Polar varieties and optimization

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Séminaire ALGO - INRIA
2011/01/31
**General problem.** Let \( f \in \mathbb{Q}[X_1, \ldots, X_n] \) and \( \mathcal{I} \subset \mathbb{R}^n \) the semialgebraic set defined by

\[
f_1 = \cdots = f_p = 0, g_1 > 0, \cdots, g_s > 0
\]

where \( f_i \) and \( g_i \) are polynomials in \( \mathbb{Q}[X_1, \ldots, X_n] \)

**Problem:** compute \( f^* := \inf_{x \in \mathcal{I}} f(x) \)
General problem. Let $f \in \mathbb{Q}[X_1, \ldots, X_n]$ and $\mathcal{S} \subset \mathbb{R}^n$ the semialgebraic set defined by

$$f_1 = \cdots = f_p = 0, g_1 > 0, \cdots, g_s > 0$$

where $f_i$ and $g_i$ are polynomials in $\mathbb{Q}[X_1, \ldots, X_n]$

Problem: compute $f^* := \inf_{x \in \mathcal{S}} f(x)$

Subproblems.

$\rightsquigarrow$ infimum encoded by a polynomial + isolating interval;

$\rightsquigarrow$ lower bounds on $f^*$;

$\rightsquigarrow$ test of the reachability of the infimum.
General problem. Let $f \in \mathbb{Q}[X_1, \ldots, X_n]$ and $\mathcal{I} \subset \mathbb{R}^n$ the semialgebraic set defined by

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Subproblems.

$\leadsto$ infimum encoded by a polynomial + isolating interval;
$\leadsto$ lower bounds on $f^*$;
$\leadsto$ test of the reachability of the infimum.

Motivations. Engineering sciences, control theory (Henrion), program verification (Jouannaud, Monniaux), polynomial system solving with noise (Hutton, Kaltofen, Zhi), etc.
Methodology

Unconstrained case \( (f^* = \inf_{x \in \mathbb{R}^n} f(x)) \).

Real Algebra

Algebraic certificate when \( S = \emptyset \).

Examples:

- \( S = \{ f < 0 \} \)
  - Positivstellensatz: \( f > 0 \) on \( \mathbb{R}^n \) if and only if \( \exists s, t \) s.t. \( sf = 1 + t \); no degree bounds.
  - Sums of squares: \( f = \text{sos} \).
  - Based on numerical solvers.
  - Rationalization of numerical sos decomposition.
  - Lower bound on \( f^* \).

Real Geometry

Algebraic certificate when \( S \neq \emptyset \).

Example:

- \( S_t = \{ f - t = 0 \} \)
  - Certificates: points in \( S_t \);
  - Reduction to the 0-dim case (critical point method).
Unconstrained case \((f^* = \inf_{x \in \mathbb{R}^n} f(x))\).

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  \(\therefore\) no degree bounds.
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Goal: compute \textit{sos} representation
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\[ f \geq 0 \not\Rightarrow f = \text{sos} \] (Artin, Motzkin)
Real Algebra

Goal: compute \textit{sos} representation

\[ f \geq 0 \nRightarrow f = \text{sos} \ (\text{Artin, Motzkin}) \]

\[ \frac{\text{vol}(\text{sos})}{\text{vol}(f \geq 0)} \rightarrow 0 \text{ when } n \rightarrow \infty \]

(Blekherman)
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*Example* (Nie, Demmel, Sturmfels):

When \( f^\star \) is reached,

\[ f - f^\star + \varepsilon = \text{sos mod } \nabla f; \]
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When \( f^* \) is reached,
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Real Geometry

Goal: compute solutions of \( \nabla f = 0 \),
where \( \nabla f = \mathbb{V} \left( \frac{\partial f}{\partial X_1}, \ldots, \frac{\partial f}{\partial X_n} \right) \).
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\[ \rightsquigarrow \text{Problem when } \dim \nabla f > 0 \text{ and } \nabla f \text{ can be non radical.} \]
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Example (Berg):
\( f(x, y) = x^2y^2(x^2 + y^2 - 1) \).
\( f^* = -1/27 \) but \( \dim \nabla(f) = 1 \).
Real Algebra

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\[ f \geq 0 \not\Rightarrow f = \text{sos} \quad \text{(Artin, Motzkin)} \]
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\frac{\text{vol(sos)}}{\text{vol}(f \geq 0)} \to 0 \quad \text{when} \quad n \to \infty
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When \( f^* \) is reached, \[ f - f^* + \epsilon = \text{sos} \text{ mod } \nabla f; \]

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\[ f(x, y) = x^2y^2 \left( x^2 + y^2 - 1 \right) \]

\[ f^* = -1/27 \text{ but dim } \nabla( f ) = 1. \]
\[ \leadsto \text{Problem when } f^* \not\in f \left( \nabla( f ) \right). \]
Asymptotic phenomena

Example

\[ f := (xy - 1)^2 + x^2 \geq 0. \]
\[ f(1/\ell, \ell) \xrightarrow{\ell \to \infty} 0 \sim f^* = 0 \text{ whereas } V(\nabla f) = \{(0,0)\} \text{ and } f(0,0) = 1 \neq f^*. \]
Real Algebra

**Goal**: more general certificates

\( \Rightarrow \) valid even if \( f^* \) not reached;

\( \Rightarrow \) valid in the **constrained** case
Real Algebra

**Goal:** more general certificates
⇝ valid even if $f^*$ not reached;
⇝ valid in the constrained case

**Problem:** $V$ variety, $f^* = \inf_{x \in V} f(x)$.
Find an ideal $I$ such that:
- $f - f^* + \epsilon = \text{sos mod } I$;
- $I$ has “small” dimension;
- $\inf_{x \in V} f(x) = \inf_{x \in V(I)} f(x)$. 
Real Algebra

**Goal**: more general certificates
\[ \Rightarrow \text{valid even if } f^* \text{ not reached}; \]
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Real Geometry

**Goal**: \( f^* = \inf_{x \in \mathbb{R}^n} f(x) \), compute a point in \( \mathbb{V}(f - f^*) \cap \mathbb{R}^n \).

\( \Rightarrow \) “smaller” varieties with enough information about the image.
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Find an ideal \( I \) such that:
- \( f - f^* + \varepsilon = \text{sos mod } I \);
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Real Geometry

**Goal:** \( f^* = \inf_{x \in \mathbb{R}^n} f(x) \), compute a point in \( \mathbb{V}(f - f^*) \cap \mathbb{R}^n \).

**Problem:** Find a variety \( W \) of “small” dimension such that

\[
f^* = \inf_{x \in \mathbb{R}^n} f(x) \text{ reached } \iff \exists x^* \in W, f(x^*) = f^*.
\]

⇝ “smaller” varieties with enough information about the image
\[
\inf_{x \in V} f(x) = \inf_{x \in \mathbb{V}(I)} f(x).
\]

\[f = x_1, \quad \mathbf{F} = x_1^2 + x_2^2 + (x_3 - 1)^2 - 1, \quad V = \mathbb{V}(\mathbf{F}).\]

\[\mathbb{V}(I) = \text{red circle} \ni \text{bad choice:} \]
\[\inf_{x \in V} f(x) \neq \inf_{x \in \mathbb{V}(I)} f(x) \text{ because} \]
\[f^{-1}(t) \cap V \cap \mathbb{R}^n \neq \emptyset \]
\[f^{-1}(t) \cap \mathbb{V}(I) \cap \mathbb{R}^n \neq \emptyset.\]
\[ \inf_{x \in V} f(x) = \inf_{x \in \mathbb{V}(I)} f(x). \]

\[ f = x_1, \quad F = x_1^2 + x_2^2 + (x_3 - 1)^2 - 1, \quad V = \mathbb{V}(F). \]

\begin{itemize}
  \item \[ \mathbb{V}(I) = \text{blue circle} \implies \text{OK} \]
  \item \[ \implies f^{-1}(t) \cap V \cap \mathbb{R}^n \neq \emptyset \iff f^{-1}(t) \cap \mathbb{V}(I) \cap \mathbb{R}^n \neq \emptyset. \]
\end{itemize}

Sufficient condition: for almost all \( t \), \( \mathbb{V}(f - t) \cap \mathbb{V}(I) \cap \mathbb{R}^n \) meets every connected component of \( \mathbb{V}(f - t) \cap V \cap \mathbb{R}^n \).
Bank, Giusti, Heintz, Mbakop (2001):
- \( V = \mathbb{V}(f_1, \ldots, f_s) \) smooth and compact;
- for all \( i \), \( \langle f_1, \ldots, f_i \rangle \) radical of codimension \( i \);
\[ \Leftrightarrow \text{union of polar varieties meets each connected component of } V \cap \mathbb{R}^n. \]
Bank, Giusti, Heintz, Mbakop (2001):
- $V = \mathbb{V}(f_1, \ldots, f_s)$ smooth and compact;
- for all $i$, $\langle f_1, \ldots, f_i \rangle$ radical of codimension $i$;
$\leadsto$ union of polar varieties meets each connected component of $V \cap \mathbb{R}^n$.

Safey, Schost (2003):
- $V = \mathbb{V}(f_1, \ldots, f_s)$ smooth and equidimensional;
- $\langle f_1, \ldots, f_s \rangle$ radical;
$\leadsto$ union of polar varieties meets each connected component of $V \cap \mathbb{R}^n$ in generic coordinates.
Polar varieties: geometric idea
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A. Greuet

Polar varieties and optimization
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Polar varieties and optimization
\[ \tilde{W}_2 = V; \]
\[ \tilde{W}_2 \cap V(x, y): \text{ red points.} \]
\[ \tilde{W}_1 = \text{critical points of the projection } (x, y, z) \mapsto (x, y); \]
Polar varieties: geometric idea

\[ \tilde{W}_1 = \text{critical points of the projection } (x, y, z) \mapsto (x, y); \]

\[ \tilde{W}_1 \cap V(x): \text{red points.} \]
Polar varieties: geometric idea

\[ \tilde{W}_0 = \text{critical points of the projection } (x, y, z) \mapsto x \]
\[ \sim \rightarrow \text{red points.} \]
Polar varieties: definition and properties

- \( X_{\leq i} = \{X_1, \ldots, X_i\} \);
- \( F = \{f_1, \ldots, f_s\}, \ V = \mathbb{V}(F) \) of dimension \( d \);

\[
J = \begin{bmatrix}
\frac{\partial f_1}{\partial X_1} & \cdots & \frac{\partial f_1}{\partial X_{i+2}} & \cdots & \frac{\partial f_1}{\partial X_n} \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
\frac{\partial f_s}{\partial X_1} & \cdots & \frac{\partial f_s}{\partial X_{i+2}} & \cdots & \frac{\partial f_s}{\partial X_n}
\end{bmatrix}
\]

- \( J \) the Jacobian matrix

\( \tilde{W}_i = \mathbb{V}(F, \text{MaxMinors}(J, X \geq i+2)) \);

\( \tilde{W}_d = \mathbb{V} \).

Theorem (Safey/Schost)

\( V = \mathbb{V}(F) \) smooth, equidimensional, \( \langle F \rangle \) radical;

\( A \in \text{GL}_n(\mathbb{Q}) \) generic change of variables.

The sets \( \tilde{W}_A \cap \mathbb{V}(X \leq i) \) are empty or 0-dimensional; their reunion meets every connected component of \( \mathbb{V}_A \cap \mathbb{R}^n \).
Polar varieties: definition and properties

- \( X_{\leq i} = \{X_1, \ldots, X_i\} \);
- \( F = \{f_1, \ldots, f_s\} \), \( V = \mathbb{V}(F) \) of dimension \( d \);
- \( J \) the Jacobian matrix \( J = \begin{bmatrix} \frac{\partial f_1}{\partial X_1} & \cdots & \frac{\partial f_1}{\partial X_{i+2}} & \cdots & \frac{\partial f_1}{\partial X_n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \frac{\partial f_s}{\partial X_1} & \cdots & \frac{\partial f_s}{\partial X_{i+2}} & \cdots & \frac{\partial f_s}{\partial X_n} \end{bmatrix} \);
- \( 0 \leq i \leq d - 1 \), \( i \)-th polar variety: 
  \( \tilde{W}_i = \mathbb{V}(F, \text{MaxMinors}(J, X_{\geq i+2})) \);
- \( \tilde{W}_d = V \).
Polar varieties: definition and properties

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\( V = \mathbb{V}(F) \) smooth, equidimensional, \( \langle F \rangle \) radical;
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- The sets \( \tilde{W}_i^A \cap \mathbb{V}(X_{\leq i}) \) are empty or 0-dimensional;
- their reunion meets every connected component of \( V^A \cap \mathbb{R}^n \).
Main results

- Extension to the constrained case with equations of the sums of squares approach (joint work with F. Guo, M. Safey El Din and L. Zhi):

$V = V(F)$ smooth and equidimensional s.t. $\langle F \rangle$ is radical.

$\Rightarrow$ varieties $W_i = V(F, G_i)$ such that $f^{\star}$ can be computed by solving the sos problems restricted to the $W_i$.

$\Rightarrow$ Generalization of Nie, Demmel, Sturmfels;

$\Rightarrow$ Efficient in practice.

Algorithm to decide whether a infimum is reached in the unconstrained case in time $\tilde{O}(n^{6}(nL+n^{4})(D-1)^{3}n^3)$ (joint work with M. Safey El Din).
Extension to the constrained case with equations of the sums of squares approach (joint work with F. Guo, M. Safey El Din and L. Zhi):

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Main results

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  \[ \Rightarrow \text{varieties } W_i = \mathbb{V}(F, G_i) \text{ such that } f^* \text{ can be computed by solving the sos problems restricted to the } W_i. \]
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  \[ \Rightarrow \text{Efficient in practice.} \]

- Algorithm to decide whether a infimum is reached in the unconstrained case in time \( \tilde{O}(n^6(nL + n^4)(D - 1)^{3n}) \) (joint work with M. Safey El Din).
\[ F = \{ f_1, \ldots, f_s \} \subset \mathbb{Q}[X_1, \ldots, X_n] \] such that
\[ \langle f_1, \ldots, f_s \rangle \text{ is radical}; \]
\[ V = \mathbb{V}(F) \text{ is smooth and equidimensional}; \]
\[ f^* = \inf_{x \in V} f(x); \]
Sums of squares in the constrained case

\[ F = \{ f_1, \ldots, f_s \} \subset \mathbb{Q}[X_1, \ldots, X_n] \] such that

- \( \langle f_1, \ldots, f_s \rangle \) is radical;
- \( V = \mathbb{V}(F) \) is smooth and equidimensional;
- \( f^* = \inf_{x \in V} f(x) \);

**Goal:** construct algebraic varieties \( W_i = \mathbb{V}(F, G_i) \) such that

\[ f^* := \inf_{x \in V} f(x) = \inf_{x \in \bigcup W_i} f(x) \]

\[ \leftrightarrow \text{condition: for almost all } t, \ f^{-1}(t) \cap \bigcup W_i \cap \mathbb{R}^n \text{ meets every connected component of } f^{-1}(t) \cap V \cap \mathbb{R}^n; \]
Sums of squares in the constrained case

- $\mathbf{F} = \{f_1, \ldots, f_s\} \subset \mathbb{Q}[X_1, \ldots, X_n]$ such that
  - $\langle f_1, \ldots, f_s \rangle$ is radical;
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**Goal:** construct algebraic varieties $W_i = \mathbb{V}(\mathbf{F}, G_i)$ such that

1. $f^* := \inf_{x \in V} f(x) = \inf_{x \in \bigcup W_i} f(x)$
   - $\leadsto$ condition: for almost all $t$, $f^{-1}(t) \cap \bigcup W_i \cap \mathbb{R}^n$ meets every connected component of $f^{-1}(t) \cap V \cap \mathbb{R}^n$;

2. $f - f^* + \epsilon = \text{sos} \mod \langle \mathbf{F}, G_i \rangle$.
   - $\leadsto$ $f^*_{|W_i} = f_{|W_i}^{\text{sos}} := \sup \{a \in \mathbb{R} | f - a = \text{sos} \mod \langle \mathbf{F}, G_i \rangle \}$;
   - $\leadsto$ $f_{|W_i}^{\text{sos}}$ can be computed using SDP.
A ∈ GL_n(\mathbb{Q}) generic change of variables, \( J^A \) jacobian matrix of \{F^A, f^A\};
\( A \in GL_n(\mathbb{Q}) \) generic change of variables, \( J^A \) jacobian matrix of \( \{F^A, f^A\} \);

**Definition**

for \( 0 \leq i \leq d \):

\[
W^A_i = \mathbb{V} \left( X_{\leq i}, F^A, \text{MaxMinors}(J^A, X_{\geq i+2}) \right).
\]
Sums of squares (constrained case)

\[ A \in GL_n(\mathbb{Q}) \] generic change of variables, \( J^A \) jacobian matrix of \( \{F^A, f^A\} \);

**Definition**

for \( 0 \leq i \leq d \):

\[ W^A_i = \mathbb{V} \left( X_{\leq i}, F^A, \text{MaxMinors}(J^A, X_{\geq i+2}) \right). \]

\( \sim \) \( (f^A)^{-1}(t) \cap W^A_i \) intersection of \( \mathbb{V}(X_{\leq i}) \) and the \( i \)-th polar variety associated with \( \mathbb{V}(F^A, f^A - t) = (f^A)^{-1}(t) \cap V \).
A ∈ \( GL_n(\mathbb{Q}) \) generic change of variables, \( J^A \) jacobian matrix of \( \{ F^A, f^A \} \);

**Definition**

for \( 0 \leq i \leq d \):

\[
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\]

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\( \leadsto \) \((f^A)^{-1}(t) \cap \bigcup W_i^A \cap \mathbb{R}^n\) meets every c.c. of \( (f^A)^{-1}(t) \cap V \cap \mathbb{R}^n \);

\( \leadsto \) \( \inf_{x \in V} f(x) = \inf_{x \in \bigcup W_i^A} f^A(x) \).
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\( \sim \bigcup W_i^A \) has dimension \( \leq 1. \)
Schweighofer’s goal: fix numerical issues due to asymptotic phenomena.

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- Asymptotic values:

\[ R_\infty(f, S) = \{ y \in \mathbb{R} \mid \exists (x_k)_k \subset S \text{ s.t. } \lim_{k \to \infty} \|x_k\| = \infty, \lim_{k \to \infty} f(x_k) = y \}. \]

**Theorem (Schweighofer)**

\( f, g_1, \ldots, g_m \in \mathbb{R}[X_1, \ldots, X_n] \), \( S = \{ x \in \mathbb{R}^n \mid g_1(x) \geq 0, \ldots, g_m(x) \geq 0 \} \) and

- \( f > 0 \) on \( S \);
- \( f \) bounded on \( S \);
- \( R_\infty(f, S) \) finite subset of \( ]0, +\infty[ \).

Then \( f = \text{sos} \mod \langle g_1, \ldots, g_m \rangle \).
Theorem (Schweighofer)

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\[ \leadsto \text{With } W_i = \mathbb{V}(F, G_i): \]

\- \( f|_{W_i} - f^*|_{W_i} + \varepsilon > 0; \)
# Workplan

## Theorem (Schweighofer)

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Then \( f = \text{sos mod} \langle g_1, \ldots, g_m \rangle \).

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- \( f|_{W_i} - f^*|_{W_i} + \varepsilon > 0 \);
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Theorem (Schweighofer)

\[ f, g_1, \ldots, g_m \in \mathbb{R}[X_1, \ldots, X_n], \quad S = \{ x \in \mathbb{R}^n \mid g_1(x) \geq 0, \ldots, g_m(x) \geq 0 \} \text{ and } \]
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\[ \rightsquigarrow \text{ With } W_i = \mathbb{V}(F, G_i): \]
\begin{itemize}
  \item \( f|_{W_i} - f^*_{|W_i} + \varepsilon > 0 \);
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\end{itemize}

\[ \rightsquigarrow f|_{W_i} - f^*_{|W_i} + \varepsilon = \text{sos} \mod \langle F, G_i \rangle \]
Sums of squares (constrained case)

Theorem (G. / Guo / Safey / Zhi)

\[ \mathbf{F} = \{f_1, \ldots, f_s\} \subset \mathbb{Q}[X_1, \ldots, X_n], \langle f_1, \ldots, f_s \rangle \text{ radical}, \ V = \mathbb{V}(\mathbf{F}) \text{ smooth and equidimensional}; \]

- \( J \) jacobian matrix of \( \{ \mathbf{F}^A, f^A \} \);
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- \( W_i = \mathbb{V}(\mathbf{F}^A, \mathbf{G}^A_i) \).

Numerical experiments:

- equivalent as existent methods when \( f^\star \) is reached;
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- \( W_i = \mathbb{V}(\mathbf{F}^A, \mathbf{G}_i^A). \)

\[ \Rightarrow f^A \mid_{W_i} - f^* \mid_{W_i} + \varepsilon = \text{sos mod } \langle \mathbf{F}^A, \mathbf{G}_i^A \rangle; \]

\[ \Rightarrow f^* = \min \left\{ f^* \mid_{W_i} \right\}. \]
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\[ f^A |_{W_i} - f^* |_{W_i} + \varepsilon = \text{sos} \mod \langle F^A, G^A_i \rangle; \]

\[ f^* = \min \left\{ f^* |_{W_i} \right\}. \]

Numerical experiments:

- equivalent as existent methods when \( f^* \) is reached;
- better results when \( f^* \) is not reached.
Deciding whether the infimum is reached or not

- $f \in \mathbb{Q}[X_1, \ldots, X_n]$
- $f^* := \inf_{x \in \mathbb{R}^n} f(x)$.

**Goal:** construct an algebraic variety $V$ such that
- $f^*$ is reached if and only if there exists $x^* \in V$ such that $f(x^*) = f^*$;
- computations can be done in $V$. 

First idea: find $x^* \in V (\nabla f) \cap \mathbb{R}^n = V (\partial f/\partial X_1, \ldots, \partial f/\partial X_n) \cap \mathbb{R}^n$.

Obstruction: $V (\nabla f)$ can have dimension $> 0$, can be non radical.

Solution: construction of $0$-dimensional sets meeting every connected component of $V (\nabla f)$. 

A. Greuet  
Polar varieties and optimization
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\( \varepsilon \) infinitesimal, \( A \in GL_n(\mathbb{Q}) \) generic change of coordinates;
- $\varepsilon$ infinitesimal, $A \in GL_n(\mathbb{Q})$ generic change of coordinates;
- $\tilde{W}_i(\varepsilon)$ polar varieties associated with $\mathbb{V}(f^A - f^* + \varepsilon)$ (smooth variety).

$\sim \bigcup \tilde{W}_i(\varepsilon) \cap \mathbb{V}(X_{\leq i})$ meets every connected component of $\mathbb{V}(f^A - f^* + \varepsilon)$;
Deciding whether the infimum is reached: idea

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\[ \leadsto \text{so does } \bigcup \tilde{W}_i(\varepsilon) \setminus \mathbb{V} \left( \frac{\partial f^\mathbf{A}}{\partial X_1} \right)^Y \cap \mathbb{V}(X_{\leq i}); \]
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$\leadsto \varepsilon \to 0 \leadsto$ a point in each connected component of $\mathbb{V} \left( f^A - f^* \right)$;

$\leadsto$ projection to $[X_1, \ldots, X_n]$. 
Deciding whether the infimum is reached

- \( \mathcal{C}_i^A = \mathbb{V} \left( X_{\leq i-1}, \frac{\partial f^A}{\partial X_{i+1}}, \ldots, \frac{\partial f^A}{\partial X_n} \right) \setminus \mathbb{V} \left( \frac{\partial f^A}{\partial X_1} \right) \); 
- \( \mathcal{F}_i^A = \mathcal{C}_i^A \cap \mathbb{V} \left( \frac{\partial f^A}{\partial X_1}, \ldots, \frac{\partial f^A}{\partial X_n} \right) \).
Deciding whether the infimum is reached

\[ \mathcal{C}_i^A = \bigvee \left( X_{\leq i-1}, \frac{\partial f^A}{\partial X_{i+1}}, \ldots, \frac{\partial f^A}{\partial X_n} \right) \setminus \bigvee \left( \frac{\partial f^A}{\partial X_1} \right) ; \]

\[ \mathcal{T}_i^A = \mathcal{C}_i^A \cap \bigvee \left( \frac{\partial f^A}{\partial X_1}, \ldots, \frac{\partial f^A}{\partial X_n} \right). \]

**Theorem 1 (G. / Safey)**

For a generic \( A \in GL_n(\mathbb{Q}) \),

\[ \bigcup_{i=1}^{n} \mathcal{T}_i^A \text{ meets every connected component of } V(f^A - f^*) \cap \mathbb{R}^n. \]
Deciding whether the infimum is reached

- $C_i^A = \mathbb{V} \left( X_{\leq i-1}, \frac{\partial f^A}{\partial X_{i+1}}, \ldots, \frac{\partial f^A}{\partial X_n} \right) \setminus \mathbb{V} \left( \frac{\partial f^A}{\partial X_1} \right)^\mathbb{L}$;

- $\mathcal{F}_i^A = C_i^A \cap \mathbb{V} \left( \frac{\partial f^A}{\partial X_1}, \ldots, \frac{\partial f^A}{\partial X_n} \right)$.

**Theorem 1 (G. / Safey)**

For a generic $A \in GL_n(\mathbb{Q})$,

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**Theorem 2 (G. / Safey)**

For a generic $A$ and $1 \leq i \leq n$,

- the sets $C_i^A$ have dimension $\leq 1$;
- the sets $\mathcal{F}_i^A$ have dimension $\leq 0$. 
Example

\[ f(x, y) = x^2 y^2 (x^2 + y^2 - 1) \mapsto f^A(x, y) = y^2 x^4 - 4y^3 x^3 + 7y^4 x^2 - x^2 y^2 - 6y^5 x + 2xy^3 + 2y^6 - y^4. \]
Example

\[ f(x, y) = x^2 y^2 (x^2 + y^2 - 1) \sim \]
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- \( \nabla \left( \frac{\partial f^A}{\partial x_2} \right) \);
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Example

\[ f(x, y) = x^2y^2(x^2 + y^2 - 1) \rightrightarrows \]
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- \( \nabla \left( \frac{\partial f^A}{\partial x_2} \right) \);
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- \( \nabla \left( \nabla(f^A) \right) = \nabla \left( \frac{\partial f^A}{\partial x_2} \right) \cap \nabla \left( \frac{\partial f^A}{\partial x_1} \right) \);
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\[ f(x, y) = x^2y^2(x^2 + y^2 - 1) \Rightarrow f^A(x, y) = y^2x^4 - 4y^3x^3 + 7y^4x^2 - x^2y^2 - 6y^5x + 2xy^3 + 2y^6 - y^4. \]

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The algorithm

Input: $f \in \mathbb{Q}[X_1, \ldots, X_n]$ bounded below. A real interval $I$ and $P \in \mathbb{Q}[T]$ encoding $f^* = \inf_{x \in \mathbb{R}^n} f(x)$.

Output: a boolean which equals true if $f^*$ is reached.
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   1. compute the points in $F_i$;
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3. return false.

$\leadsto$ complexity using geometric resolution (Giusti, Lecerf, Salvy): $\tilde{O}\left(n^6(nL + n^4)(D - 1)^{3n}\right)$ arithmetic operations in $\mathbb{Q}$. 
Benchmarks

Computations performed on a PC under Scientific Linux OS release 5.5 on Intel(R) Xeon(R) CPUs E5420 at 2.50 GHz with 20.55G RAM using FGb.

<table>
<thead>
<tr>
<th>D</th>
<th>n</th>
<th>#Terms</th>
<th>Time</th>
<th>CAD</th>
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<tr>
<td>Sot1</td>
<td>24</td>
<td>4</td>
<td>677</td>
<td>3 h.</td>
</tr>
<tr>
<td>Vor1</td>
<td>6</td>
<td>8</td>
<td>63</td>
<td>&lt; 1 min.</td>
</tr>
<tr>
<td>Vor2</td>
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<td>18</td>
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<td>5 h.</td>
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<tr>
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<tr>
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<td>4</td>
<td>8</td>
<td>53</td>
<td>&lt; 1 min.</td>
</tr>
<tr>
<td>K3</td>
<td>4</td>
<td>8</td>
<td>67</td>
<td>&lt; 1 min.</td>
</tr>
<tr>
<td>K4</td>
<td>4</td>
<td>8</td>
<td>45</td>
<td>&lt; 1 min.</td>
</tr>
</tbody>
</table>
Contributions

- Extension to the constrained case with equations of the sums of squares approach;
- Algorithm to decide whether an infimum is reached in the unconstrained case in time $\tilde{O}(n^6(nL + n^4)(D - 1)^{3n})$. 

Further work

- Remove assumptions of regularity and decrease the number of minors involved in the sos approach;
- Extend the sos approach to the constrained case with inequalities;
- Investigate numerical stability;
- Implement the algorithm deciding whether an infimum is reached or not in Mathemagix.
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