

Hyperexponential-Hypergeometric Functions: Multiplicative Structure and Termination of Creative Telescoping

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Outline

Introduction

Structure theorems

Termination criteria

Summary

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Summary

Multivariate hyperexponential-hypergeometric functions

- ▶ $\mathbf{x} = (x_1, \dots, x_s)$: continuous vars $\mathbf{n} = (n_1, \dots, n_t)$: discrete vars
- ▶ D_i : derivation $\partial/\partial x_i$ E_j : shift operator $n_j \rightarrow n_j + 1$

Hyper-Hyper: $h(\mathbf{x}, \mathbf{n})$ is hyperexponential-hypergeometric over $k(\mathbf{x}, \mathbf{n})$ if

all $\frac{D_i(h)}{h}$ and $\frac{E_j(h)}{h}$ are rational functions in $k(\mathbf{x}, \mathbf{n})$.

Certificates:

$\frac{D_i(h)}{h}$: x_i -certificate of h , $\frac{E_j(h)}{h}$: n_j -certificate of h .

Examples:

$\frac{1}{x_1 x_2 + n_1 + n_2}$, $\exp\left(\frac{1}{x_1 + x_2}\right)$, $(x_1 x_2 + 1)^{\sqrt{2}}$, $x_1^{n_1} x_2^{n_2}$, $(n_1 + n_2)!$, etc.

The story of creative telescoping

Claim in Apéry's proof that $\zeta(3) \notin \mathbb{Q}$: $A_n = \sum_m \binom{n}{m}^2 \binom{n+m}{m}^2$ satisfies

$$n^3 A_n - (34n^3 - 51n^2 + 27n - 5)A_{n-1} + (n-1)^3 A_{n-2} = 0.$$

Proof by Zagier and Cohen: Let $a(n, m) = \binom{n}{m}^2 \binom{n+m}{m}^2$ and

$$L(n, E_n) := n^3 - (34n^3 - 51n^2 + 27n - 5)E_n^{-1} + (n-1)^3 E_n^{-2}.$$

1. There exists $b(n, m)$ such that $b/a \in \mathbb{Q}(n, m)$ and

$$L(n, E_n) \cdot a(n, m) = \Delta_m \cdot b(n, m), \quad \text{where } \Delta_m = E_m - 1.$$

2. Note that \sum_m commutes with $L(n, E_n)$, then

$$L(n, E_n) \cdot A_n = \sum_m \Delta_m \cdot b(n, m) = 0.$$

Telescoping problems: the **bivariate** case

Discrete case:

$$\sum_m h(n, m) \rightsquigarrow \underbrace{L(n, E_n)}_{\text{Telescoper}}(h) = \Delta_m(g)$$

Continuous case:

$$\int_a^b h(x, y) dy \rightsquigarrow L(x, D_x)(h) = D_y(g)$$

Mixed case:

$$\sum_n h(x, n) \rightsquigarrow L(x, D_x)(h) = \Delta_n(g)$$

$$\int_a^b h(x, n) dx \rightsquigarrow L(n, E_n)(h) = D_x(g)$$

From Gosper's algorithm to Zeilberger's algorithm

Gosper's algorithm: (Gosper 1978) For hypergeometric $h(n)$, decide

whether $h(n) = \Delta_n(g(n))$, for some hypergeometric $g(n)$.

Zeilberger's algorithm \mathcal{Z} : (Zeilberger 1990) For hypergeometric $h(n)$, find (L, g) , such that

$L(n, E_n)(h) = \Delta_m(g)$, for some hypergeometric $g(n)$.

0. Initialize $\rho := 0$;
1. Set $L_\rho := \sum_{i=0}^{\rho} \ell_i(n) E_n^i$;
2. Solve $L_\rho(h) = \Delta_m(g_\rho)$ via **ParaGosper**;
3. If find a nontrivial solution in Step 2, return;
otherwise increase ρ by 1 and go to Step 1.

Termination of Zeilberger's algorithm

Problem: When does Zeilberger's algorithm **terminate**?

Termination \Leftrightarrow Existence of telescopers

Sufficient conditions: h is holonomic \Rightarrow \mathcal{Z} terminates on h

Holonomicity:

1. $\mathcal{W}_s := k\langle x_1, \dots, x_s, D_1, \dots, D_s \rangle$, in which

$$D_i x_j = x_j D_i + 1, \quad \text{for all } i = 1, \dots, s.$$

2. $h(\mathbf{x})$ is holonomic if $\text{ann}(h) \subseteq \mathcal{W}_s$ has Hilbert dimension s .
3. $h(\mathbf{x}, \mathbf{n})$ is holonomic if the generating function

$$H(\mathbf{x}, \mathbf{y}) := \sum_{n_1, \dots, n_t \geq 0} h(\mathbf{x}, \mathbf{n}) y_1^{n_1} \cdots y_t^{n_t} \quad \text{is holonomic in } \mathcal{W}_{s+t}.$$

Detecting holonomicity and the WZ conjecture

During the discussion of structure theorems, k is algebraically closed.

Wilf and Zeilberger's conjecture: Let $h(\mathbf{x}, \mathbf{n})$ be hyper-hyper, then

$$h(\mathbf{x}, \mathbf{n}) \text{ is holonomic} \iff h(\mathbf{x}, \mathbf{n}) \text{ is proper.}$$

Properness: A hyper-hyper function is proper if it has the form

$$P \exp(g_0) \prod_{\ell=1}^L g_{\ell}^{c_{\ell}} \prod_{j=1}^t \beta_j^{n_j} \prod_{p=1}^P \underbrace{(\phi_{p,1}n_1 + \cdots + \phi_{p,t}n_t + \varphi_p)}_{\text{Integer-linear polynomial}}!^{e_p}$$

where $P \in k[\mathbf{x}, \mathbf{n}]$, $g_0 \in k(\mathbf{x})$, $g_{\ell}, \beta_j \in k(\mathbf{x})$, $c_{\ell}, \varphi_p \in k$, and $\phi_{p,j}, e_p \in \mathbb{Z}$.

The discrete case of WZ conjecture is solved!

Discrete case of the WZ conjecture: If $h(\mathbf{x})$ is hypergeometric,

$$h(\mathbf{x}) \text{ is holonomic} \Leftrightarrow h(\mathbf{x}) \text{ is proper.}$$

Ore-Sato theorem (Ore1930, Sato1990?): Hypergeometric term $h(\mathbf{n})$ can be written as

$$\lambda \cdot f(\mathbf{n}) \cdot \prod_{i=1}^I u_i^{n_i} \cdot \prod_{p=1}^P (\phi_{p,1} n_1 + \cdots + \phi_{p,t} n_t + \varphi_p)^{e_p},$$

where $\lambda \in k$, $f \in k(\mathbf{n})$, $u_i, \varphi_p \in k$, and $\phi_{j,p} \in \mathbb{Z}$.

Historical note:

- ▶ Bivariate case: Abramov–Petkovšek 2001, Hou 2001.
- ▶ Multivariate case: Payne 1997, Abramov–Petkovšek 2002.

Existence criteria: the bivariate hyper-hyper case

Simple situation:

- ▶ Differential case (Bernstein 1971, Lipshitz 1988):

Hyperexponential \Rightarrow Holonomic \Rightarrow Telescoper exists

Involved situation:

- ▶ Shift case (Abramov 2002), q -Shift case (Chen–Hou–Mu 2005):

$$L(n, E_n)(h) = \Delta_m(g) \quad \Leftrightarrow \quad h = \Delta_m(h_1) + \text{proper term}$$

- ▶ Differential-shift case (Chen–Chyzak–Feng–Li 2010): **new!**

$$L(x, D_x)(h) = \Delta_n(g) \quad \Leftrightarrow \quad h = \Delta_n(h_1) + \text{proper term}$$

$$L(n, E_n)(h) = D_x(g) \quad \Leftrightarrow \quad h = D_x(h_1) + \text{proper term}$$

Our contributions

1. Extension of the Ore–Sato theorem to the multivariate continuous-discrete case.
2. A proof that all hyperexponential functions are proper.
3. Criteria for the existence of telescopers in the bivariate continuous-discrete case.

Introduction

Structure theorems

Termination criteria

Summary

First-order fully integrable system

First-order systems (FOS):

$$D_1(z) = a_1z, \dots, D_s(z) = b_sz, E_1(z) = b_1z, \dots, E_t(z) = b_tz$$

where $a_i, b_j \in k(\mathbf{x}, \mathbf{n})$. The FOS is fully integrable if it satisfies the

integrability conditions (ICs):

$$\text{D-D: } D_i(a_j) = D_j(a_i) \quad \iff \quad D_i D_j(h) = D_j D_i(h)$$

$$\text{S-S: } \frac{E_i(b_j)}{b_j} = \frac{E_j(b_i)}{b_i} \quad \iff \quad E_i E_j(h) = E_j E_i(h)$$

$$\text{D-S: } \frac{D_i(b_j)}{b_j} = E_j(a_i) - a_i \quad \iff \quad D_i E_j(h) = E_j D_i(h)$$

For brevity, denote the fully integrable FOS by $(a_1, \dots, a_s, b_1, \dots, b_t)$.

Correspondence between functions and systems

Theorem (Bronstein–Li–Wu 2005): If k is algebraically closed, the solution space of a fully integrable FOS is one-dimensional over k .

Dictionary

$$h(\mathbf{x}, \mathbf{n}) \quad \leftrightarrow \quad (a_1, \dots, a_s, b_1, \dots, b_t)$$

$$\exp(f(\mathbf{x})) \quad \leftrightarrow \quad (D_1(f), \dots, D_s(f), 1, \dots, 1)$$

$$\beta(\mathbf{x})^\lambda \quad \leftrightarrow \quad \left(\lambda \frac{D_1(\beta)}{\beta}, \dots, \lambda \frac{D_s(\beta)}{\beta}, 1, \dots, 1\right)$$

$$\beta(\mathbf{x})^{n_j} \quad \leftrightarrow \quad \left(n_j \frac{D_1(\beta)}{\beta}, \dots, n_j \frac{D_s(\beta)}{\beta}, 1, \dots, 1, \beta, 1, \dots, 1\right)$$

$$L(\mathbf{n})!^\star \quad \leftrightarrow \quad \left(0, \dots, 0, \prod_{\ell=1}^{\phi_1} L(\mathbf{n}) + \ell, \dots, \prod_{\ell=1}^{\phi_t} L(\mathbf{n}) + \ell\right)$$

$\star L(\mathbf{x})$ is an integer-linear polynomial $\phi_1 n_1 + \dots + \phi_t n_t + \varphi$.

Multiplicative structure

Correspondence:

$$h(\mathbf{x}, \mathbf{n}) \cdot h'(\mathbf{x}, \mathbf{n}) \quad \leftrightarrow \quad (a_1 + a'_1, \dots, a_s + a'_s, b_1 b'_1, \dots, b_t b'_t)$$

Theorem (new): $(a_1, \dots, a_s, b_1, \dots, b_t)$ has a solution of the form

$$f \cdot \exp(g_0) \prod_{\ell=1}^L g_\ell^{c_\ell} \cdot \prod_{j=1}^t \beta_j^{n_j} \cdot \prod_{p=1}^P (\phi_{p,1} n_1 + \dots + \phi_{p,t} n_t + \varphi_p)!^{e_p},$$

where $f \in k(\mathbf{x}, \mathbf{n})$, $g_0, g_\ell, \beta_j \in k(\mathbf{x})$, $c_\ell, \varphi_p \in k$, and $\phi_{p,j}, e_p \in \mathbb{Z}$.



$$a_i = \frac{D_i(f)}{f} + D_i(g_0) + \sum_{\ell=1}^L c_\ell \frac{D_i(g_\ell)}{g_\ell} + \sum_{j=1}^t n_j \frac{D_i(\beta_j)}{\beta_j}, \quad 1 \leq i \leq s,$$

$$b_j = \frac{E_j(f)}{f} \cdot \beta_j \cdot \alpha_j, \quad 1 \leq j \leq t, \quad \alpha_j \in k(\mathbf{n}) \text{ factor into integer-linear polys.}$$

Separating continuous and discrete parts

Lemma (Feng–Singer–Wu 2010): For $\frac{D(b)}{b} = E(a) - a$ in $k(x, n)$, there exist $f \in k(x, n)$, $\beta, \gamma \in k(x)$ and $\alpha \in k(n)$ s.t.

$$a = \frac{D_x(f)}{f} + n \frac{D_x(\beta)}{\beta} + \gamma \quad \text{and} \quad b = \frac{E_n(f)}{f} \beta \alpha.$$

Theorem (new): Certificates of a hyper-hyper function can be written as

$$a_i = \frac{D_i(f)}{f} + \sum_{j=1}^t n_j \frac{D_i(\beta_j)}{\beta_j} + A_i(\mathbf{x}), \quad 1 \leq i \leq s,$$

$$b_j = \frac{E_j(f)}{f} \cdot \beta_j \cdot B_j(\mathbf{n}), \quad 1 \leq j \leq t,$$

where $f \in k(\mathbf{x}, \mathbf{n})$, $g_j, A_i \in k(\mathbf{x})$, $B_j \in k(\mathbf{n})$, and A_i, B_j satisfy ICs.

The Ore–Sato theorem and its differential analogue

Ore–Sato Theorem (shift case, certificate version):

$$\frac{E_i(B_j)}{B_j} = \frac{E_j(B_i)}{B_i} \quad \Rightarrow \quad B_j = \frac{E_j(f)}{f} \alpha_j, \quad 1 \leq j \leq t,$$

where $f \in k(\mathbf{n})$ and $\alpha_j \in k(\mathbf{n})$'s factor into **integer-linear** polys.

Theorem (differential case, new):

$$D_i(A_j) = D_j(A_i) \quad \Rightarrow \quad a_i = D_i(f) + \frac{D_i(g_0)}{g_0} + \sum_{\ell=1}^L c_\ell \frac{D_i(g_\ell)}{g_\ell}, \quad 1 \leq i \leq s,$$

where $f, g_0 \in k(\mathbf{x})$, $c_1, \dots, c_L \in k$, and $g_1, \dots, g_L \in k(\mathbf{x})$.

Continuous case of the WZ conjecture

Cor: Multivariate hyperexponential functions are **proper**.

$$\frac{P}{Q} \exp(g_0) \prod_{\ell=1}^L g_\ell^{c_\ell} \iff P \exp(g_0) \prod_{\ell=1}^{L+1} g_\ell^{c_\ell}, \quad c_{L+1} = -1, g_{L+1} = Q$$

Theorem (Kashiwara1978, Takayama1992):

$$h(\mathbf{x}) \text{ is } D\text{-finite} \implies h(\mathbf{x}) \text{ is holonomic.}$$

Cor: Multivariate hyperexponential functions are **holonomic**.

This makes the WZ conjecture self-evident in the continuous setting!

Towards a proof of the general case of WZ conjecture

Recall: $h(\mathbf{x}, \mathbf{n})$ is holonomic iff

$$H(\mathbf{x}, \mathbf{y}) = \sum_{n_1, \dots, n_s \geq 0} h(\mathbf{x}, \mathbf{n}) y_1^{n_1} \cdots y_s^{n_s} \quad \text{is holonomic over } k[\mathbf{x}, \mathbf{y}].$$

Need to show or clarify the literature:

1. Closure properties: product, antiderivative;
2. Holonomic \Rightarrow Telescopier exists;
3. If $f \in k(\mathbf{x}, \mathbf{n})$ is holonomic,

$$f = \frac{P(\mathbf{x}, \mathbf{n})}{Q_1(\mathbf{x})Q_2(\mathbf{n})}, \quad \text{where } Q_2 \text{ factors into integer-linear polys.}$$

Introduction

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Termination criteria

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Termination of Creative telescoping: the mixed case

Problem: Given a bivariate hyper-hyper function $h(x, n)$, decide:

whether $\exists L \in k(x)\langle D_x \rangle \setminus \{0\}$ s.t. $L(x, D_x)(h) = \Delta_n(g)$?

whether $\exists L \in k(n)\langle E_n \rangle \setminus \{0\}$ s.t. $L(n, E_n)(h) = D_x(g)$?

where g is hyper-hyper over $k(x, n)$.

$\mathcal{H}(a, b)$: The solution space of the fully integrable system

$$D_x(h) = ah, \quad E_n(h) = bh.$$

Split polynomials

Splitness: A polynomial $p \in k[x, n]$ is **split** if

$$p = p_1(x) \cdot p_2(n) \text{ with } p_1 \in k[x] \text{ and } p_2 \in k[n].$$

By the structure theorem, $h(x, n)$ can be written as

$$f(x, n) \cdot \beta(x)^n \cdot h'(x, n), \quad (\text{multiplicative form})$$

where $f \in k(x, n)$, $\beta \in k(x)$, and $h' \in \mathcal{H}(\gamma, \alpha)$ with $\gamma \in k(x)$, $\alpha \in k(n)$.

Proposition:

$$h(x, n) \text{ is proper} \quad \Leftrightarrow \quad \text{denom. of } f \text{ is split}$$

Proof:

$$\frac{P}{A(x)B(n)} \cdot \beta(x)^n \cdot \mathcal{H}(\gamma, \alpha) \quad \Leftrightarrow \quad P \cdot \beta(x)^n \cdot \mathcal{H}\left(\gamma - \frac{D_x A}{A}, \alpha \frac{B}{E_n B}\right)$$

Introductory example

$$f(x, n) = \frac{1}{x + n}.$$

Claim 1: f has no telescoper w.r.t. x .

Proof. For any nonzero $L = \sum_{i=0}^r e_i(n) E_n^i$,

$$L(f) = \sum_{i=0}^r \frac{e_i(n)}{x + n + i} = \frac{P}{(x + n + r + 1) \cdots (x + n)}.$$

1. $L(f)$ is proper w.r.t. x + its denom. is **square free** w.r.t. x .
2. If $L(f) = D_x(g)$, then $L(f) = 0$.
3. The denominator of a rational solution of $L(y) = 0$ is **split**, but $x + n$ is not split, contradiction!

Introductory example (continued)

Claim 2: f has no telescoper w.r.t. n .

Proof. For any nonzero $L = \sum_{i=0}^r d_i(x)D_x^i$,

$$L(f) = \sum_{i=0}^r \frac{p_i}{(x+n)^{i+1}} = \frac{P}{(x+n)^{r+1}}.$$

1. $L(f)$ is proper w.r.t. n + its denom. is **shift-free** w.r.t. n .
2. If $L(f) = \Delta_n(g)$, then $L(f) = 0$.
3. The denominator of a rational solution of $L(y) = 0$ is **split**, but $x+n$ is not split, contradiction!

Additive decomposition (**discrete** case)

$$h(x, n) = f(x, n) \cdot \beta(x)^n \cdot \mathcal{H}(\gamma(x), \alpha(n))$$

Abramov-Petkovšek decomposition: $\exists h_1$ with $h/h_1 \in k(x, n)$, s.t.,

$$h = \Delta_n(h_1) + \frac{v}{u} \cdot \beta(x)^n \cdot \mathcal{H}(\tilde{\gamma}(x), \tilde{\alpha}(n)),$$

1. $v, u \in k(x)[n]$ and $\gcd(v, u) = 1$;
2. u is **shift-free** w.r.t. n ;
3. $\beta(x)\tilde{\alpha}(n)$ is **shift-reduced** w.r.t. n ;
4. $\gcd(u, E_n^{-\ell}(\text{num}(\tilde{\alpha}))) = \gcd(u, E_n^{\ell}(\text{den}(\tilde{\alpha}))) = 1$ for all $\ell \in \mathbb{N}$.

Lemma (Abramov-Petkovšek, 2002):

$$h = \Delta_n(g) \Rightarrow u \in k(x).$$

Additive decomposition (**continuous** case)

$$h(x, n) = f(x, n) \cdot \beta(x)^n \cdot \mathcal{H}(\gamma(x), \alpha(n))$$

Geddes-Le-Li decomposition: $\exists h_1$ with $h/h_1 \in k(x, n)$, s.t.,

$$h = D_x(h_1) + \frac{v}{u} \cdot \beta(x)^n \cdot \mathcal{H}(\tilde{\gamma}(x), \alpha(n)),$$

1. $v, u \in k(n)[x]$ and $\gcd(v, u) = 1$;
2. u is **square-free** w.r.t. x ;
3. $\tilde{\gamma} + n \frac{D_x(\beta)}{\beta}$ is **differential-reduced** w.r.t. x ;
4. $\gcd(u, \text{den}(\tilde{\gamma}(x) + n \frac{D_x(\beta)}{\beta})) = 1$.

Lemma (Geddes-Le-Li, 2004):

$$h = D_x(g) \Rightarrow u \in k(n).$$

Applying operators to additive decompositions

Applying differential operators: For all $L(x, D_x) \in k(x)\langle D_x \rangle$,

$$h = \Delta_n(h_1) + \frac{v}{u} \cdot \beta(x)^n \cdot \mathcal{H}(\tilde{\gamma}(x), \tilde{\alpha}(n)) \quad \text{add. decomp.}$$

$$\Downarrow$$

$$L(h) = \Delta_n(L(h_1)) + \frac{V}{U} \cdot \beta(x)^n \cdot \mathcal{H}(\tilde{\gamma}(x), \tilde{\alpha}(n)) \quad \text{add. decomp.}$$

Applying recurrence operators: For all $L(n, E_n) \in k(n)\langle E_n \rangle$,

$$h = D_x(h_1) + \frac{v}{u} \cdot \beta(x)^n \cdot \mathcal{H}(\tilde{\gamma}(x), \alpha(n)) \quad \text{add. decomp.}$$

$$\Downarrow$$

$$L(h) = D_x(L(h_1)) + \frac{V}{U \text{den}(\beta)^\rho} \cdot \beta(x)^n \cdot \mathcal{H}(\tilde{\gamma}(x), \alpha(n)) \quad \text{add. decomp.}$$

Two criteria for existence

$$h(x, n) = f(x, n) \cdot \beta(x)^n \cdot \mathcal{H}(\gamma(x), \alpha(n))$$

Discrete case: $h = \Delta_n(h_1) + h_2$ with $h_2 = \frac{v}{u} \cdot \beta(x)^n \cdot \mathcal{H}(\tilde{\gamma}(x), \tilde{\alpha}(n))$,

h has a telescoper w.r.t. $n \iff u$ is **split** $\iff h_2$ is **proper**.

Continuous case: $h = D_x(h_1) + h_2$ with $h_2 = \frac{v}{u} \cdot \beta(x)^n \cdot \mathcal{H}(\tilde{\gamma}(x), \alpha(n))$,

h has a telescoper w.r.t. $x \iff u$ is **split** $\iff h_2$ is **proper**.

Sufficient part: h has a telescoper if it is proper.

Wilf and Zeilberger's **fundamental** theorem + Wegschaider's trick.

Two observations

Continuous case: Let p be a polynomial in $k[x, n]$ and

$$L(x, n, D_x) = e_n(x, n)D_x^\rho + e_n(x, n)D_x^{\rho-1} + \cdots + e_0(x, n)$$

$f \in k(x, n)$ s.t. $L(f) = p$ and e_n is split \Rightarrow denom. of f is **split**.

Discrete case: Let p be a polynomial in $k[x, n]$ and

$$L(x, n, E_n) = e_n(x, n)E_n^\rho + e_n(x, n)E_n^{\rho-1} + \cdots + e_0(x, n).$$

$f \in k(x, n)$ s.t. $L(f) = p$ and e_n is split \Rightarrow denom. of f is **split**.

Proof of necessary part (the **discrete** case)

Assumption:

A hypergeometric $h(x, n)$ has a telescoper L in $k(x)\langle D_x \rangle$.

A sketch:

1. Additive decomposition:

$$h = \Delta_n(h_1) + h_2$$

where

$$h_2 = \frac{v}{u} \cdot \tilde{h}_2 \quad \text{with} \quad \tilde{h}_2 = \beta(x)^n \cdot \mathcal{H}(\tilde{\gamma}(x), \alpha(n)).$$

2. Applying L :

$$L(n, E_n)(h) = D_x(g) \quad \Rightarrow \quad L(h_2) = D_x(g - L(h_1)).$$

Goal: Prove u split.

3 Rewriting $L(h_2)$:

$\forall i \in \mathbb{N}, \exists M_i \in k(x)[n]\langle D_x \rangle$ with leading coefficient **1** s.t.

$$D_x^i(h_2) = M_i \left(\frac{v}{u} \right) \cdot \tilde{h}_2$$

\Downarrow

$$L(h_2) = M \left(\frac{v}{u} \right) \cdot \tilde{h}_2 = \frac{V}{U} \cdot \tilde{h}_2 \Rightarrow M \left(\frac{v}{u} \right) = \frac{V}{U}.$$

where $M \in k(x)[n]\langle D_x \rangle$ with leading coefficient in $k(x)$, and $U, V \in k[x, n]$.

4 Applying Lemma (Abramov-Petkovšek, 2002):

Since $L(h_2)$ is hypergeometric summable w.r.t. n ,

$$L(h_2) = \frac{V}{U} \cdot \tilde{h}_2 \Rightarrow U \in k[x].$$

5. Applying the first observation:

Since v/u is a rational solution of $(U \cdot M)(y) = \tilde{V}$, u is split.

Two criteria are not redundant!

Example:

$$h(x, n) = \frac{1}{(x+n)^s},$$

- ▶ $s = 1$: h has no telescoper w.r.t. n or x ;
- ▶ $s > 1$: h has no telescoper w.r.t. n , but it has a telescoper w.r.t. x ,

$$h = D_x \left(\frac{-1}{(s-1)(x+n)^{s-1}} \right).$$

Non-proper example

Example:

$$h(x, n) = \frac{-x + 2nx + 2n^2}{(x+n)^2 x} \cdot x^n \cdot e^{-x}.$$

Though h is not proper, it still has a telescoper w.r.t. x since

$$h = D_x \left(\frac{1}{x+n} x^n e^{-x} \right) + \frac{1}{x} \cdot x^n \cdot e^{-x}.$$

Introduction

Structure theorems

Termination criteria

Summary

Summary

Present:

1. Structure theorem for hyper-hyper functions;
2. The continuous case of WZ conjecture is true;
3. Two criteria for the existence of telescopers:

h has a telescoper w.r.t. $n \iff h = \Delta_n(h_1) + \text{proper term.}$

h has a telescoper w.r.t. $x \iff h = D_x(h_1) + \text{proper term.}$

Future:

- ▶ The general case of WZ conjecture;
- ▶ The existence criterion for multivariate hyper-hyper functions;
- ▶ Implement Zeilberger's algorithm with termination criteria;
- ▶ The mixed differential and q -shift cases.