

Symbolic summation for Feynman integrals

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A Feynman parameter integral

$$I = \int_0^1 \int_0^1 \frac{(1-w)^{-1-\epsilon/2} z^{\epsilon/2} (1-z)^{-\epsilon/2}}{(1-wz)^{1-\epsilon}} (1-w)^{N+1} dw dz,$$

where $N \in \mathbb{N}$ and $\epsilon > 0$.

- ▶ Symbolic summation methods
- ▶ Feynman integrals as multiple sums
- ▶ Feynman integrals as Mellin-Barnes integrals

Part 1: Symbolic summation methods

Hypergeometric series

A simple example

$${}_3F_2 \left(\begin{matrix} a, b, c \\ d, e \end{matrix} ; x \right) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n (c)_n}{(d)_n (e)_n} \frac{x^n}{n!}$$

$$= \frac{\Gamma(d)\Gamma(e)}{\Gamma(b)\Gamma(c)\Gamma(d-b)\Gamma(e-c)} \int_0^1 \int_0^1 \frac{t^{b-1}(1-t)^{d-b-1}u^{c-1}(1-u)^{e-c-1}}{(1-utx)^a} dt du$$

$${}_3F_2 \left(\begin{matrix} a, b, c \\ d, e \end{matrix} ; x \right) = \sum_{n=0}^{\infty} \frac{\overbrace{(a)_n (b)_n (c)_n}^{t_n} x^n}{(d)_n (e)_n n!}$$

where the Pochhammer symbol / rising factorial is

$$(a)_n := a(a+1) \dots (a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}$$

if $a \in \mathbb{C}$ and $a+n \neq 0, -1, -2, \dots$

$$\frac{t_{n+1}}{t_n} = \frac{(a+n)(b+n)(c+n)}{(d+n)(e+n)} \frac{x}{n+1}$$

An illustrative application for summation methods¹

For $n \geq k \geq 1$ prove the following identity:

$$1 + (-1)^{n+k} \sum_{m \geq 0} (-1)^m \binom{k-1-n}{m} \binom{n}{k-1-m} = 2^k \sum_{m \geq k} \binom{-k}{n-m}$$

¹question asked by P. van der Kamp

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Initial values:

- ▶ if $n = 1$ then $k = 1$
- ▶ if $n = 2$ then $k \in \{1, 2\}$

¹question asked by P. van der Kamp

Zeilberger's Algorithm^{2,3}

Given the summation problem

$$\text{SUM}[n] := \sum_{m=b}^B f(m, n)$$

Zeilberger's Algorithm delivers **certificate recurrences** of the form

$$\sum_{i \in S} c_i(n) f(m, n+i) = \Delta_m [g(m, n)]$$

where $g(m, n) = \text{rat}(m, n) f(m, n)$ and

$$\Delta_m [g(m, n)] := g(m+1, n) - g(m, n).$$

²D. Zeilberger *A fast algorithm for proving terminating hypergeometric identities*. Discrete Mathematics, 1990.

³P. Paule, M. Schorn *A Mathematica version of Zeilberger's algorithm for proving binomial coefficient identities*. J. Symbolic Comput. 1995.

Summing over the **certificate recurrences**

$$\sum_{i \in \mathbb{S}} c_i(n) f(m, n+i) = \Delta_m [g(m, n)]$$

we obtain recurrences for the sum $\text{SUM}[n] := \sum_{m=b}^B f(m, n)$

$$\sum_{i \in \mathbb{S}} c_i(n) \text{SUM}[n+i] = g(B+1, n) - g(b, n).$$

Remark: Zeilberger's algorithm also deals with **definite summation** problems \rightarrow the bounds b, B can linearly depend on n .

Back to the illustrative example

$$1 + (-1)^{k+n} \sum_{m=0}^{k-1} (-1)^m \binom{k-1-n}{m} \binom{n}{k-1-m} = 2^k \underbrace{\sum_{m=k}^n \binom{-k}{n-m}}_{T(n)}$$

We extend the sum $T(n)$ to a summation problem with standard boundary conditions:

$$T(n) := \sum_{m=k}^n \binom{-k}{n-m} = \lim_{\epsilon \rightarrow 0} \underbrace{\sum_{m=-\infty}^{\infty} \binom{-k}{n-m} \frac{(m-k+\epsilon)!}{(m-k)!}}_{t(n,\epsilon)}$$

Zeilberger's algorithm delivers a recurrence for $t(n, \epsilon)$

$$(\epsilon + n + 1)t(n, \epsilon) + (\epsilon - k + 1)t(n + 1, \epsilon) + (k - n - 2)t(n + 2, \epsilon) = 0$$

Taking here $\epsilon \rightarrow 0$, we come to the conclusion that $T(n)$ satisfies

$$(n + 1)T(n) + (1 - k)T(n + 1) + (k - n - 2)T(n + 2) = 0$$

while for the LHS of the identity we computed the same recurrence

$$(n + 1)\text{SUM}[n] + (1 - k)\text{SUM}[n+1] + (k - n - 2)\text{SUM}[n+2] = 0.$$

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A multisum example⁵

For $r \geq k \geq 0$, $s \geq 0$ prove that:

$$\begin{aligned} \sum_{i=0}^{r-k} \sum_{j=0}^s \binom{r-j}{i} \binom{r+k-i}{j} \binom{k+i}{s-j} \binom{r-s+j}{r-k-i} &= \\ = \sum_{i=0}^{r-k} \sum_{j=0}^s \binom{r-j+2}{i} \binom{r+k-i}{j} \binom{k+i}{s-j} \binom{r-s+j-2}{r-k-i} \end{aligned}$$

MultiSum⁴ delivers the same recurrence for both sides:

$$\begin{aligned} (2k+r+2)(2r-s+2)\text{SUM}[s,r] - (2r^2+4kr+8r+2k-3ks+8) \\ \text{SUM}[s+1,r+1] - (k-r-2)(s+2)\text{SUM}[s+2,r+2] = 0 \end{aligned}$$

⁴K. Wegschaider, *Computer generated proofs of binomial multi-sum identities*, Diploma thesis, RISC, Johannes Kepler University, 1997

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More general: WZ-summation methods

WZ-summation methods deliver recurrences in elements of $\mu = (\mu_1, \dots, \mu_l)$ for the multiple sum:

$$\text{Sum}(\mu) := \sum_{\kappa_1} \cdots \sum_{\kappa_r} \mathcal{F}(\mu, \kappa_1, \dots, \kappa_r)$$

where $\mathcal{F}(\mu, \kappa)$ is a hypergeometric term in the variables $\mu = (\mu_1, \dots, \mu_l)$ and $\kappa = (\kappa_1, \dots, \kappa_r)$.

Remark: for $r = 1$ and $l = 1$ we have Zeilberger's Algorithm

Given a κ -free recurrence⁶ satisfied by the summand

$$\sum_{(u,v) \in \mathbb{S}} a_{u,v}(\mu, \alpha) \mathcal{F}(\mu + u, \kappa + v) = 0$$

one computes a **certificate recurrence**⁷

$$\sum_{m \in \mathbb{S}'} c_m(\mu) \mathcal{F}(\mu + m, \kappa) = \sum_{i=1}^r \Delta_{\kappa_i} \left(\sum_{(j,k) \in \mathbb{S}_i} p_{j,k}(\mu, \kappa) \mathcal{F}(\mu + j, \kappa + k) \right),$$

where

$$\Delta_{\kappa_i} \mathcal{F}(\mu, \kappa) := \mathcal{F}(\mu, \kappa_1, \dots, \kappa_i + 1, \dots, \kappa_r) - \mathcal{F}(\mu, \kappa).$$

⁶M.C. Fassenmyer, *Some generalized hypergeometric polynomials*, PhD thesis, Univ. of Michigan, 1945

⁷H.S. Wilf and D. Zeilberger, *An algorithmic proof theory for hypergeometric (ordinary and q) multisum/integral identities*, *Inventiones Mathematicae*, 1992

Summary of Part 1

For the summation problem

$$\text{Sum}(\mu) := \sum_{\kappa_1} \cdots \sum_{\kappa_r} \mathcal{F}(\mu, \kappa_1, \dots, \kappa_r)$$

WZ-methods give a certificate recurrence of the form

$$\sum_{m \in \mathbb{S}} c_m(\mu) \mathcal{F}(\mu + m, \kappa) = \sum_{i=1}^r \Delta_{\kappa_i} \left(\sum_{(j,k) \in \mathbb{S}_i} p_{j,k}(\mu, \kappa) \mathcal{F}(\mu + j, \kappa + k) \right)$$

By summing over the certificate recurrence we obtain a recurrence for the sum

$$\sum_{m \in \mathbb{S}} c_m(\mu) \text{Sum}(\mu + m) = \text{RHS}$$

Remark:

$$\Delta_{\kappa_i} \mathcal{F}(\mu, \kappa) := \mathcal{F}(\mu, \kappa_1, \dots, \kappa_i + 1, \dots, \kappa_r) - \mathcal{F}(\mu, \kappa)$$

Part 2: Feynman integrals as multiple sums ⁸

Back to the Feynman integral example

$$I = \int_0^1 \int_0^1 \frac{(1-w)^{-1-\epsilon/2} z^{\epsilon/2} (1-z)^{-\epsilon/2}}{(1-wz)^{1-\epsilon}} (1-w)^{N+1} dw dz,$$

where $N \in \mathbb{N}$ and $\epsilon > 0$.

.. seen as a hypergeometric series

$$I = \int_0^1 \int_0^1 \frac{(1-w)^{N-\epsilon/2} z^{\epsilon/2} (1-z)^{-\epsilon/2}}{(1-wz)^{1-\epsilon}} dw dz$$
$$= \frac{\Gamma(1-\frac{\epsilon}{2}) \Gamma(1+\frac{\epsilon}{2})}{2(N+1-\frac{\epsilon}{2})} \underbrace{{}_3F_2 \left(\begin{matrix} 1, 1-\epsilon, 1+\frac{\epsilon}{2} \\ 2, N+2-\frac{\epsilon}{2} \end{matrix} ; 1 \right)}_{\sum_{\sigma \geq 0} \frac{(1-\epsilon)_\sigma (1+\frac{\epsilon}{2})_\sigma}{(2)_\sigma (N+2-\frac{\epsilon}{2})_\sigma}}$$

SUM[N]

MultiSum⁹ computes a certificate recurrence satisfied by its summand

Out[0]=

$$\text{certRec} = (N+1)(2N-\epsilon+2)F[N, \sigma] - (N-\epsilon+1)(2N+\epsilon+2)F[N+1, \sigma] = \Delta_{\sigma} [(-\sigma-1)(2N-\epsilon+2)F[N, \sigma]]$$

The sum representation of I , denoted by $\text{SUM}[N]$, satisfies the recurrence

$$\text{Out[0]} = \text{rec} = (N+1)(2N-\epsilon+2)\text{SUM}[N] - (N-\epsilon+1)(2N+\epsilon+2)\text{SUM}[N+1] == \Gamma\left(1 - \frac{\epsilon}{2}\right) \Gamma\left(\frac{\epsilon}{2} + 1\right).$$

⁹K. Wegschaider, *Computer generated proofs of binomial multi-sum identities*, Diploma thesis, RISC, Johannes Kepler University, 1997

Unfolding recurrences

Given the first order inhomogeneous recurrence

$$\text{SUM}[N+1] - c(n)\text{SUM}[N] = h(n)$$

one can write

$$\begin{aligned}\text{SUM}[N+1] &= c(n)\text{SUM}[N] + h(n) \\ &= c(n)(c(n-1)\text{SUM}[N-1] + h(n-1)) + h(n) \\ &= \dots\end{aligned}$$

$$\text{SUM}[N+1] = \left(\prod_{i=1}^n c(i) \right) \text{SUM}[1] + \sum_{i=1}^{n-1} h(n-i) \prod_{j=0}^i c(j) + h(n)$$

$\Pi\Sigma$ extensions ¹⁰

$$\sum_{i=1}^n iH_i = \sum_{i=1}^n i \sum_{j=1}^i \frac{1}{j} = ?$$

$$\begin{array}{c} K(i, H_i) \\ | \\ K(i) \\ | \\ K = \mathbb{Q} \end{array} \quad \sigma(H_i) = H_i + \frac{1}{i+1}$$
$$\sigma(i) = i + 1$$

Solve in the field $K(i, H_i), \sigma$
the difference equation:

$$\Delta(g) = \sigma(g) - g = i \cdot H_i$$

Sigma delivers the solution:

$$g = \frac{i(i-1)}{4} \cdot (2H_i - 1)$$

$$\sum_{i=1}^n iH_i = g(n+1) - g(1) = \frac{n(n-1)}{4} \cdot (2H_n - 1)$$

Higher order recurrences

$$\text{SUM}[n+2] - \alpha(n)\text{SUM}[n+1] - \beta(n)\text{SUM}[n] = h(n)$$

can be rewritten using operator notation

$$(S_n^2 - \alpha(n)S_n - \beta(n)\mathcal{I})\text{SUM}[n] = h(n)$$

Using [Hyper](#)¹¹ we find a factorization of the form

$$(S_n - c(n)\mathcal{I})(S_n - d(n)\mathcal{I})\text{SUM}[n] = h(n)$$

if it exists.

¹¹[M. Petkovsek](#) *Hypergeometric solutions of linear recurrences with polynomial coefficients* J. Symbolic Comp. 1992

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¹¹M. Petkovsek *Hypergeometric solutions of linear recurrences with polynomial coefficients* J. Symbolic Comp. 1992

A realistic example

$$\begin{aligned} \mathcal{U}(N, \epsilon) &:= (-1)^N \sum_{\sigma_0=0}^{\infty} \sum_{j_0=0}^{N-3} \sum_{j_1=0}^{N-j_0-3} \sum_{j_2=0}^{j_0+1} \binom{j_0+1}{j_2} \binom{N-j_0-3}{j_1} \\ &\times \frac{\left(\frac{\epsilon}{2}+1\right)_{\sigma_0} (-\epsilon)_{\sigma_0} (j_1+j_2+3)_{\sigma_0} \left(3-\frac{\epsilon}{2}\right)_{j_1}}{(j_1+4)_{\sigma_0} \left(-\frac{\epsilon}{2}+j_1+j_2+4\right)_{\sigma_0} \left(4-\frac{\epsilon}{2}\right)_{j_1+j_2}} \\ &\times \frac{\Gamma(j_1+j_2+2)\Gamma(j_1+j_2+3)\Gamma(N-j_0-1)\Gamma(N-j_1-j_2-1)}{\Gamma(\sigma_0+1)\Gamma(j_1+4)\Gamma(N-j_0-2)} \end{aligned}$$

where $N \geq 3$ is the Mellin moment and $\epsilon > 0$ is the dimension regularization parameter.

Setting up the inhomogeneous recurrence

E.g. $\boxed{\text{SUM}[N] = \sum_{\sigma=0}^{\infty} \sum_{j_0=0}^N \sum_{j_1=0}^{j_0} F[N, \sigma, j_0, j_1]}$ and `MultiSum` delivers

the certificate recurrence

$$F[N + 1, \sigma, j_0, j_1] + (N + 2)F[N, \sigma, j_0, j_1] =$$

$$= \Delta_{j_0} [(j_1 - j_0 + 1)F[N + 1, \sigma, j_0, j_1] + F[N, \sigma, j_0, j_1]]$$

$$\left| \begin{array}{ccc} \infty & N & j_0 \\ \sum_{\sigma=0} & \sum_{j_0=0} & \sum_{j_1=0} \end{array} \right.$$

$$\sum_{\sigma=0}^{\infty} \sum_{j_0=0}^N \sum_{j_1=0}^{j_0} F(N+1, \sigma, j_0, j_1) = \text{SUM}[N+1]$$

$$- \underbrace{\sum_{\sigma=0}^{\infty} \sum_{j_1=0}^{N+1} F(N+1, \sigma, N+1, j_1)}_{\text{shift compensation}}$$

$$\sum_{\sigma=0}^{\infty} \sum_{j_1=0}^{j_0} \Delta_{j_0} [(j_1 - j_0 + 1)F(N + 1, \sigma, j_0, j_1)] \Bigg|_{k=0}^{j_0=N+1} =$$

$$= \underbrace{\sum_{\sigma=0}^{\infty} \sum_{j_1=0}^{j_0} (j_1 - j_0 + 1)F(N + 1, \sigma, j_0, j_1)}_{\Delta\text{-boundary sums}} \Bigg|_{j_0=0}^{j_0=N+1}$$

$$- \underbrace{\sum_{\sigma=0}^{\infty} \sum_{j_0=1}^{N+1} F(N + 1, \sigma, j_0, j_0)}_{\text{telescoping compensation}}$$

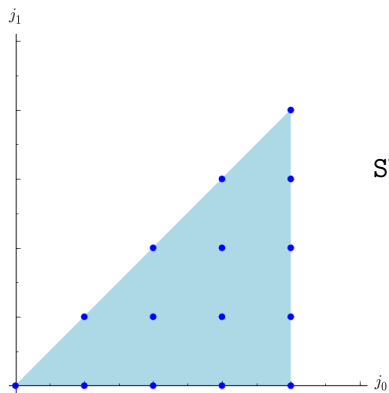
The inhomogeneous recurrence

$$\text{SUM}[N + 1] + (N + 2)\text{SUM}[N] = \sum_{\sigma=0}^{\infty} \sum_{j_1=0}^{N+1} F[N + 1, \sigma, N + 1, j_1]$$

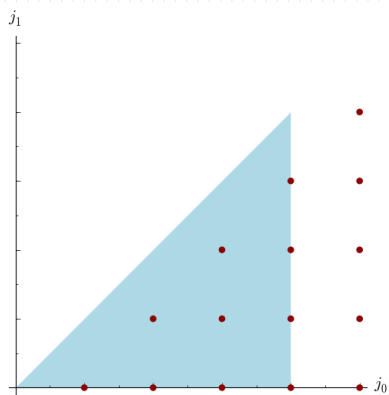
$$+ \sum_{\sigma=0}^{\infty} \sum_{j_1=0}^{j_0} (j_1 - j_0 + 1)F[N + 1, \sigma, j_0, j_1] \Bigg|_{j_0=0}^{j_0=N+1} - \sum_{\sigma=0}^{\infty} \sum_{j_0=1}^{N+1} F[N, \sigma, j_0, j_0]$$

$$+ \sum_{\sigma=0}^{\infty} \sum_{j_1=0}^{j_0} F[N, \sigma, j_0, j_1] \Bigg|_{j_0=0}^{j_0=N+1} - \sum_{\sigma=0}^{\infty} \sum_{j_0=1}^{N+1} F[N + 1, \sigma, j_0, j_0]$$

where the **RHS** of the recurrence contains **shift and telescoping compensating sums** as well as **delta boundary sums**.

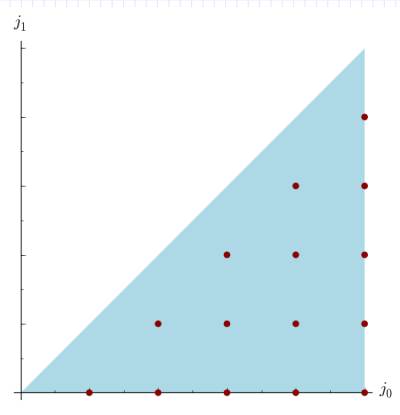


$$\text{SUM}[N] := \sum_{\sigma=0}^{\infty} \sum_{j_0=0}^N \sum_{j_1=0}^{j_0} F[N, \sigma, j_0, j_1]$$



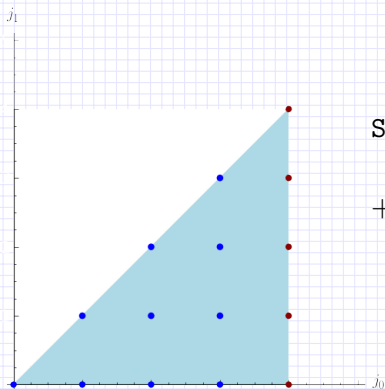
$$\sum_{\sigma=0}^{\infty} \sum_{j_0=0}^N \sum_{j_1=0}^{j_0} F[N, \sigma, j_0 + 1, j_1]$$

$$= \sum_{\sigma=0}^{\infty} \sum_{j_0=1}^{N+1} \sum_{j_1=0}^{j_0-1} F[N, \sigma, j_0, j_1]$$



$$\sum_{\sigma=0}^{\infty} \sum_{j_0=0}^N \sum_{j_1=0}^{j_0} F[N+1, \sigma, j_0+1, j_1]$$

$$= \sum_{\sigma=0}^{\infty} \sum_{j_0=1}^{N+1} \sum_{j_1=0}^{j_0-1} F[N+1, \sigma, j_0, j_1]$$



$$\text{SUM}[N] = \sum_{\sigma=0}^{\infty} \sum_{j_0=0}^{N-1} \sum_{j_1=0}^{j_0} F[N, \sigma, j_0, j_1]$$

$$+ \sum_{\sigma=0}^{\infty} \sum_{j_1=0}^N F[N, \sigma, N, j_1]$$

Part 3:

Feynman integrals as Mellin-Barnes integrals

The Mellin transform

The **Mellin transform** of a locally integrable function $f : (0, \infty) \rightarrow \mathbb{C}$ is

$$\tilde{f}(s) = \int_0^{\infty} x^{s-1} f(x) dx =: M[f; s]$$

defined usually on an infinite strip $a < \operatorname{Re}(s) < b$.

The **inversion formula**

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{-s} \tilde{f}(s) ds$$

has a contour of integration placed in the strip of analyticity $a < c < b$.

For example

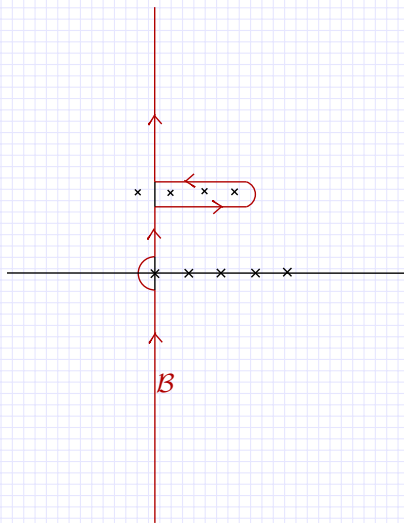
$$\frac{1}{(1-x)^a} = \frac{1}{2\pi i} \int_{\mathcal{B}} \frac{\Gamma(-s)\Gamma(a+s)}{\Gamma(a)} (-x)^s ds$$

where the contour \mathcal{B} separates the set of ascending poles

$$\{0, 1, 2, 3, \dots\}$$

from the set of descending poles of the integrand

$$\{-a, -a-1, -a-2, \dots\}$$



Let's ask the question

$$\int_{\mathcal{B}} \underbrace{\frac{\Gamma(a+s)\Gamma(-s)}{\Gamma(a)}}_{\Psi_{z,a}(s)} (-z)^s ds = ?$$

\mathcal{B} is a Barnes contour of integration putting the poles of $\Gamma(a+s)$ to left and those of $\Gamma(-s)$ to the right

$$\int_{\mathcal{B}} \underbrace{\frac{\Gamma(a+s)\Gamma(-s)}{\Gamma(a)}}_{\Psi_{z,a}(s)} (-z)^s ds = ?$$

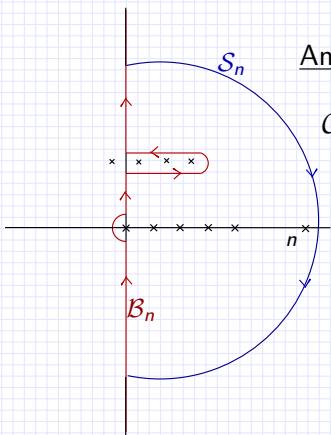
Answer:

$$\mathcal{C}_n := \mathcal{B}_n \cup \mathcal{S}_n \text{ for all } n \geq 0 \text{ and}$$

$$\lim_{n \rightarrow \infty} \int_{\mathcal{S}_n} \Psi_{z,a}(s) ds = 0$$

then

$$\int_{\mathcal{B}} \Psi_{z,a}(s) ds = \lim_{n \rightarrow \infty} \underbrace{\int_{\mathcal{C}_n} \Psi_{z,a}(s) ds}_{(-2\pi i) \sum_{k=0}^n \text{Res}(\Psi(t), k)}$$



$${}_1F_0 \left(\begin{matrix} a \\ - \end{matrix} ; z \right) = \frac{1}{(1-z)^a} = \frac{1}{2\pi i} \int_{\mathcal{B}} \frac{\Gamma(a+s)\Gamma(-s)}{\Gamma(a)} (-z)^s ds$$

More general,

$$\begin{aligned} & \frac{1}{2\pi i} \int_{\mathcal{B}} \frac{\prod_{l=1}^p \Gamma(a_l + s)\Gamma(-s)(-z)^s}{\prod_{j=1}^q \Gamma(b_j + s)} ds = \\ & = \frac{\prod_{l=1}^p \Gamma(a_l)}{\prod_{j=1}^q \Gamma(b_j)} {}_pF_q \left(\begin{matrix} a_1, a_2, \dots, a_p \\ b_1, b_2, \dots, b_q \end{matrix} ; z \right) \end{aligned}$$

Back to our Feynman integral

$$I = \int_0^1 \int_0^1 \frac{(1-w)^{-1-\epsilon/2} z^{\epsilon/2} (1-z)^{-\epsilon/2}}{(1-wz)^{1-\epsilon}} (1-w)^{N+1} dw dz,$$

where $N \in \mathbb{N}$ and $\epsilon > 0$.

$$I = \int_0^1 dw \int_0^1 dz (1-w)^{N-\epsilon/2} z^{\epsilon/2} (1-z)^{-\epsilon/2} \times$$

$$\underbrace{\frac{1}{2\pi i} \int_{C_s} \frac{\Gamma(-s)\Gamma(1-\epsilon+s)}{\Gamma(1-\epsilon)} (-wz)^s ds}_{1/(1-wz)^{1-\epsilon}}$$

$$= \frac{1}{2\pi i} \int_{C_s} \frac{\Gamma(-s)\Gamma(1-\epsilon+s)}{\Gamma(1-\epsilon)} (-1)^s \times$$

$$\underbrace{\left(\int_0^1 z^{\epsilon/2+s} (1-z)^{-\epsilon/2} dz \right)}_{B(\epsilon/2+s+1, -\epsilon/2+1)} \underbrace{\left(\int_0^1 w^s (1-w)^{N-\epsilon/2} dw \right)}_{B(s+1, N-\epsilon/2+1)} ds$$

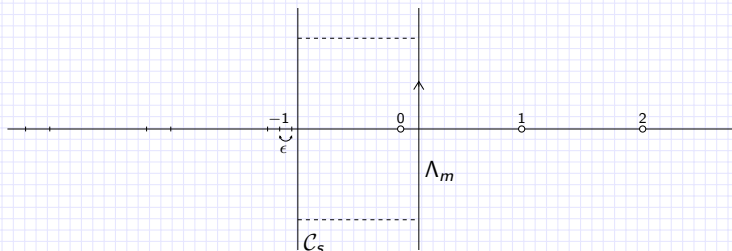
$$I = \frac{\Gamma\left(1 - \frac{\epsilon}{2}\right) \Gamma\left(N - \frac{\epsilon}{2} + 1\right)}{4\pi i \Gamma(1 - \epsilon)} \times \int_{c_s} \frac{\Gamma(s + 1 - \epsilon) \Gamma\left(s + 1 + \frac{\epsilon}{2}\right)}{(s + 1) \Gamma\left(N + s - \frac{\epsilon}{2} + 2\right)} \Gamma(-s) (-1)^s ds$$

MultiSum computes a certificate recurrence satisfied by its integrand

Out[0]=

$$\mathit{certRec} = (N+1)(2N-\epsilon+2)F[N, s] - (N-\epsilon+1)(2N+\epsilon+2)F[N+1, s] = \Delta_s[(-s-1)(2N-\epsilon+2)F[N, s]]$$

The contour \mathcal{C}_s separates the ascending chain of poles of $\Gamma(-s)$ from the descending chains coming from $\frac{\Gamma(s+1-\epsilon)\Gamma(s+1+\frac{\epsilon}{2})}{(s+1)}$.



On the RHS we have an improper integral

$$\int_{\mathcal{C}_s} \Delta_s [(-s-1)(2N-\epsilon+2)F[N,s]] ds$$

$$= -(2N-\epsilon+2) \lim_{m \rightarrow \infty} \int_{\Lambda_m} (s+1)F[N,s] ds = \Gamma\left(1-\frac{\epsilon}{2}\right) \Gamma\left(\frac{\epsilon}{2}+1\right)$$

where $F[N,s]$ denotes the integrand.

For the Feynman parameter integral

$$I = \int_0^1 \int_0^1 \frac{(1-w)^{-1-\epsilon/2} z^{\epsilon/2} (1-z)^{-\epsilon/2}}{(1-wz)^{1-\epsilon}} (1-w)^{N+1} dw dz,$$

the sum representation satisfies the recurrence

$$\begin{aligned} \text{Out}[0] = \text{rec} &= (N+1)(2N-\epsilon+2)\text{SUM}[N] \\ &\quad - (N-\epsilon+1)(2N+\epsilon+2)\text{SUM}[N+1] == \Gamma\left(1-\frac{\epsilon}{2}\right)\Gamma\left(\frac{\epsilon}{2}+1\right) \end{aligned}$$

and the Mellin-Barnes integral representation satisfies the same

`In[1]:= rec/.SUM → INT`

$$\begin{aligned} \text{Out}[1] = & (N+1)(2N-\epsilon+2)\text{INT}[N] \\ & - (N-\epsilon+1)(2N+\epsilon+2)\text{INT}[N+1] == \Gamma\left(1-\frac{\epsilon}{2}\right)\Gamma\left(\frac{\epsilon}{2}+1\right). \end{aligned}$$

Recurrences for Mellin-Barnes integrals

For the integration problem

$$\text{Int}(\mu) := \int_{\mathcal{B}_{\kappa_1}} \dots \int_{\mathcal{B}_{\kappa_r}} \mathcal{F}(\mu, \kappa_1, \dots, \kappa_r) d\kappa$$

WZ-methods give a certificate recurrence of the form

$$\sum_{m \in \mathbb{S}} c_m(\mu) \mathcal{F}(\mu + m, \kappa) = \sum_{i=1}^r \Delta_{\kappa_i} \left(\sum_{(j,k) \in \mathbb{S}_i} p_{j,k}(\mu, \kappa) \mathcal{F}(\mu + j, \kappa + k) \right)$$

By integrating over the certificate recurrence we obtain a recurrence for the integral

$$\sum_{m \in \mathbb{S}} c_m(\mu) \text{Int}(\mu + m) = \text{RHS}$$

One more example

Identity **7.512.2** from the Table¹²

$$\int_0^1 x^{\rho-1}(1-x)^{\beta-\gamma-n} {}_2F_1\left(\begin{matrix} -n, \beta \\ \gamma \end{matrix}; x\right) dx \\ = \frac{\Gamma(\gamma)\Gamma(\rho)\Gamma(\beta-\gamma+1)\Gamma(\gamma-\rho+n)}{\Gamma(\gamma+n)\Gamma(\gamma-\rho)\Gamma(\beta-\gamma+\rho+1)},$$

where $n \in \mathbb{N}$, $\operatorname{Re} \rho > 0$ and $\operatorname{Re}(\beta - \gamma) > n - 1$.

Recall the Mellin transform slide..

The **Mellin transform** of a locally integrable function $f : (0, \infty) \rightarrow \mathbb{C}$ is

$$\tilde{f}(s) = \int_0^{\infty} x^{s-1} f(x) dx =: M[f; s]$$

defined usually on an infinite strip $a < \operatorname{Re}(s) < b$.

Remark: For a polynomial function $f(x) = (1-x)^n$ we have

$$a = 0 \quad \text{and} \quad b = -n$$

\Rightarrow no strip of analyticity.

A generalized definition of the Mellin transform¹³

Decomposing $f(x)$ such that $f(x) = f_1(x) + f_2(x)$, e.g.,

$$f_1(x) = \begin{cases} f(x), & x \in [0, 1) \\ 0, & x \in [1, \infty) \end{cases}, \quad f_2(x) = \begin{cases} 0, & x \in [0, 1) \\ f(x), & x \in [1, \infty) \end{cases}$$

we have

$$M[f; z] = M[f_1; z] + M[f_2; z].$$

In our case, $f(x) = (1-x)^n$ and its Mellin transform is

$$M[f; z] = \Gamma(n+1) \left[\frac{\Gamma(z)}{\Gamma(n+z+1)} + (-1)^n \frac{\Gamma(-n-z)}{\Gamma(1-z)} \right].$$

..we arrive at the Mellin-Barnes integral representation:

$$\begin{aligned} {}_2F_1 \left(\begin{matrix} -n, \beta \\ \gamma \end{matrix} ; x \right) &= \\ &= \frac{\Gamma(\gamma)\Gamma(n+1)}{\Gamma(\beta)} \frac{1}{2\pi i} \left[\int_{\delta-i\infty}^{\delta+i\infty} \frac{\Gamma(s)}{\Gamma(n+s+1)} \frac{\Gamma(\beta-s)}{\Gamma(\gamma-s)} x^{-s} ds \right. \\ &\quad \left. + (-1)^n \int_{\eta-i\infty}^{\eta+i\infty} \frac{\Gamma(-n-s)}{\Gamma(1-s)} \frac{\Gamma(\beta-s)}{\Gamma(\gamma-s)} x^{-s} ds \right] \end{aligned}$$

where $\operatorname{Re} \beta > \delta > 0$ and $\eta < -n$.

Using the Mellin transform and WZ-methods we can now prove identities of the form¹⁴

$$\int_0^{\infty} e^{-x} x^{\alpha+\beta} L_m^{\alpha}(x) L_n^{\beta}(x) dx = (-1)^{m+n} (\alpha+\beta)! \binom{\alpha+m}{n} \binom{\beta+n}{m}$$

where $L_m^{\alpha}(x)$ are the Laguerre polynomials.

Conclusions

- ▶ real-world applications of symbolic summation methods
- ▶ hypergeometric summation with non-standard bounds
- ▶ computing recurrences for nested Mellin-Barnes integrals