Polar varieties and optimization

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Optimization problem

- **General problem.** Let $f \in \mathbb{Q}[X_1, \ldots, X_n]$ and $\mathscr{S} \subset \mathbb{R}^n$ the semialgebraic set defined by

$$f_1 = \cdots = f_p = 0, g_1 > 0, \cdots, g_s > 0$$

where f_i and g_i are polynomials in $\mathbb{Q}[X_1, \ldots, X_n]$
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- **Subproblems.**
  $\rightsquigarrow$ infimum encoded by a polynomial + isolating interval;
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- **Motivations.** Engineering sciences, control theory (Henrion), program verification (Jouannaud, Monniaux), polynomial system solving with noise (Hutton, Kaltofen, Zhi), etc.

$\rightsquigarrow$ Unconstrained case ($f^\star = \inf_{x \in \mathbb{R}^n} f(x)$).

Real Algebra

Algebraic certificate when $S = \varnothing$.

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Examples:
- $S = \{f < 0\}$

$\rightsquigarrow$ Positivstellensatz:
$f > 0$ on $\mathbb{R}^n \Leftrightarrow \exists s, t\, \textbf{sos},\ sf = 1 + t$;
$\rightsquigarrow$ no degree bounds.

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$\rightsquigarrow$ based on numerical solvers;
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Example:

- $S_t = \{f - t = 0\}$

$\rightsquigarrow$ certificates: points in S_t;

$\rightsquigarrow$ reduction to the 0-dim case (critical point method).

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Goal: compute **sos** representation

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Example (Nie, Demmel, Sturmfels):

When $f^\star$ is reached,

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Example (Berg):
$f(x,y) = x^2y^2(x^2+y^2-1)$.
$f^\star = -1/27$ but $\dim \nabla(f) = 1$.

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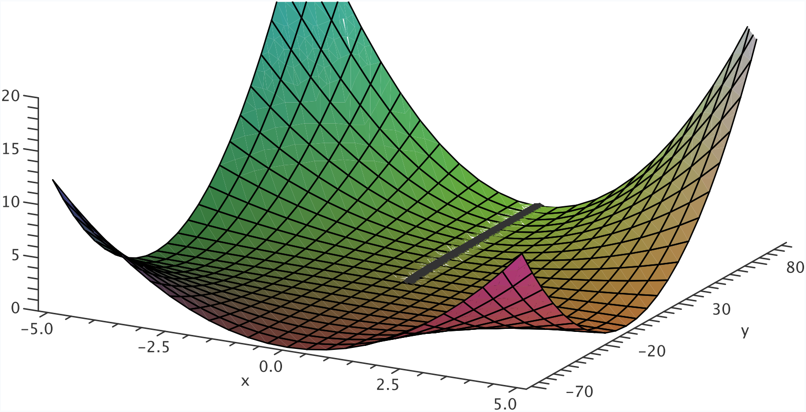
$\rightsquigarrow$ Problem when $f^\star \notin f(\nabla(f))$.

Example

$f := (xy-1)^2 + x^2 \geq 0.$

$f(1/\ell, \ell) \underset{\ell\to\infty}{\longrightarrow} 0 \rightsquigarrow f^\star = 0$ whereas $V(\nabla f) = \{(0,0)\}$ and $f(0,0) = 1 \neq f^\star$.



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Problem: V variety, $f^\star = \inf_{x \in V} f(x)$.
Find an ideal I such that:
- $f - f^\star + \varepsilon = \mathbf{sos}$ mod I;
- I has "small" dimension;
- $\inf_{x \in V} f(x) = \inf_{x \in \mathbb{V}(I)} f(x)$.

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Problem: Find a variety W of "small" dimension such that

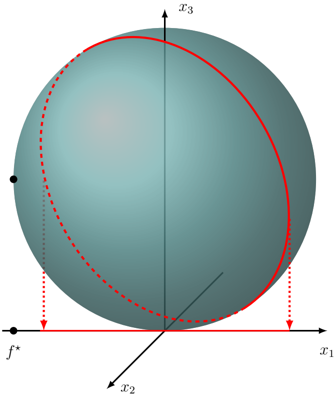
$$f^\star = \inf_{x \in \mathbb{R}^n} f(x) \text{ reached}$$

$$\Leftrightarrow \exists x^\star \in W,\ f(x^\star) = f^\star.$$

$\rightsquigarrow$ "smaller" varieties with enough information about the image

$\inf_{x \in V} f(x) = \inf_{x \in \mathbb{V}(I)}$.

$$f = x_1,\ \mathbf{F} = x_1^2 + x_2^2 + (x_3 - 1)^2 - 1,\ V = \mathbb{V}(\mathbf{F}).$$



- $\mathbb{V}(I)$ = red circle $\rightsquigarrow$ bad choice: $\inf_{x \in V} f(x) \neq \inf_{x \in \mathbb{V}(I)} f(x)$ because $f^{-1}(t) \cap V \cap \mathbb{R}^n \neq \varnothing \not\Leftrightarrow f^{-1}(t) \cap \mathbb{V}(I) \cap \mathbb{R}^n \neq \varnothing$.

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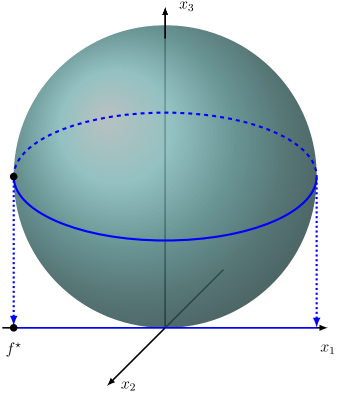

- $\mathbb{V}(I)$ = blue circle $\rightsquigarrow$ OK $\rightsquigarrow f^{-1}(t) \cap V \cap \mathbb{R}^n \neq \varnothing \Leftrightarrow f^{-1}(t) \cap \mathbb{V}(I) \cap \mathbb{R}^n \neq \varnothing$.

Sufficient condition: for almost all t, $\mathbb{V}(f-t) \cap \mathbb{V}(I) \cap \mathbb{R}^n$ meets every connected component of $\mathbb{V}(f-t) \cap V \cap \mathbb{R}^n$.

Bank, Giusti, Heintz, Mbakop (2001):

- $V = \mathbb{V}(f_1, \ldots, f_s)$ smooth and compact;
- for all i, $\langle f_1, \ldots, f_i \rangle$ radical of codimension i;

$\rightsquigarrow$ union of polar varieties meets each connected component of $V \cap \mathbb{R}^n$.

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Safey, Schost (2003):

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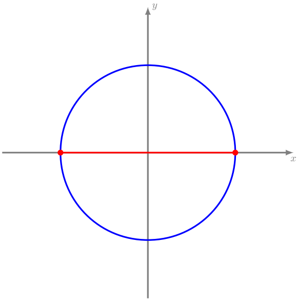
- $\langle f_1, \ldots, f_s \rangle$ radical;

$\rightsquigarrow$ union of polar varieties meets each connected component of $V \cap \mathbb{R}^n$ in generic coordinates.

Polar varieties: geometric idea



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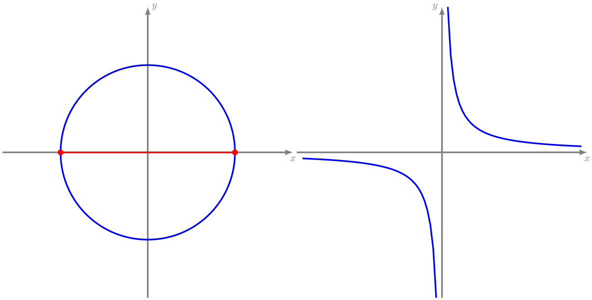

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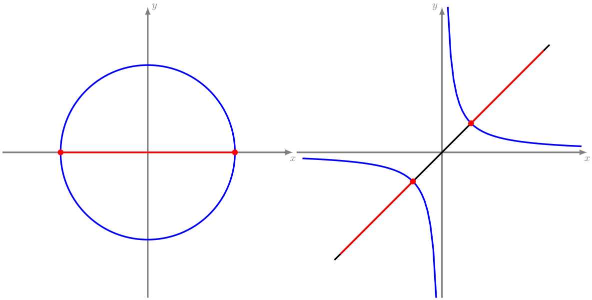

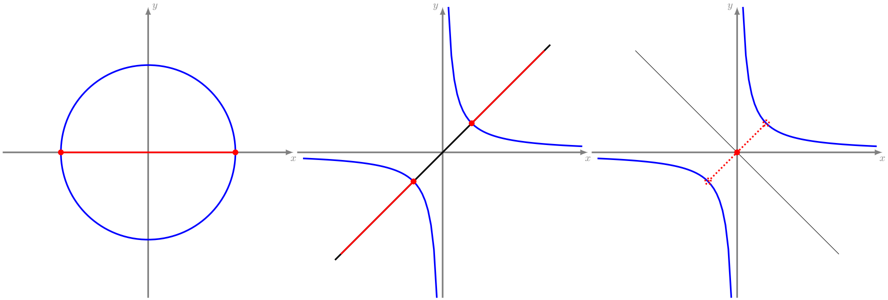
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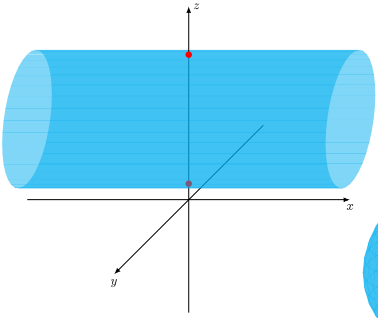

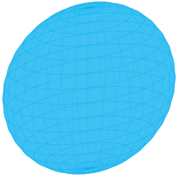

$\widetilde{W}_2 = V$;

$\widetilde{W}_2 \cap \mathbb{V}(x, y)$: red points.

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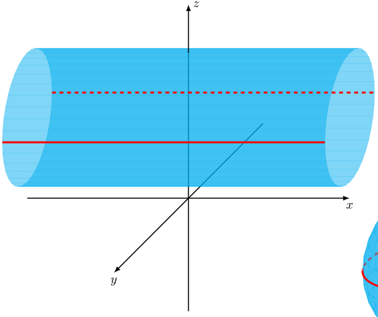
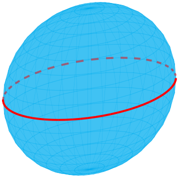
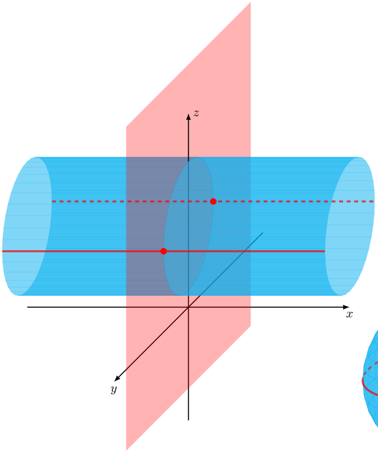
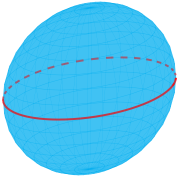




$\widetilde{W}_1$ = critical points of the projection $(x, y, z) \mapsto (x, y)$;





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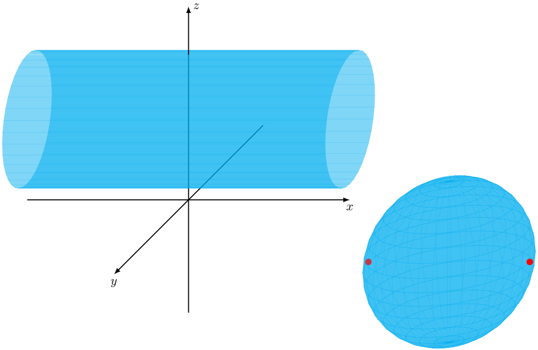

$\widetilde{W}_0$ = critical points of the projection $(x, y, z) \mapsto x$ $\rightsquigarrow$ red points.

Polar varieties: definition and properties

- $\mathbf{X}_{\leq i} = \{X_1, \ldots, X_i\}$;
- $\mathbf{F} = \{f_1, \ldots, f_s\}$, $V = \mathbb{V}(\mathbf{F})$ of dimension d;
- J the Jacobian matrix $J = \begin{bmatrix} \frac{\partial f_1}{\partial X_1} & \cdots & \frac{\partial f_1}{\partial X_{i+2}} & \cdots & \frac{\partial f_1}{\partial X_n} \\ \vdots & & \vdots & & \vdots \\ \frac{\partial f_s}{\partial X_1} & \cdots & \frac{\partial f_s}{\partial X_{i+2}} & \cdots & \frac{\partial f_s}{\partial X_n} \end{bmatrix}$

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- $0 \leq i \leq d-1$, i-th polar variety: $\widetilde{W}_i = \mathbb{V}(\mathbf{F}, \mathrm{MaxMinors}(J, \mathbf{X}_{\geq i+2}))$;
- $\widetilde{W}_d = V$.

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Theorem (Safey/Schost)

$V = \mathbb{V}(\mathbf{F})$ smooth, equidimensional, $\langle \mathbf{F} \rangle$ radical;
$\mathbf{A} \in GL_n(\mathbb{Q})$ generic change of variables.

- The sets $\widetilde{W}_i^{\mathbf{A}} \cap \mathbb{V}(\mathbf{X}_{\leq i})$ are empty or 0-dimensional;
- their reunion meets every connected component of $V^{\mathbf{A}} \cap \mathbb{R}^n$.

- Extension to the constrained case with equations of the sums of squares approach (joint work with F. Guo, M. Safey El Din and L. Zhi):

Main results

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  $\rightsquigarrow$ varieties $W_i = \mathbb{V}(\mathbf{F}, \mathbf{G}_i)$ such that $f^\star$ can be computed by solving the **sos** problems restricted to the W_i.

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  $\rightsquigarrow$ Efficient in practice.

- Algorithm to decide whether a infimum is reached in the unconstrained case in time $\tilde{O}\left(n^6(nL+n^4)(D-1)^{3n}\right)$ (joint work with M. Safey El Din).

Sums of squares in the constrained case

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Goal: construct algebraic varieties $W_i = \mathbb{V}(\mathbf{F}, \mathbf{G}_i)$ such that

1. $f^\star := \inf_{x \in V} f(x) = \inf_{x \in \bigcup W_i} f(x)$
   $\rightsquigarrow$ condition: for almost all t, $f^{-1}(t) \cap \bigcup W_i \cap \mathbb{R}^n$ meets every connected component of $f^{-1}(t) \cap V \cap \mathbb{R}^n$;

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   $\rightsquigarrow$ condition: for almost all t, $f^{-1}(t) \cap \bigcup W_i \cap \mathbb{R}^n$ meets every connected component of $f^{-1}(t) \cap V \cap \mathbb{R}^n$;

2. $f - f^\star + \varepsilon = \mathbf{sos}$ mod $\langle \mathbf{F}, \mathbf{G}_i \rangle$.
   $\rightsquigarrow$ $f^\star_{|W_i} = f^{\mathbf{sos}}_{|W_i} := \sup\{a \in \mathbb{R} \mid f - a = \mathbf{sos} \text{ mod } \langle \mathbf{F}, \mathbf{G}_i \rangle\}$;
   $\rightsquigarrow$ $f^{\mathbf{sos}}_{|W_i}$ can be computed using SDP.

Sums of squares (constrained case)

$\mathbf{A} \in GL_n(\mathbb{Q})$ generic change of variables, $J^{\mathbf{A}}$ jacobian matrix of $\{\mathbf{F}^{\mathbf{A}}, f^{\mathbf{A}}\}$;

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for $0 \leq i \leq d$:

$$W_i^{\mathbf{A}} = \mathbb{V}\left(\mathbf{X}_{\leq i}, \mathbf{F}^{\mathbf{A}}, \mathsf{MaxMinors}(J^{\mathbf{A}}, X_{\geq i+2})\right).$$

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- Asymptotic values:

$$R_\infty(f,S) = \{y \in \mathbb{R} \mid \exists (x_k)_k \subset S \text{ s.t. } \lim_{k\to\infty} \|x_k\| = \infty, \lim_{k\to\infty} f(x_k) = y\}.$$

Theorem (Schweighofer)

$f, g_1, \ldots, g_m \in \mathbb{R}[X_1, \ldots, X_n]$, $S = \{x \in \mathbb{R}^n \mid g_1(x) \geq 0, \ldots, g_m(x) \geq 0\}$ and

- $f > 0$ on S;
- f bounded on S;
- $R_\infty(f,S)$ finite subset of $]0, +\infty[$.

Then $f = \mathbf{sos} \bmod \langle g_1, \ldots, g_m \rangle$.

Workplan

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Sums of squares (constrained case)

Theorem (G. / Guo / Safey / Zhi)

$\mathbf{F} = \{f_1, \ldots, f_s\} \subset \mathbb{Q}[X_1, \ldots, X_n]$, $\langle f_1, \ldots, f_s \rangle$ radical, $V = \mathbb{V}(\mathbf{F})$ smooth and equidimensional;

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Numerical experiments:

- equivalent as existent methods when $f^{\star}$ is reached;
- better results when $f^{\star}$ is not reached.

Deciding whether the infimum is reached or not

- $f \in \mathbb{Q}[X_1, \ldots, X_n]$
- $f^\star := \inf_{x \in \mathbb{R}^n} f(x)$.

Goal: construct an algebraic variety V such that

- $f^\star$ is reached if and only if there exists $x^\star \in V$ such that $f(x^\star) = f^\star$;
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First idea: find $x^\star$ in $\mathbb{V}(\nabla f) \cap \mathbb{R}^n = \mathbb{V}\left(\frac{\partial f}{\partial X_1}, \ldots, \frac{\partial f}{\partial X_n}\right) \cap \mathbb{R}^n$.

Obstruction: $\mathbb{V}(\nabla f)$ can have dimension > 0, can be non radical.

Solution: construction of 0-dimensional sets meeting every connected component of $\mathbb{V}(\nabla(f))$.

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$\rightsquigarrow \varepsilon \to 0 \rightsquigarrow$ a point in each connected component of $\mathbb{V}\left(f^{\mathbf{A}} - f^\star\right)$;

$\rightsquigarrow$ projection to $[X_1, \ldots, X_n]$.

Deciding whether the infimum is reached

- $\mathfrak{C}_i^{\mathbf{A}} = \overline{\mathbb{V}\left(\mathbf{X}_{\leq i-1}, \frac{\partial f^{\mathbf{A}}}{\partial X_{i+1}}, \ldots, \frac{\partial f^{\mathbf{A}}}{\partial X_n}\right) \setminus \mathbb{V}\left(\frac{\partial f^{\mathbf{A}}}{\partial X_1}\right)}^{\mathscr{Z}}$;
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Theorem 1 (G. / Safey)

For a generic $\mathbf{A} \in GL_n(\mathbb{Q})$,

$\bigcup_{i=1}^{n} \mathscr{F}_i^{\mathbf{A}}$ meets every connected component of $V(f^{\mathbf{A}} - f^{\star}) \cap \mathbb{R}^n$.

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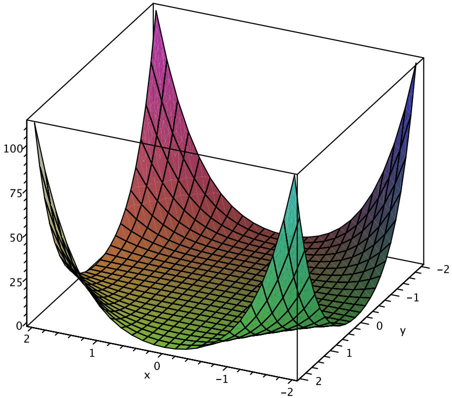
Theorem 2 (G. / Safey)

For a generic $\mathbf{A}$ and $1 \leq i \leq n$,

- the sets $\mathfrak{C}_i^{\mathbf{A}}$ have dimension ≤ 1;
- the sets $\mathscr{F}_i^{\mathbf{A}}$ have dimension ≤ 0.

Example

$$f(x,y) = x^2y^2(x^2+y^2-1) \rightsquigarrow$$
$$f^{\mathbf{A}}(x,y) = y^2x^4 - 4y^3x^3 + 7y^4x^2 - x^2y^2 - 6y^5x + 2xy^3 + 2y^6 - y^4.$$



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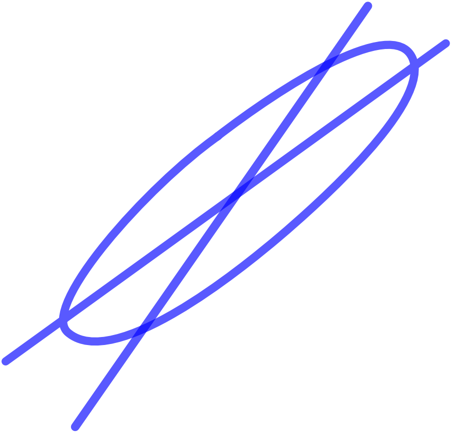

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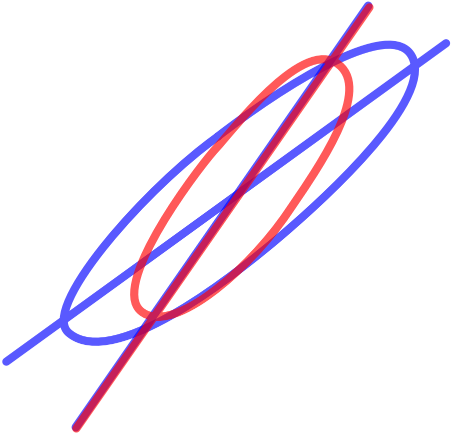

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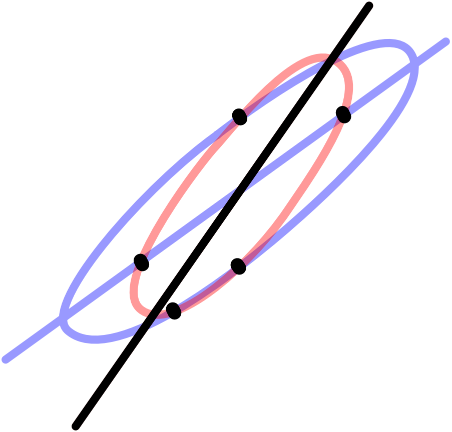

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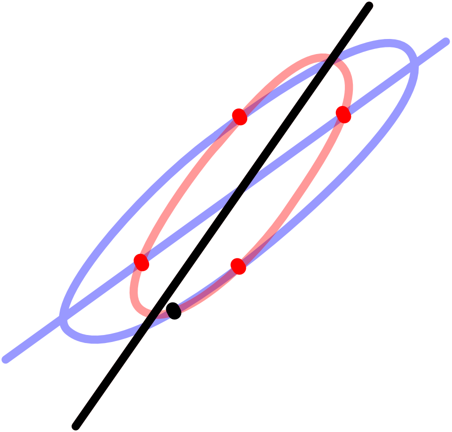

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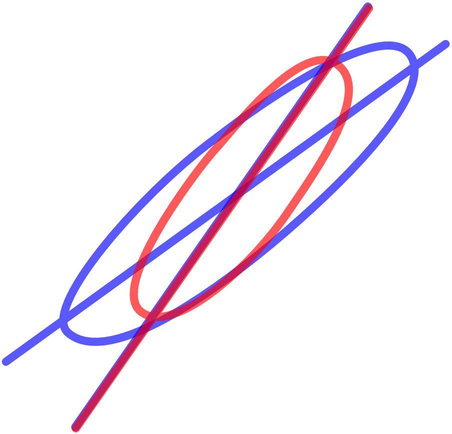

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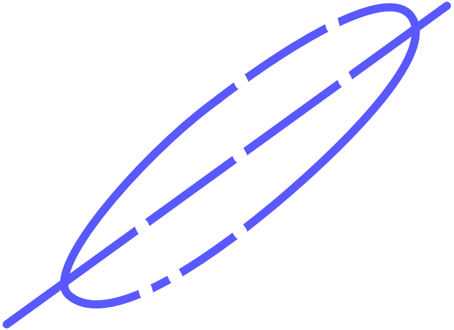

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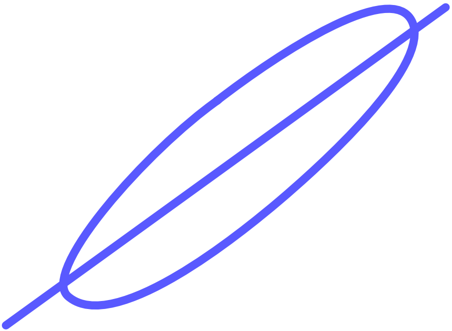

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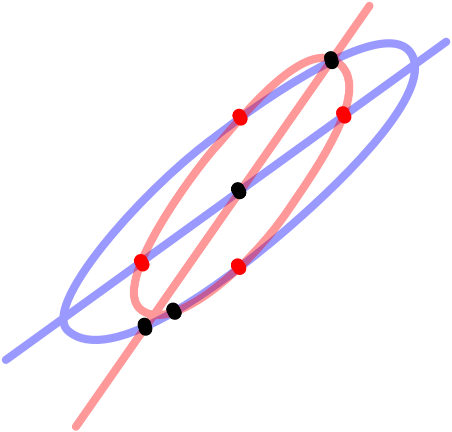

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The algorithm

Input: $f \in \mathbb{Q}[X_1, \ldots, X_n]$ bounded below. A real interval I and $P \in \mathbb{Q}[T]$ encoding $f^\star = \inf_{\mathbf{x} \in \mathbb{R}^n} f(\mathbf{x})$.

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1. choose randomly $\mathbf{A} \in GL_n(\mathbb{Q})$.
2. For $1 \leq i \leq n$
    1. compute the points in $\mathscr{F}_i$;
    2. for $x \in \mathscr{F}_i$, if $f(x) = f^\star$ then return `true`;

The algorithm

Input: $f \in \mathbb{Q}[X_1, \dots, X_n]$ bounded below. A real interval I and $P \in \mathbb{Q}[T]$ encoding $f^\star = \inf_{\mathbf{x} \in \mathbb{R}^n} f(\mathbf{x})$.

Output: a boolean which equals `true` if $f^\star$ is reached.

1. choose randomly $\mathbf{A} \in GL_n(\mathbb{Q})$.
2. For $1 \leq i \leq n$
   1. compute the points in $\mathscr{F}_i$;
   2. for $x \in \mathscr{F}_i$, if $f(x) = f^\star$ then return `true`;
3. return `false`.

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$\rightsquigarrow$ complexity using geometric resolution (Giusti, Lecerf, Salvy): $\tilde{O}\left(n^6(nL+n^4)(D-1)^{3n}\right)$ arithmetic operations in $\mathbb{Q}$.

Computations performed on a PC under Scientific Linux OS release 5.5 on Intel(R) Xeon(R) CPUs E5420 at 2.50 GHz with 20.55G RAM using FGb.

	D	**n**	$\sharp$Terms	Time	CAD
Sot1	24	4	677	3 h.	∞
Vor1	6	8	63	< 1 min.	∞
Vor2	5	18	253	5 h.	∞
K1	4	8	77	< 1 min.	∞
K2	4	8	53	< 1 min.	∞
K3	4	8	67	< 1 min.	∞
K4	4	8	45	< 1 min.	∞

- Contributions
  - Extension to the constrained case with equations of the sums of squares approach;
  - Algorithm to decide whether an infimum is reached in the unconstrained case in time $\tilde{O}\left(n^6(nL+n^4)(D-1)^{3n}\right)$.

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  - Extension to the constrained case with equations of the sums of squares approach;
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- Further work
  - Remove assumptions of regularity and decrease the number of minors involded in the **sos** approach;
  - Extend the **sos** approach to the contrained case with inequalities;
  - Investigate numerical stability;
  - Implement the algorithm deciding wheter an infimum is reached or not in Mathemagix.