On the Structure of Compatible Rational Functions

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joint work with S. Chen, R. Feng, and G. Fu.
Outline

1. Bivariate compatible rational functions
2. General case
3. A structure theorem
4. Multiplicative decomposition of hyperexp.-hypergeom. elements
5. Applications
6. Algebraic dependence of hyperexp.-hypergeom. elements

Note. \( \mathbb{F} \) is an algebraically closed field of characteristic zero.
Bivariate case 1: differential and difference

Let

\[ \delta : \mathbb{F}(t, x) \to \mathbb{F}(t, x) \quad \text{and} \quad \sigma : \mathbb{F}(t, x) \to \mathbb{F}(t, x) \]

\[ f(t, x) \mapsto \frac{\partial f(t, x)}{\partial t} \quad \text{and} \quad f(t, x) \mapsto f(t, x + 1) \]

A first-order system

\[
\begin{cases}
\delta(z) = u z, \\
\sigma(z) = v z,
\end{cases}
\]

where \( u, v \in \mathbb{F}(t, x) \) with \( v \neq 0 \),

has a nonzero solution iff the following compatibility condition (CC) holds:

\[ \frac{\delta(v)}{v} = \sigma(u) - u. \]

\[
\begin{cases}
\sigma \circ \delta(z) = \sigma(u) v z \\
\delta \circ \sigma(z) = (\delta(v) + uv) z
\end{cases} \quad \Rightarrow \quad \text{the CC.}
\]
**Definition.** \( u, v \in \mathbb{F}(t, x) \) are compatible w.r.t. \( \{\delta, \sigma\} \) if \( v \neq 0 \) and
\[
\frac{\delta(v)}{v} = \sigma(u) - u.
\]

**Lemma.** (R. Feng, M. Singer, and M. Wu, 2010)
\( u, v \in \mathbb{F}(t, x) \) are compatible w.r.t. \( \{\delta, \sigma\} \) iff \( \exists f \in \mathbb{F}(t, x), \alpha, \beta \in \mathbb{F}(t), \lambda \in \mathbb{F}(x) \) s.t.
\[
u = \frac{\delta(f(t, x))}{f(t, x)} + x\frac{\delta(\alpha(t))}{\alpha(t)} + \beta(t)
\]
and
\[
u = \frac{\sigma(f(t, x))}{f(t, x)} \alpha(t) \lambda(x).
\]

**Corollary.** A solution of \( \{\delta(z) = u z, \sigma(z) = v z\} \) can be written as
\[
f(t, x) \alpha(t)^x \exp \left( \int \beta(t) dt \right) T(x),
\]
where \( \delta(T) = 0 \) and \( \sigma(T) = \lambda T \).
Bivariate case 2: differential and q-difference

Let
\[ \delta : \mathbb{F}(t, y) \to \mathbb{F}(t, y) \quad \text{and} \quad \tau : \mathbb{F}(t, y) \to \mathbb{F}(t, y) \]
\[ f(t, y) \mapsto \frac{\partial f(t, y)}{\partial t} \quad \text{and} \quad f(t, y) \mapsto f(t, qy), \]
where \( q \in \mathbb{F} \) is not a root of unity.

A first-order system:
\[
\begin{align*}
\delta(z) &= uz, \\
\tau(z) &= wz,
\end{align*}
\]
\[ u, w \in \mathbb{F}(t, y) \text{ with } w \neq 0, \]
has a nonzero solution iff the following CC holds:
\[ \frac{\delta(w)}{w} = \tau(u) - u. \]
**Definition.** $u, w \in \mathbb{F}(t, y)$ are compatible w.r.t. $\{\delta, \tau\}$ if $w \neq 0$ and

$$\frac{\delta(w)}{w} = \tau(u) - u.$$

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**Lemma.** (a $q$-analogue of FSW Lemma)

$u, w \in \mathbb{F}(t, y)$ are compatible w.r.t. $\{\delta, \tau\}$ iff $\exists f \in \mathbb{F}(t, y), \beta \in \mathbb{F}(t), \mu \in \mathbb{F}(y)$ s.t.

$$u = \frac{\delta(f(t, y))}{f(t, y)} + \beta(t) \quad \text{and} \quad v = \frac{\tau(f(t, y))}{f(t, y)} \mu(y).$$

---

**Corollary.** A solution of $\{\delta(z) = u z, \tau(z) = w z\}$ can be written as

$$f(t, y) \exp \left( \int \beta(t) dt \right) Q(y),$$

where $\delta(Q) = 0$ and $\tau(Q) = \mu Q$. 
Comparison

Differential and difference.

**Fact.** For $g \in \mathbb{F}(t, x)$,

$$\sigma(g) - g \in \mathbb{F}(t) \iff g \in \mathbb{F}(t)[x] \text{ and } \deg_x g \leq 1.$$ 

$$\downarrow$$

$$u = \frac{\delta(f(t, x))}{f(t, x)} + x \frac{\delta(\alpha(t))}{\alpha(t)} + \beta(t) \quad \text{and} \quad v = \frac{\sigma(f(t, x))}{f(t, x)} \alpha(t) \lambda(x).$$

Differential and $q$-difference.

**Fact.** For $g \in \mathbb{F}(t, y)$,

$$\tau(g) - g \in \mathbb{F}(t) \iff g \in \mathbb{F}(t).$$

$$\downarrow$$

$$u = \frac{\delta(f(t, y))}{f(t, y)} + \beta(t) \quad \text{and} \quad w = \frac{\tau(f(t, y))}{f(t, y)} \mu(y).$$
Bivariate Case III: difference and q-difference

Let

\[ \sigma : \mathbb{F}(x, y) \to \mathbb{F}(x, y) \quad \tau : \mathbb{F}(x, y) \to \mathbb{F}(x, y) \]

and

\[ f(x, y) \mapsto f(x + 1, y) \quad f(x, y) \mapsto f(x, qy), \]

where \( q \in \mathbb{F} \) is not a root of unity.

For \( v, w \in \mathbb{F}(x, y) \) with \( vw \neq 0 \), the system:

\[
\begin{align*}
\sigma(z) &= vz \\
\tau(z) &= wz
\end{align*}
\]

has a nonzero solution iff the following \( CC \) holds:

\[ \frac{\sigma(w)}{w} = \frac{\tau(v)}{v}. \]
Definition. $v, w \in \mathbb{F}(t, x)$ are compatible w.r.t. $\{\sigma, \tau\}$ if $vw \neq 0$ and

$$\frac{\sigma(w)}{w} = \frac{\tau(v)}{v}.$$ 

Lemma. $v$ and $w$ are compatible w.r.t. $\{\sigma, \tau\}$ iff there exist $f \in \mathbb{F}(x, y)$, $\lambda \in \mathbb{F}(x)$, $\mu \in \mathbb{F}(y)$ s.t.

$$v = \frac{\sigma(f(x, y))}{f(x, y)} \lambda(x) \quad \text{and} \quad w = \frac{\tau(f(x, y))}{f(x, y)} \mu(y).$$

Corollary. A solution of

$$\begin{cases} 
\sigma(z) = vz \\
\tau(z) = wz 
\end{cases}$$

can be written as

$$c f(x, y) T(x) Q(y),$$

where $c \in \mathbb{F}$, $\sigma(T) = \lambda T$ and $\tau(Q) = \mu Q$. 
### Survey on mixed cases

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<th>Operators</th>
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<th>Rational solutions</th>
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<td>$\delta = \frac{\partial}{\partial t}$</td>
<td>$\frac{\delta(v)}{v} = \sigma(u) - u$</td>
<td>$u = \frac{\delta(f)}{f} + x\frac{\delta(\alpha)}{\alpha} + \beta$, $v = \frac{\sigma(f)}{f} \alpha \lambda$</td>
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<tr>
<td>$\sigma : x \mapsto x + 1$</td>
<td></td>
<td>where $f \in \mathbb{F}(t,x)$, $\alpha, \beta \in \mathbb{F}(t)$, $\lambda \in \mathbb{F}(x)$</td>
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<td></td>
<td></td>
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</table>
General case

Let $t = (t_1, \ldots, t_\ell)$, $x = (x_1, \ldots, x_m)$ and $y = (y_1, \ldots, y_n)$.

For $i = 1, \ldots, \ell$,

$$
\delta_i : \mathbb{F}(t, x, y) \longrightarrow \mathbb{F}(t, x, y) \\
f \mapsto \frac{\partial f}{\partial t_i}.
$$

For $j = 1, \ldots, m$,

$$
\sigma_j : \mathbb{F}(t, x, y) \longrightarrow \mathbb{F}(t, x, y) \\
f(t, x_1, \ldots, x_m, y) \mapsto f(t, x_1, \ldots, x_{j-1}, x_j + 1, x_{j+1}, \ldots, x_m, y).
$$

For $k = 1, \ldots, n$,

$$
\tau_k : \mathbb{F}(t, x, y) \longrightarrow \mathbb{F}(t, x, y) \\
f(t, x, y_1, \ldots, y_n) \mapsto f(t, x, y_1, \ldots, y_{k-1}, q_k y_k, y_{k+1}, \ldots, y_n),
$$

where $q_1, \ldots, q_n \in \mathbb{F}^\times$ s.t. $q_1^{e_1} \cdots q_n^{e_n} \neq 1$ unless $e_1 = \cdots = e_n = 0$. 


**Compatible rational functions**

\[ \Delta = \{ \delta_1, \ldots, \delta_\ell, \sigma_1, \ldots, \sigma_m, \tau_1, \ldots, \tau_n \} . \]

**Definition.** A sequence of rational functions

\[ u_1, \ldots, u_\ell, v_1, \ldots, v_m, w_1, \ldots, w_n \]

is compatible w.r.t. \( \Delta \) if

\[ v_1 \cdots v_m w_1 \cdots w_n \neq 0 , \]

and the following six sets of CC’s hold:

<table>
<thead>
<tr>
<th>Unmixed</th>
<th>Mixed</th>
</tr>
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<tr>
<td>[ \delta_i(u_j) = \delta_j(u_i), \quad 1 \leq i &lt; j \leq \ell ]</td>
<td>[ \frac{\delta_i(v_j)}{v_j} = \sigma_j(u_i) - u_i, \quad i = 1, \ldots, \ell, \quad j = 1, \ldots, m ]</td>
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<td>[ \frac{\sigma_i(v_j)}{v_j} = \frac{\sigma_j(v_i)}{v_i}, \quad 1 \leq i &lt; j \leq m ]</td>
<td>[ \frac{\delta_i(w_k)}{w_k} = \tau_k(u_i) - u_i, \quad i = 1, \ldots, \ell, \quad k = 1, \ldots, n ]</td>
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<td>[ \frac{\sigma_j(w_k)}{w_k} = \tau_k(v_j), \quad j = 1, \ldots, m, \quad k = 1, \ldots, n ]</td>
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</table>
A structure theorem

**Theorem.** A sequence of rational functions $u_1, \ldots, u_\ell, v_1, \ldots, v_m, w_1, \ldots, w_n$ is compatible iff, for all $i, j, k$ with $1 \leq i \leq \ell, 1 \leq j \leq m$ and $1 \leq k \leq n$,

$$u_i = \frac{\delta_i(f(t, x, y))}{f(t, x, y)} + \sum_{j=1}^{m} x_j \frac{\delta_i(\alpha_j(t))}{\alpha_j(t)} + \beta_i(t),$$

$$v_j = \frac{\sigma_j(f(t, x, y))}{f(t, x, y)} \alpha_j(t) \lambda_j(x),$$

$$w_k = \frac{\tau_k(f(t, x, y))}{f(t, x, y)} \mu_k(y),$$

where $f \in \mathbb{F}(t, x, y)$, $\alpha_j, \beta_i \in \mathbb{F}(t)$, $\lambda_j \in \mathbb{F}(x)$, $\mu_k \in \mathbb{F}(y)$,

$\beta_1(t), \ldots, \beta_\ell(t)$ are compatible w.r.t. $\{\delta_1, \ldots, \delta_\ell\}$,

$\lambda_1(x), \ldots, \lambda_m(x)$ are compatible w.r.t. $\{\sigma_1, \ldots, \sigma_m\}$, and

$\mu_1(y), \ldots, \mu_n(y)$ are compatible w.r.t. $\{\tau_1, \ldots, \tau_n\}$. 
**Survey on unmixed cases**

<table>
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<tr>
<th>Systems</th>
<th>CC’s</th>
<th>Structure</th>
</tr>
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</table>
| \( \left\{ \begin{array}{l} \delta_1(z) = u_1 z \\
| \vdots | \\
| \delta_\ell(z) = u_\ell z, \end{array} \right. \) | \( \delta_i(u_j) = \delta_j(u_i), \) | Christoper’s theorem: |
| \( u_1, \ldots, u_\ell \in \mathbb{F}(t) \) | \( 1 \leq i < j \leq \ell \) | \( u_i = \delta_i(g) + \sum_k c_k \frac{\delta_i(g_k)}{g_k}, \) where \( c_k \in \mathbb{F} \) and \( f, g_k \in \mathbb{F}(t) \) |

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| \( \left\{ \begin{array}{l} \sigma_1(z) = v_1 z \\
| \vdots | \\
| \sigma_m(z) = v_m z, \end{array} \right. \) | \( \frac{\sigma_i(v_j)}{v_j} = \frac{\sigma_j(v_i)}{v_i} \) | Ore-Sato’s theorem: |
| \( v_1, \ldots, v_m \in \mathbb{F}(x) \) | \( 1 \leq i < j \leq m \) | \( v_j = \frac{\sigma_j(g)}{g} \prod_{s \in S} \prod_{k=0}^{s_j} r_s(s \cdot x + k) \) where \( f \in \mathbb{F}(x), S \subset \mathbb{Z}^m \) and \( r_s \in \mathbb{F}(z). \) |

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| \( \left\{ \begin{array}{l} \tau_1(z) = w_1 z \\
| \vdots | \\
| \tau_n(z) = w_n z, \end{array} \right. \) | \( \frac{\tau_i(w_j)}{w_j} = \frac{\tau_j(w_i)}{w_i} \) | A \( q \)-analogue |
| \( w_1, \ldots, w_n \in \mathbb{F}(y) \) | \( 1 \leq i < j \leq n \) | by I. Gel’fand, M. Graev, and V. Retakh. when \( q_1 = \cdots = q_n \) |
Hyperexponential-Hypergeometric elements

Definition. A first-order system

\[ \{ \delta_i(z) = u_i z, \ \sigma_j(z) = v_j z, \ \tau_k(z) = w_k z \mid i = 1, \ldots, \ell, \ j = 1, \ldots, m, \ k = 1, \ldots, n \} \quad (*) \]

is compatible if \( u_1, \ldots, u_\ell, v_1, \ldots, v_m, w_1, \ldots, w_n \) are compatible w.r.t. \( \Delta \).

Definition. A nonzero solution of (*) is called a hyperexponential-hypergeometric element, abbr. as \( H \)-element.
Corollary. An $H$-element can be written as

\[ a \text{ rational function} \times (\alpha_1(t)^{x_1} \cdots \alpha_m(t)^{x_m}) \times E(t) \mathcal{G}(x) \mathcal{Q}(y), \]

where

- $E$ is hyperexp. w.r.t. $\{\delta_1, \ldots, \delta_\ell\}$,
- $\mathcal{G}$ hypergeom. w.r.t. $\{\sigma_1, \ldots, \sigma_m\}$,
- $\mathcal{Q}$ $q$-hypergeom. w.r.t. $\{\tau_1, \ldots, \tau_n\}$.
Standard representations

Let $\prec$ be a monomial ordering in $\mathbb{F}[t, x, y]$.

**Definition.** Given a sequence $R$ of compatible rational functions

$$u_1, \ldots, u_\ell, v_1, \ldots, v_m, w_1, \ldots, w_n$$

in $\mathbb{F}(t, x, y)$, a sequence $S$ of rational functions:

$$f, \alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_\ell, \lambda_1, \ldots, \lambda_m, \mu_1, \ldots, \mu_n$$

is the **standard representation** of $R$ if

(i) the members of $R$ and $S$ satisfy the relations in the structure theorem;

(ii) $f$ has no nontrivial factors in $\mathbb{F}[t] \cup \mathbb{F}[x] \cup \mathbb{F}[y]$;

(iii) $f, \alpha_1, \ldots, \alpha_m$ are monic w.r.t. $\prec$. 
Computing a representation

For brevity, let $n = 0$.

**Algorithm.** Given compatible rational functions $u_1, \ldots, u_\ell, v_1, \ldots, v_m$, compute $f \in \mathbb{F}(t, x)$, $\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_\ell \in \mathbb{F}(t)$, and $\lambda_1, \ldots, \lambda_m \in \mathbb{F}(x) \in \mathbb{F}(x)$ s.t.

$$u_i = \frac{\delta_i(f)}{f} + \sum_{j=1}^{m} x_j \frac{\delta_i(\alpha_j)}{\alpha_j} + \beta_i \quad \text{for } 1 \leq i \leq \ell,$$

and

$$v_j = \frac{\sigma_j(f)}{f} \alpha_j \lambda_j \quad \text{for } 1 \leq j \leq m.$$
(1) [Determine $\alpha_j$ and $\lambda_j$.]

(1.1) For all $j$, compute $f_j, a_j, b_j$ s.t.

$$v_j = f_j a_j b_j,$$

where $f_j \in \mathbb{F}(t, x)$ has no factors in $\mathbb{F}(t) \cup \mathbb{F}(x)$, $a_j \in \mathbb{F}(t)$ is monic, and $b_j \in \mathbb{F}(x)$;

(1.2) Compute $g_j, r_j \in \mathbb{F}(t, x, y)$ s.t.

$$f_j = \frac{\sigma_j(g_j)}{g_j} r_j \text{ with } r_j \text{ being } \sigma_j\text{-reduced};$$

(1.3) Set $\alpha_j = a_j$ and $\lambda_j = r_j b_j$;

(2) [Determine $f$.] Set $f$ to be a nonzero solution of the system:

$$\left\{ \sigma_j(z) = \frac{\sigma_j(g_j)}{g_j} z \mid j = 1, \ldots, m \right\}.$$

(3) [Determine $\beta_i$] For all $i$, set

$$\beta_i = u_i - \frac{\delta_i(f)}{f} - \sum_{j=1}^{m} x_j \frac{\delta_i(\alpha_j)}{\alpha_j};$$

(4) Return: $f, \alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_\ell, \lambda_1, \ldots, \lambda_m$. 
Example

Consider an $H$-element $h(t, x_1, x_2, x_3, x_4)$ with certificates

\[ u = \frac{2x_1 + x_2 t - x_2}{t^2 - 1} \quad u = x_1 \frac{\delta(\alpha_1)}{\alpha_1} + x_2 \frac{\delta(\alpha_2)}{\alpha_2} \]

\[ v_1 = \frac{(t-1)(x_2-x_1)(x_2+4-x_1)}{(x_3+x_1+1)(x_1+1)(t+1)} \quad v_1 = \alpha_1 \lambda_1 \]

\[ v_2 = \frac{(t+1)(x_2+x_4+1)(x_2+x_3+1)}{2(x_2-x_1+1)(x_2+x_4-x_1+1)} \quad \sim \quad v_2 = \alpha_2 \cdot \lambda_2 \]

\[ v_3 = \frac{x_2+x_3+1}{x_3+x_1+1} \quad v_3 = \lambda_3 \]

\[ v_4 = \frac{x_2+x_4+1}{x_2+x_4-x_1+1} \quad v_4 = \lambda_4 \]

where

\[ \alpha_1 = \frac{t-1}{t+1}, \quad \alpha_2 = \frac{t+1}{2}, \]

\[ \lambda_1 = \frac{(x_2-x_1)(x_2+x_4-x_1)}{(x_3+x_1+1)(x_1+1)}, \quad \lambda_2 = \frac{(x_2+x_4+1)(x_2+x_3+1)}{(x_2-x_1+1)(x_2+x_4-x_1+1)}, \quad \lambda_3 = \frac{x_2+x_3+1}{x_3+x_1+1}, \quad \lambda_4 = \frac{x_2+x_4+1}{x_2+x_4-x_1+1}. \]

A multiplicative form of $h$ is

\[ h = \left( \frac{t - 1}{t + 1} \right)^{x_1} \left( \frac{t + 1}{2} \right)^{x_2} (x_2 + x_3) (x_2 + x_4). \]
Is \( h \) in \( \mathbb{F}(t, x, y) \)?

Let \( h \) be an \( H \)-element of the form:

\[
h = f(t, x, y) \alpha_1(t)^{x_1} \cdots \alpha_m(t)^{x_m} \mathcal{E}(t) \mathcal{G}(x) \mathcal{Q}(y),
\]

where the \( \alpha_j \)'s are monic

**Prop.** \( h \in \mathbb{F}(t, x, y) \iff \alpha_1 = \cdots = \alpha_m = 1, \ \mathcal{E}(t) \in \mathbb{F}(t), \ \mathcal{G}(x) \in \mathbb{F}(x) \) and \( \mathcal{Q}(y) \in \mathbb{F}(y) \).

**Facts:**

1. \( \mathcal{E}(t) \in \mathbb{F}(t) \iff \frac{\delta_i(\mathcal{E})}{\delta_i} = \frac{\delta_i(g_i)}{g_i} \) for some \( g_i \in \mathbb{F}(t) \), \( i = 1, \ldots \ell \).
2. \( \mathcal{G}(x) \in \mathbb{F}(x) \iff \frac{\sigma_j(\mathcal{G})}{\sigma_j} = \frac{\sigma_j(g_j)}{g_j} \) for some \( g_j \in \mathbb{F}(x) \), \( j = 1, \ldots, m \).
3. \( \mathcal{Q}(y) \in \mathbb{F}(y) \iff \frac{\tau_k(\mathcal{Q})}{\tau_k} = \frac{\tau_k(g_k)}{g_k} \) for some \( g_k \in \mathbb{F}(y) \), \( k = 1, \ldots, n \).
Is $h$ algebraic over $\mathbb{F}(t, x, y)$?

**Prop.** $h$ is algebraic over $\mathbb{F}(t, x, y)$ iff $h^d \in \mathbb{F}(t, x, y)$ for some $d \in \mathbb{Z}$ with $d \neq 0$.

**Corollary.** Let $h$ be an $H$-element of the form:

$$h = f(t, x) \alpha_1(t)^{x_1} \cdots \alpha_m(t)^{x_m} \mathcal{E}(t) \mathcal{G}(x),$$

where the $\alpha_j$’s are monic. Let $\mathbb{U}$ denote the set of roots of unity in $\mathbb{F}$. Then $h$ is algebraic over $\mathbb{F}(t, x)$ iff

(i) $\alpha_1 = \cdots = \alpha_m = 1$,

(ii) $\frac{\delta_i(\mathcal{E})}{\mathcal{E}} = r_i \frac{\delta_i(\mathcal{G}(t))}{g_i(t)}$, where $r_i \in \mathbb{Q}, g_i \in \mathbb{F}(t), i = 1, \ldots, \ell$;

(iii) $\frac{\sigma_j(\mathcal{G})}{\mathcal{G}} = c_j \frac{\sigma_j(\mathcal{G}(x))}{g_j(x)}$, where $c_j \in \mathbb{U}, g_j \in \mathbb{F}(x), j = 1, \ldots, m$;
Example

Let $h$ satisfy $\delta_1(h) = u_1 h$, $\delta_2(h) = u_2 h$ and $\sigma(h) = v h$, where

$$u_1 = \frac{3t_1 + 1}{2t_1(t_1 + 1)}, \quad u_2 = \frac{x - 2t_2}{3t_2(x + t_2)}, \quad v = -\frac{x + t_2}{x + t_2 + 1}$$

- By the structure theorem,

$$f = \frac{1}{x + t_2}, \quad \alpha = 1, \quad \beta_1 = \frac{3t_1 + 1}{2t_1(t_1 + 1)}, \quad \beta_2 = \frac{1}{3t_2}, \quad \lambda = -1.$$

- Christopher’s theorem

$$\beta_1 = \frac{\delta_1(t_1 + 1)}{t_1 + 1} + \frac{1}{2} \frac{\delta_1(t_1)}{t_1}, \quad \beta_2 = \frac{\delta_2(t_1 + 1)}{t_1 + 1} + \frac{1}{3} \frac{\delta_2(t_2)}{t_2}.$$

- Observe that $\lambda = -1$.

So $h$ is algebraic over $\mathbb{F}(t_1, t_2, x)$.

$h$ is of the form

$$h = c \frac{1}{x + t_2} \left( (t_1 + 1)t_1^{\frac{1}{2}} t_2^{\frac{1}{3}} \right) (-1)^x.$$
Algebraic dependence (ongoing project)

**Problem.** Given $H$-elements $h_1, \ldots, h_s$ over $\mathbb{F}(t, x, y)$, determine whether they are algebraically dependent over $\mathbb{F}(t, x, y)$.

For $i = 1, \ldots, s$, write

$$h_i = f_i \alpha_{i,1}^{x_1} \cdots \alpha_{i,m_i}^{x_m} \mathcal{E}_i(t) \mathcal{G}_i(x) \mathcal{Q}_i(y),$$

where $\alpha_{i,j} \in \mathbb{F}(t)$ is monic, $\mathcal{E}_i(t)$ hyperexp., $\mathcal{G}_i(x)$ hypergeom., and $\mathcal{Q}_i q$-hypergeom..

**Prop.** $h_1, \ldots, h_s$ are a.d. over $\mathbb{F}(t, x, y)$ iff there exist $e_1, \ldots, e_s \in \mathbb{Z}$, not all zero, s.t.

(i)$$\alpha_{1,j}^{e_1} \cdots \alpha_{s,j}^{e_s} = 1 \quad \text{for } 1 \leq j \leq m;$$

(ii)$$\mathcal{E}_1^{e_1} \cdots \mathcal{E}_s^{e_s} \in \mathbb{F}(t), \quad \mathcal{G}_1^{e_1} \cdots \mathcal{G}_s^{e_s} \in \mathbb{F}(x), \quad \mathcal{Q}_1^{e_1} \cdots \mathcal{Q}_s^{e_s} \in \mathbb{F}(y).$$
Symbolic power functions

**Problem.** Given $\alpha_{i,j} \in \mathbb{F}(t)$ is monic, where $i = 1, \ldots, s$ and $j = 1, \ldots, m$, compute the $\mathbb{Z}$-submodule $M_{\alpha,j}$ of $\mathbb{Z}^s$ s.t.

$$\alpha_{1,j}^{e_{1,j}} \cdots \alpha_{s,j}^{e_{s,j}} = 1$$

for all $(e_{1,j}, \ldots, e_{s,j}) \in \mathbb{Z}^s$.

---

**A method**

1. Determine $M_{\alpha,j}$ — computing polynomial gccls and solving linear systems over $\mathbb{Z}$.
2. $M_{\alpha} = M_{\alpha,1} \cap \cdots \cap M_{\alpha,m}$. 
Example

Let
\[ f_1 = \frac{1}{(t^2 - 1)(t^3 - 1)} \quad \text{and} \quad f_2 = \frac{(t - 1)^2}{(t + 1)^3(t^2 + t + 1)}. \]

Decide if \( f_1^x \) and \( f_2^x \) are a.d. over \( \mathbb{F}(t, x) \).

By gcd-computation
\[ f_1 = \frac{1}{(t - 1)^2(t + 1)(t^2 + t + 1)} \quad \text{and} \quad f_2 = \frac{(t - 1)^2}{(t + 1)^3(t^2 + t + 1)}. \]

\( f_1^{e_1}f_2^{e_2} = 1 \) iff
\[
\begin{align*}
2e_1 - 2e_2 &= 0 \\
e_1 + 3e_2 &= 0 \\
e_1 + e_2 &= 0
\end{align*}
\]
iff \( e_1 = e_2 = 0 \).

**Answer.** \( f_1^x \) and \( f_2^x \) are a.i. over \( \mathbb{F}(t, x) \).
Hyperexponential case

Let $\delta = d/dz$

**Lemma.** Let $\mathcal{E}_1, \ldots, \mathcal{E}_s$ be hyperexponential over $\mathbb{F}(z)$, and

$$\frac{\delta(\mathcal{E}_i)}{\mathcal{E}_i} = \beta_i, \quad i = 1, \ldots, s.$$  

Then

$$\exists e_1, \ldots, e_s \in \mathbb{Z}, \quad \prod_{i=1}^{s} \mathcal{E}_i^{e_i} \in \mathbb{F}(z)$$

is a logarithmic derivative of some element in $\mathbb{F}(z)$. 

$$\sum_{i=1}^{s} e_i \beta_i$$
Example

Let

\[ E_1 = \exp\left( \int \left( \frac{1}{z} - \frac{1}{z-1} + 4z \right) \, dz \right) \quad \text{and} \quad E_2 = \exp\left( \int \left( \frac{1}{z} - \frac{2z}{z^2 + 1} - 2z \right) \, dt \right). \]

 Decide if \( E_1 \) and \( E_2 \) are a.d. over \( \mathbb{F}(z) \).

By partial fraction decomposition,

\[ \frac{\delta(E_1)}{E_1} = \left( \frac{1}{z} - \frac{1}{z-1} \right) + 4z \quad \text{and} \quad \frac{\delta(E_2)}{E_2} = \left( \frac{1}{z} - \frac{2z}{z^2 + 1} \right) - 2z. \]

\( E^{e_1}E^{e_2} \in \mathbb{F}(z) \) iff

\[ e_1 \frac{\delta(E_1)}{E_1} + e_2 \frac{\delta(E_2)}{E_2} \]

is a logarithemic derivative

iff \( 4e_1 - 2e_2 = 0 \).

Answer. \( E_1 \) and \( E_2 \) are a.d. over \( \mathbb{F}(z) \).
Hypergeometric case

Let \( \sigma(f(z)) = f(z + 1) \) for all \( f \in \mathbb{F}(z) \).

**Lemma.** Let \( G_1, \ldots, G_s \) be hypergeometric over \( \mathbb{F}(z) \), and

\[
\frac{\sigma(G_i)}{G_i} = \lambda_i, \quad i = 1, \ldots, s.
\]

Write \( \lambda_i = c_i \tilde{\lambda}_i \), where \( c_i \) is in \( \mathbb{F} \) and \( \tilde{\lambda}_i \) is monic. Then

\[
\exists e_1, \ldots, e_s \in \mathbb{Z}, \quad \prod_{i=1}^{s} G_i^{e_i} \in \mathbb{F}(z)
\]

\[
\iff \prod_{i=1}^{s} c_i^{e_i} = 1 \quad \text{and} \quad \sum_{i=1}^{s} e_s \frac{\delta(\lambda_i)}{\lambda_i} = \sigma \left( \frac{\delta(g)}{g} \right) - \frac{\delta(g)}{g}
\]

for some \( g \in \mathbb{F}(z) \).

**Reference.** M. Singer. *Deciding if solutions of \( \sigma(y_1) = f_1 y_1, \ldots, \sigma y_n = f_n y_n \) are algebraically dependent over \( \mathbb{C}(x) \).* A note on discussions at the 2nd NCSU-China Symb. Comput. Collaboration Workshop, Hangzhou, 2007.
Subproblems

1. Given \( c_1, \ldots, c_s \) in a finite algebraic extension over \( \mathbb{Q} \), find all vectors \((e_1, \ldots, e_s) \in \mathbb{Z}^s\) s.t.

\[
\prod_{i=1}^{s} c_i^{e_i} = 1.
\]


2. Given \( \lambda_1, \ldots, \lambda_s \in \mathbb{F}(z) \), find all vectors \((e_1, \ldots, e_s) \in \mathbb{Z}^s\) s.t.

\[
\sum_{i=1}^{s} e_i \frac{\delta(\lambda_i)}{\lambda_i} = \sigma \left( \frac{\delta(g)}{g} \right) - \frac{\delta(g)}{g}
\]

for some \( g \in \mathbb{F}(z) \).

Reference. Singer’s note
\(q\)-Hypergeometric case

Let \(\tau(f(z)) = f(qz)\) for all \(f \in \mathbb{F}(z)\), where \(q\) is not a root of unity.

Let \(Q_1, \ldots, Q_s\) be \(q\)-hypergeometric over \(\mathbb{F}(z)\), and

\[
\frac{\tau(Q_i)}{Q_i} = \mu_i, \quad i = 1, \ldots, s.
\]

Write \(\mu_i = c_i\tilde{\mu}_i\), where \(c_i\) is in \(\mathbb{F}\) and \(\tilde{\mu}_i\) is monic.

**Lemma.** Assume that there exist \(e_1, \ldots, e_s \in \mathbb{Z}\) s.t. \(\prod_{i=1}^{s} Q_i^{e_i} \in \mathbb{F}(z)\). Then

(i) there exists \(g \in \mathbb{F}(z)\) s.t.

\[
\sum_{i=1}^{s} e_i \frac{\delta(\mu_i)}{\mu_i} = \tau \left( \frac{\delta(g)}{g} \right) - \frac{\delta(g)}{g};
\]

(ii)

\[
\prod_{i=1}^{s} c_i^{e_i} = q^d,
\]

where \(d = \deg \text{num}(g) - \deg \text{den}(g)\).
Summary

\[ h = \text{a rational function} \times \alpha_1(t)^{x_1} \cdots \alpha_m(t)^{x_m} \mathcal{E}(t) \mathcal{G}(x) \mathcal{Q}(y). \]

Future work

1. Implementation of algorithms for decomposing \( H \)-elements;
2. An algorithm for determining algebraic dependence of \( H \)-elements;
3. Criteria for the existence of telescopers.
4. ...