-most bits:
The LLL-reduction is inappropriate for truncations

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\[
\begin{bmatrix}
1 & 2^{60} + 2^5 \\
-1 & 2^{60}
\end{bmatrix}
\quad \Rightarrow \quad
\begin{bmatrix}
1 & 2^{60} \\
-1 & 2^{60}
\end{bmatrix}
\]

Not reduced

\[
\begin{bmatrix}
1 & 2^{53} + 2^{-1} + 2^{-25} \\
2^{-10} & -2^{63}
\end{bmatrix}
\quad \Rightarrow \quad
\begin{bmatrix}
1 & 2^{53} + 1 \\
2^{-10} & -2^{63}
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\]

Reduced

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\]

Reduced

If \( B \in \mathbb{Z}^{n \times n} \), we may need all the bits to decide.

If \( B \in \mathbb{R}^{n \times n} \), we may not even be able to tell!
Sensitivity of the R-factor

- Take $B \in \mathbb{R}^{n \times n}$ full-rank, with $B = QR$.
- Apply a columnwise perturbation $\Delta B$, i.e., $\max_i \frac{\|\Delta b_i\|}{\|b_i\|} \leq \varepsilon$.
- If $\varepsilon$ is very small, then $B + \Delta B$ is full-rank and:

$$B + \Delta B = (Q + \Delta Q)(R + \Delta R).$$

- How large can $\Delta R$ be?
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- How large can $\Delta R$ be?

Chang-S-Villard’11

Let $\text{cond}(R) = \|\|R\|\|R^{-1}\|\|$. If $\text{cond}(R) \cdot \varepsilon \lesssim 1$, then:

$B + \Delta B$ is full-rank and $\max \frac{\|\Delta r_i\|}{\|r_i\|} \lesssim \text{cond}(R) \cdot \varepsilon$.

Furthermore, if $B$ is LLL-reduced, then $\text{cond}(R) = 2^{O(n)}$. 

Damien Stehlé  LLL-reducing in quasi-linear time  11/04/2011  21/36
Fixing the LLL-reduction

- We would like the reduction to resist perturbations.
- The bound on $\|\Delta r_j\|$ is proportional to $\|r_j\|$.
- By reducedness, $1 \leq \frac{\|r_j\|}{r_{j,j}} \leq 2^{O(n)}$.

$\Rightarrow$ $r_{i,j}$ should be related to $r_{j,j}$ instead of (only) $r_{i,i}$.
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Fixing the LLL-reduction

- We would like the reduction to resist perturbations.
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$\Rightarrow$ $r_{i,j}$ should be related to $r_{j,j}$ instead of (only) $r_{i,i}$.

Let $\Xi = (\delta, \eta, \theta)$ with $\eta \in (1/2, 1)$, $\theta > 0$ and $\delta \in (\eta^2, 1)$. A basis $B \in \mathbb{R}^{n \times n}$ with R-factor $R$ is said $\Xi$-reduced if:

- $\forall i, j : |r_{i,j}| \leq \eta \cdot r_{i,i} + \theta \cdot r_{j,j}$  
  [Modified size-reduction]
- $\forall i : \delta \cdot r_{i,i}^2 \leq r_{i,i+1}^2 + r_{i+1,i+1}^2$. 

Damien Stehlé
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Let $\Xi = (\delta, \eta, \theta)$ with $\eta \in (1/2, 1)$, $\theta > 0$ and $\delta \in (\eta^2, 1)$.
A basis $B \in \mathbb{R}^{n \times n}$ with R-factor $R$ is said $\Xi$-reduced if:

- $\forall i, j : |r_{i,j}| \leq \eta \cdot r_{i,i} + \theta \cdot r_{j,j}$ [Modified size-reduction]
- $\forall i : \delta \cdot r^2_{i,i} \leq r^2_{i,i+1} + r^2_{i+1,i+1}$.

If $B$ is balanced, this is the same as before.
The LLL-reductions, graphically

Hermite

LLL’82

Schnorr’88

Chang-S-Villard’11

(1, 1/2, 0)

(δ, 1/2, 0)

(δ, η, 0)

(δ, η, θ)
Properties of the new reduction

The new reduction is **perturbation-friendly**:

- We still have $\text{cond}(R) = 2^{\mathcal{O}(n)}$ for $\Xi$-reduced bases.

- If $B$ is reduced and $\max \frac{\|\Delta b_i\|}{\|b_i\|} \leq 2^{-\Omega(n)}$,
  then $B + \Delta B$ is reduced (for slightly weaker parameters).
Properties of the new reduction

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- We still have $\text{cond}(R) = 2^{O(n)}$ for $\Xi$-reduced bases.
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The popular properties of LLL-reduction still hold:

- Computable in polynomial time.
- $B$ reduced $\Rightarrow \prod \|b_i\| \leq 2^{O(n^2)} \cdot |\det(b_i)_i|$. 
Deformations and truncations are compatible

- \( B \) and \( \sigma_\ell B U \) reduced \( \implies \) \( U \) small.
- \( B \) reduced \( \implies \) \( B + \Delta B \) reduced.

Let \( \ell \geq 0 \), \( B \) be reduced and \( \Delta B \) s.t. \( \max \left\| \frac{\Delta b_i}{b_i} \right\| \leq 2^{-\ell - \Omega(n)} \).

If \( \sigma_\ell (B + \Delta B) U \) is reduced, then so is \( \sigma_\ell BU \).

For slightly weaker reduction factors.

The \( \ell + O(n) \) top-most bits of \( B \) suffice for finding \( U \).
Deformations and truncations are compatible

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For slightly weaker reduction factors.

The $\ell + O(n)$ top-most bits of $B$ suffice for finding $U$. 
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Plan of the talk

1. Wishful thinking.
2. Reducing by deforming.
3. Reducing by truncating.
4. The $\sim^1 L^1$ algorithm.
Overview of $\tilde{\mathcal{L}}^1$

- $\tilde{\mathcal{L}}^1$: HNF and $n$ calls to Lift-$\tilde{\mathcal{L}}^1$.
- If $B$ is reduced and $\ell \geq 0$, Lift-$\tilde{\mathcal{L}}^1$ computes $U$ unimodular such that $\sigma_\ell BU$ is reduced, in time $\mathcal{O}(n) \cdot \tilde{O}(\ell)$.

- We master “remainders/bases” truncations.
- We have “LLL quotients”.
- If the basis is balanced, the quotient has small bit-size.
Overview of $\widetilde{\text{L}}^1$

$\widetilde{\text{L}}^1$: HNF and $n$ calls to Lift-$\widetilde{\text{L}}^1$.

If $B$ is reduced and $\ell \geq 0$, Lift-$\widetilde{\text{L}}^1$ computes $U$ unimodular such that $\sigma_\ell BU$ is reduced, in time $\mathcal{P}oly(n) \cdot \widetilde{O}(\ell)$.

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A first attempt for Lift-$\tilde{L}^1$

**Inputs:** $B$ reduced, lifting target $\ell$.

**Output:** $U$ unimodular such that $\sigma_\ell BU$ reduced.

- Keep the $\ell/2 + O(n)$ top-most bits of $B$.
- Recursively compute $U_1$ s.t. $\sigma_{\ell/2} BU_1$ reduced.
- Apply $U_1$ to $\sigma_{\ell/2} B$ and keep the $\ell/2 + O(n)$ top-most bits.
- Recursively compute $U_2$ s.t. $\sigma_{\ell/2} (\sigma_{\ell/2} BU_1) U_2$ is reduced.
- Return $U_1 \cdot U_2$. 
A first attempt for Lift-$\tilde{L}^1$

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- Recursively compute $U_2$ s.t. $\sigma_{\ell/2}(\sigma_{\ell/2} BU_1)U_2$ is reduced.
- Return $U_1 \cdot U_2$. 
Some additional difficulties

1. Keep the $\ell/2 + O(n)$ top-most bits of $B$.
2. Recursively compute $U_1$ s.t. $\sigma_{\ell/2}BU_1$ reduced.
3. Apply $U_1$ to $\sigma_{\ell/2}B$ and keep the $\ell/2 + O(n)$ top-most bits.
4. Recursively compute $U_2$ s.t. $\sigma_{\ell/2}(\sigma_{\ell/2}BU_1)U_2$ is reduced.
5. Return $U_1 \cdot U_2$.

- What do we do at the recursion leaves?
- Every time we truncate, we may loosen the reduction factors...
- How do we compute $B \cdot U_1$ and $U_1 \cdot U_2$ efficiently?
Problem: Suppose we have a $\Xi$-reduced basis. How do we $\Xi'$-reduce it, for $\Xi' > \Xi$?

- Truncate, reduce, output the obtained $U$.
- This takes time $O(n^{6+\varepsilon})$ when the $r_{ii}$'s are balanced.

Otherwise, $u_{ij}$ can be as large as $r_{jj}/r_{ii}$...
Strengthening the reducedness of a basis (Morel-S-Villard)

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1. Rescale the columns of $B$: $B \mapsto BS$.
2. Do that while keeping $B$ reduced.
3. Find $U$ unimodular s.t. $(BS)U$ is reduced.
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- Used for re-strengthening the reduction factors, loosened by the truncations.
- Returns $(U, S)$ s.t.:
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Bounding the cost of $\text{Lift-}L^1$

1. Keep the $\ell/2 + O(n)$ top-most bits of $B$.
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New representations for bases and transforms:

- Easy if assuming all handled bases are “balanced”. Else...
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Final hassle:

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Sanitizing the transforms

Assume $B$ and $\sigma_\ell BU$ are reduced with $\ell \geq 0$ and $U$ unimodular.
Let $\Delta U$ s.t. $|\Delta u_{ij}| \leq 2^{-\Omega(\ell+n)}r_{jj}/r_{ii}$, then:

$$U + \Delta U \text{ unimodular and } \sigma_\ell B(U + \Delta U) \text{ reduced.}$$

- A lift-reducing $U$ may be large, but its bit-size can be made small.
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- $U \mapsto (U', D, x)$ with $U = 2^x DU'D^{-1}$. 
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Conclusion and open problems

- \( \tilde{L}^1 \) reduces in time \( O(n^{5+\epsilon} \beta + n^{\omega+1+\epsilon} \beta^{1+\epsilon}) \).
- This generalizes Knuth-Schnönhage and Schönhage-Yap to arbitrary dimensions.
- Three ingredients: deforming, truncating, Knuth-Schönhage.

1. Can we do better wrt \( n \)? [Schönhage’84, Storjohann’96, etc]
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