

Polar varieties and optimization

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2011/01/31

- **General problem.** Let $f \in \mathbb{Q}[X_1, \dots, X_n]$ and $\mathcal{S} \subset \mathbb{R}^n$ the semialgebraic set defined by

$$f_1 = \dots = f_p = 0, g_1 > 0, \dots, g_s > 0$$

where f_i and g_i are polynomials in $\mathbb{Q}[X_1, \dots, X_n]$

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 - ↪ lower bounds on f^* ;
 - ↪ test of the reachability of the infimum.
- **Motivations.** Engineering sciences, control theory (Henrion), program verification (Jouannaud, Monniaux), polynomial system solving with noise (Hutton, Kaltofen, Zhi), etc.

↪ Unconstrained case ($f^* = \inf_{x \in \mathbb{R}^n} f(x)$).

Real Algebra

Algebraic certificate when $S = \emptyset$.

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Examples:

- $S = \{f < 0\}$

↪ Positivstellensatz:

$f > 0$ on $\mathbb{R}^n \Leftrightarrow \exists s, t \text{ sos}, sf = 1 + t$;

↪ no degree bounds.

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Example:

- $S_t = \{f - t = 0\}$

↪ certificates: points in S_t ;

↪ reduction to the 0-dim case
(critical point method).

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When f^* is reached,

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Goal: compute solutions of $\nabla f = 0$,
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Example (Nie, Demmel, Sturmfels):

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Example (Berg):

$f(x, y) = x^2 y^2 (x^2 + y^2 - 1)$.

$f^* = -1/27$ but $\dim \nabla(f) = 1$.

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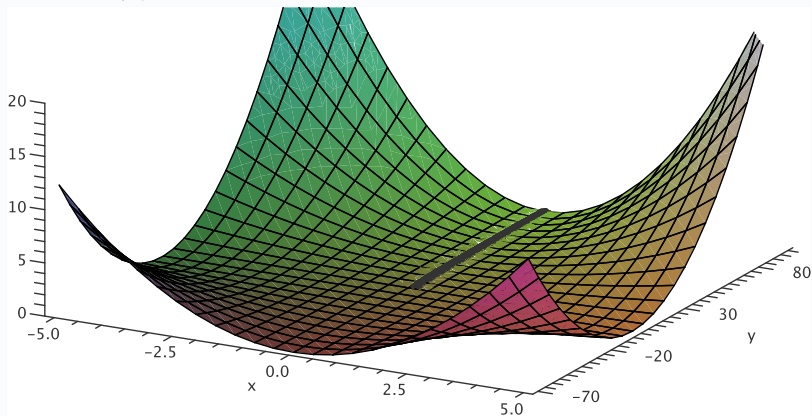
$f^* = -1/27$ but $\dim \nabla(f) = 1$.

\rightsquigarrow Problem when $f^* \notin f(\nabla(f))$.

Example

$$f := (xy - 1)^2 + x^2 \geq 0.$$

$f(1/\ell, \ell) \xrightarrow{\ell \rightarrow \infty} 0 \rightsquigarrow f^* = 0$ whereas $V(\nabla f) = \{(0,0)\}$ and $f(0,0) = 1 \neq f^*$.



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Problem: V variety, $f^* = \inf_{x \in V} f(x)$.

Find an ideal I such that:

- $f - f^* + \varepsilon = \mathbf{sos} \text{ mod } I$;
- I has “small” dimension;
- $\inf_{x \in V} f(x) = \inf_{x \in \mathbb{V}(I)} f(x)$.

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↪ “smaller” varieties with enough information about the image

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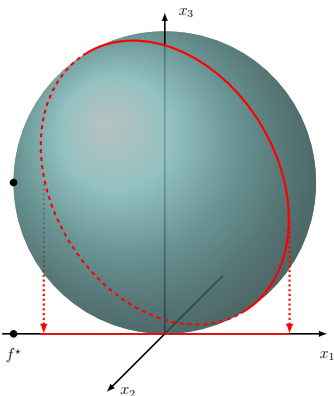
Problem: Find a variety W of “small” dimension such that

$$f^* = \inf_{x \in \mathbb{R}^n} f(x) \text{ reached}$$

$$\Leftrightarrow \exists x^* \in W, f(x^*) = f^*.$$

$$\inf_{x \in V} f(x) = \inf_{x \in \mathbb{V}(I)} f(x).$$

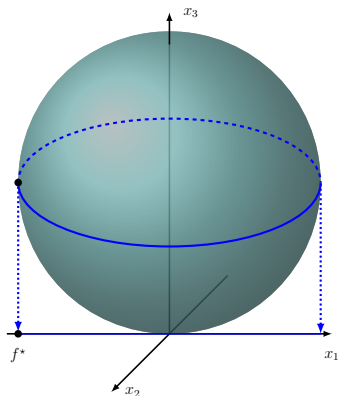
$$f = x_1, \mathbf{F} = x_1^2 + x_2^2 + (x_3 - 1)^2 - 1, V = \mathbb{V}(\mathbf{F}).$$



- $\mathbb{V}(I) = \text{red circle} \rightsquigarrow$ bad choice:
 $\inf_{x \in V} f(x) \neq \inf_{x \in \mathbb{V}(I)} f(x)$ because
 $f^{-1}(t) \cap V \cap \mathbb{R}^n \neq \emptyset \not\Leftarrow$
 $f^{-1}(t) \cap \mathbb{V}(I) \cap \mathbb{R}^n \neq \emptyset.$

$$\inf_{x \in V} f(x) = \inf_{x \in \mathbb{V}(I)} f(x).$$

$$f = x_1, \mathbf{F} = x_1^2 + x_2^2 + (x_3 - 1)^2 - 1, V = \mathbb{V}(\mathbf{F}).$$



- $\mathbb{V}(I) = \text{blue circle} \rightsquigarrow \text{OK}$
 $\rightsquigarrow f^{-1}(t) \cap V \cap \mathbb{R}^n \neq \emptyset \Leftrightarrow$
 $f^{-1}(t) \cap \mathbb{V}(I) \cap \mathbb{R}^n \neq \emptyset.$

Sufficient condition: for almost all t ,
 $\mathbb{V}(f-t) \cap \mathbb{V}(I) \cap \mathbb{R}^n$ meets every
connected component of
 $\mathbb{V}(f-t) \cap V \cap \mathbb{R}^n$.

Bank, Giusti, Heintz, Mbakop (2001):

- $V = \mathbb{V}(f_1, \dots, f_s)$ smooth and **compact**;
- for all i , $\langle f_1, \dots, f_i \rangle$ radical of codimension i ;

\rightsquigarrow union of polar varieties meets each connected component of $V \cap \mathbb{R}^n$.

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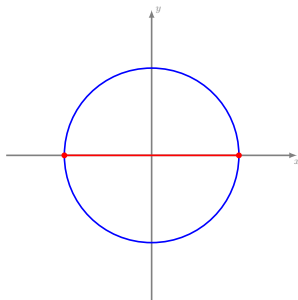
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Safey, Schost (2003):

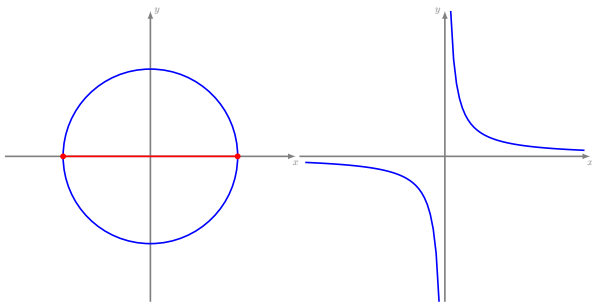
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\rightsquigarrow union of polar varieties meets each connected component of $V \cap \mathbb{R}^n$ in **generic coordinates**.

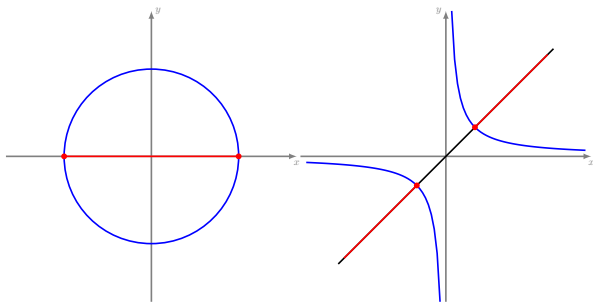
Polar varieties: geometric idea



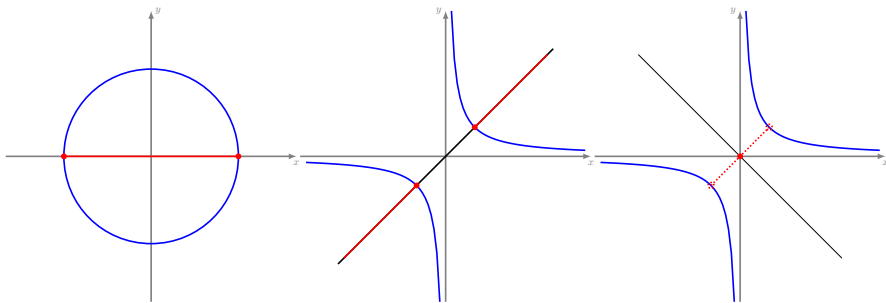
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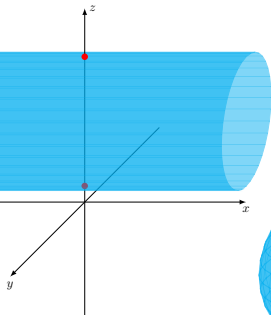
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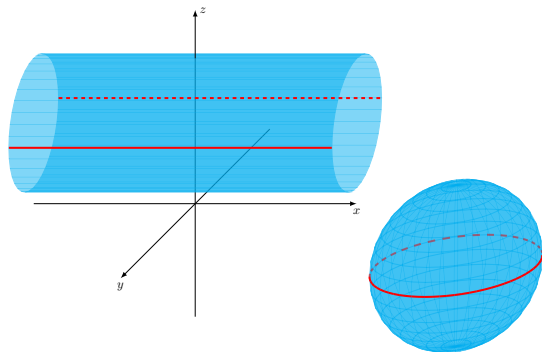
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$$\tilde{W}_2 = V;$$

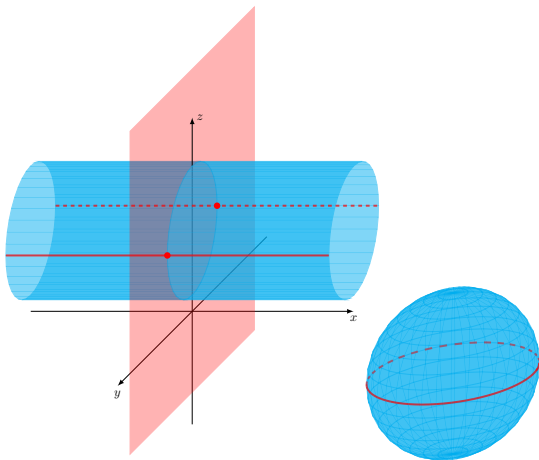
$\tilde{W}_2 \cap \nabla(x, y)$: red points.

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\tilde{W}_1 = critical points of the
projection $(x, y, z) \mapsto (x, y)$;

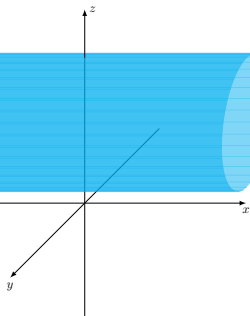
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$\tilde{W}_1 \cap \mathbb{V}(x)$: red points.

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\tilde{W}_0 = critical points of the
projection $(x, y, z) \mapsto x$
 \rightsquigarrow red points.

Polar varieties: definition and properties

- $\mathbf{X}_{\leq i} = \{X_1, \dots, X_i\}$;
- $\mathbf{F} = \{f_1, \dots, f_s\}$, $V = \mathbb{V}(\mathbf{F})$ of dimension d ;
- J the Jacobian matrix $J = \begin{bmatrix} \frac{\partial f_1}{\partial X_1} & \dots & \frac{\partial f_1}{\partial X_{i+2}} & \dots & \frac{\partial f_1}{\partial X_n} \\ \vdots & & \vdots & & \vdots \\ \frac{\partial f_s}{\partial X_1} & \dots & \frac{\partial f_s}{\partial X_{i+2}} & \dots & \frac{\partial f_s}{\partial X_n} \end{bmatrix}$

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- $0 \leq i \leq d-1$, i -th polar variety:
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Theorem (Safey/Schost)

$V = \mathbb{V}(\mathbf{F})$ smooth, equidimensional, $\langle \mathbf{F} \rangle$ radical;

$\mathbf{A} \in GL_n(\mathbb{Q})$ generic change of variables.

- The sets $\tilde{W}_i^{\mathbf{A}} \cap \mathbb{V}(\mathbf{X}_{\leq i})$ are empty or 0-dimensional;
- their reunion meets every connected component of $V^{\mathbf{A}} \cap \mathbb{R}^n$.

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\rightsquigarrow varieties $W_i = \mathbb{V}(\mathbf{F}, \mathbf{G}_i)$ such that f^* can be computed by solving the **sos** problems restricted to the W_i .

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- Algorithm to decide whether a infimum is reached in the unconstrained case in time $\tilde{O}(n^6(nL + n^4)(D - 1)^{3n})$ (joint work with M. Safey El Din).

Sums of squares in the constrained case

- $\mathbf{F} = \{f_1, \dots, f_s\} \subset \mathbb{Q}[X_1, \dots, X_n]$ such that
 - $\langle f_1, \dots, f_s \rangle$ is **radical**;
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$$\mathbf{1} \quad f^* := \inf_{x \in V} f(x) = \inf_{x \in \bigcup W_i} f(x)$$

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$$2 \quad f - f^* + \varepsilon = \mathbf{sos} \text{ mod } \langle \mathbf{F}, \mathbf{G}_i \rangle.$$

$\rightsquigarrow f|_{W_i}^* = f|_{W_i}^{\mathbf{sos}} := \sup\{a \in \mathbb{R} \mid f - a = \mathbf{sos} \text{ mod } \langle \mathbf{F}, \mathbf{G}_i \rangle\}$;

$\rightsquigarrow f|_{W_i}^{\mathbf{sos}}$ can be computed using SDP.

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for $0 \leq i \leq d$:

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$\rightsquigarrow (f^{\mathbf{A}})^{-1}(t) \cap W_i^{\mathbf{A}}$ intersection of $\mathbb{V}(\mathbf{X}_{\leq i})$ and the i -th polar variety associated with $\mathbb{V}(\mathbf{F}^{\mathbf{A}}, f^{\mathbf{A}} - t) = (f^{\mathbf{A}})^{-1}(t) \cap V$.

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Sums of squares (constrained case)

$\mathbf{A} \in GL_n(\mathbb{Q})$ generic change of variables, $J^{\mathbf{A}}$ jacobian matrix of $\{\mathbf{F}^{\mathbf{A}}, f^{\mathbf{A}}\}$;

Definition

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- Asymptotic values:

$$R_\infty(f, S) = \{y \in \mathbb{R} \mid \exists (x_k)_k \subset S \text{ s.t. } \lim_{k \rightarrow \infty} \|x_k\| = \infty, \lim_{k \rightarrow \infty} f(x_k) = y\}.$$

Theorem (Schweighofer)

$f, g_1, \dots, g_m \in \mathbb{R}[X_1, \dots, X_n]$, $S = \{x \in \mathbb{R}^n \mid g_1(x) \geq 0, \dots, g_m(x) \geq 0\}$ and

- $f > 0$ on S ;
- f bounded on S ;
- $R_\infty(f, S)$ finite subset of $]0, +\infty[$.

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$\mathbf{F} = \{f_1, \dots, f_s\} \subset \mathbb{Q}[X_1, \dots, X_n]$, $\langle f_1, \dots, f_s \rangle$ radical, $V = \mathbb{V}(\mathbf{F})$ smooth and equidimensional;

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Numerical experiments:

- equivalent as existent methods when f^* is reached;
- better results when f^* is not reached.

Deciding whether the infimum is reached or not

- $f \in \mathbb{Q}[X_1, \dots, X_n]$
- $f^* := \inf_{x \in \mathbb{R}^n} f(x)$.

Goal: construct an algebraic variety V such that

- f^* is reached if and only if there exists $x^* \in V$ such that $f(x^*) = f^*$;
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First idea: find x^* in $\mathbb{V}(\nabla f) \cap \mathbb{R}^n = \mathbb{V}\left(\frac{\partial f}{\partial X_1}, \dots, \frac{\partial f}{\partial X_n}\right) \cap \mathbb{R}^n$.

Obstruction: $\mathbb{V}(\nabla f)$ can have **dimension** > 0 , can be **non radical**.

Solution: construction of 0-dimensional sets meeting every connected component of $\mathbb{V}(\nabla(f))$.

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- ε infinitesimal, $\mathbf{A} \in GL_n(\mathbb{Q})$ generic change of coordinates;

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$\rightsquigarrow \varepsilon \rightarrow 0 \rightsquigarrow$ a point in each connected component of $\mathbb{V}(f^{\mathbf{A}} - f^{\star})$;

\rightsquigarrow projection to $[X_1, \dots, X_n]$.

Deciding whether the infimum is reached

- $\mathfrak{C}_i^{\mathbf{A}} = \overline{\mathbb{V}\left(\mathbf{X}_{\leq i-1}, \frac{\partial f^{\mathbf{A}}}{\partial X_{i+1}}, \dots, \frac{\partial f^{\mathbf{A}}}{\partial X_n}\right)} \setminus \mathbb{V}\left(\frac{\partial f^{\mathbf{A}}}{\partial X_1}\right)^{\mathcal{L}}$;
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Theorem 1 (G. / Safey)

For a generic $\mathbf{A} \in GL_n(\mathbb{Q})$,

$\bigcup_{i=1}^n \mathcal{F}_i^{\mathbf{A}}$ meets every connected component of $V(f^{\mathbf{A}} - f^*) \cap \mathbb{R}^n$.

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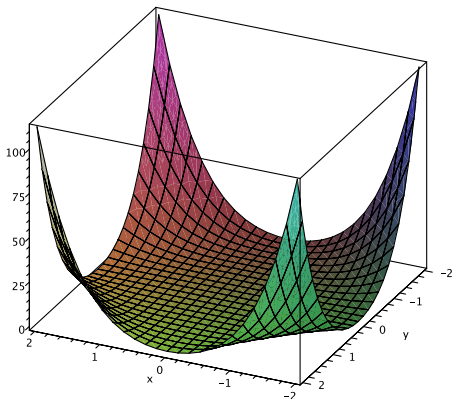
Theorem 2 (G. / Safey)

For a generic \mathbf{A} and $1 \leq i \leq n$,

- the sets $\mathfrak{C}_i^{\mathbf{A}}$ have dimension ≤ 1 ;
- the sets $\mathcal{F}_i^{\mathbf{A}}$ have dimension ≤ 0 .

Example

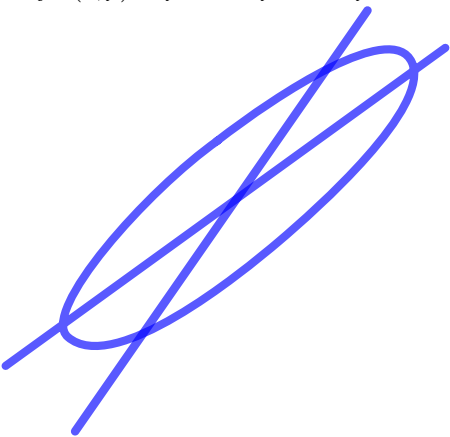
$$f(x,y) = x^2y^2(x^2 + y^2 - 1) \rightsquigarrow$$
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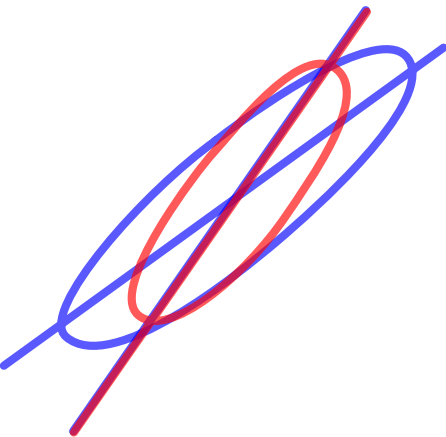


$$\blacksquare \nabla \left(\frac{\partial f^A}{\partial x_2} \right);$$

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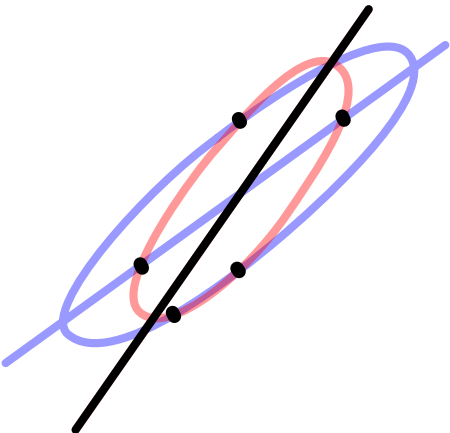
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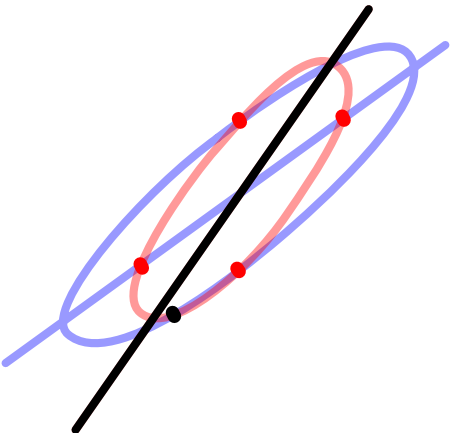


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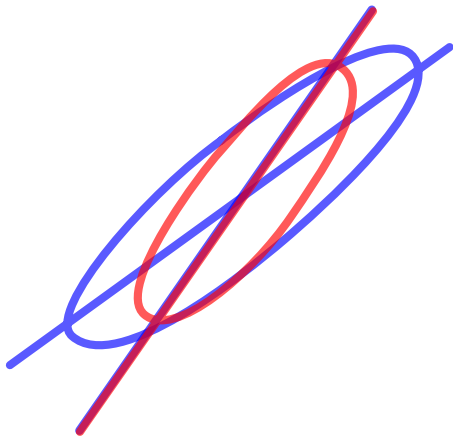


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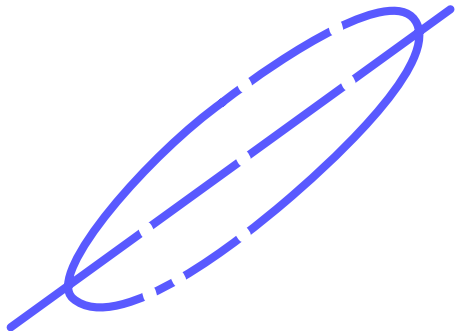
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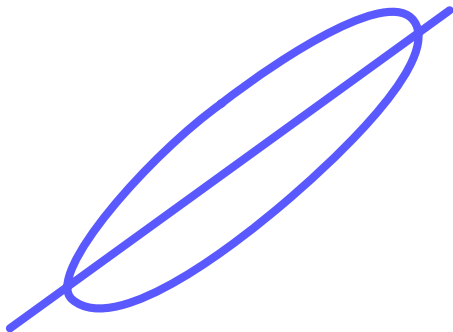


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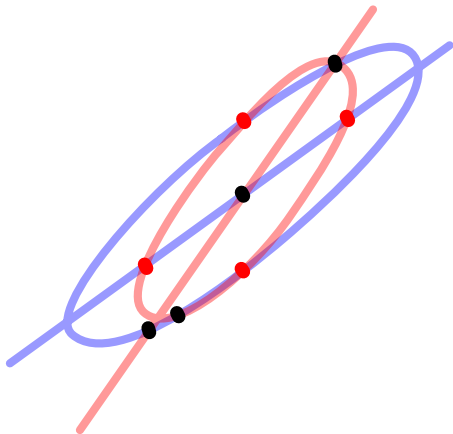
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Input: $f \in \mathbb{Q}[X_1, \dots, X_n]$ bounded below. A real interval I and $P \in \mathbb{Q}[T]$ encoding $f^* = \inf_{\mathbf{x} \in \mathbb{R}^n} f(\mathbf{x})$.

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\rightsquigarrow complexity using geometric resolution (Giusti, Lecerf, Salvy):
 $\tilde{O}(n^6(nL + n^4)(D - 1)^{3n})$ arithmetic operations in \mathbb{Q} .

Computations performed on a PC under Scientific Linux OS release 5.5 on Intel(R) Xeon(R) CPUs E5420 at 2.50 GHz with 20.55G RAM using FGb.

	D	n	#Terms	Time	CAD
Sot1	24	4	677	3 h.	∞
Vor1	6	8	63	< 1 min.	∞
Vor2	5	18	253	5 h.	∞
K1	4	8	77	< 1 min.	∞
K2	4	8	53	< 1 min.	∞
K3	4	8	67	< 1 min.	∞
K4	4	8	45	< 1 min.	∞

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■ Further work

- Remove assumptions of regularity and decrease the number of minors involved in the **sos** approach;
- Extend the **sos** approach to the constrained case with inequalities;
- Investigate numerical stability;
- Implement the algorithm deciding whether an infimum is reached or not in Mathemagix.