Hyperexponential-Hypergeometric Functions:
Multiplicative Structure and Termination of Creative Telescoping

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Outline

Multivariate hyperexponential-hypergeometric functions

- $\mathbf{x} = (x_1, \ldots, x_s)$: continuous vars $\quad$ $\mathbf{n} = (n_1, \ldots, n_t)$: discrete vars
- D_i: derivation $\partial/\partial x_i$ $\quad$ E_j: shift operator $n_j \to n_j + 1$

Hyper-Hyper: $h(\mathbf{x}, \mathbf{n})$ is hyperexponential-hypergeometric over $k(\mathbf{x}, \mathbf{n})$ if

$$\text{all } \frac{D_i(h)}{h} \text{ and } \frac{E_j(h)}{h} \text{ are rational functions in } k(\mathbf{x}, \mathbf{n}).$$

Certificates:

$$\frac{D_i(h)}{h} : x_i\text{-certificate of } h, \quad \frac{E_j(h)}{h} : n_j\text{-certificate of } h.$$

Examples:

$$\frac{1}{x_1x_2 + n_1 + n_2}, \exp\left(\frac{1}{x_1 + x_2}\right), (x_1x_2 + 1)^{\sqrt{2}}, x_1^{n_1}x_2^{n_2}, (n_1 + n_2)!, \text{etc.}$$

The story of creative telescoping

Claim in Apéry's proof that $\zeta(3) \notin \mathbb{Q}$: $A_n = \sum_m \binom{n}{m}^2 \binom{n+m}{m}^2$ satisfies

$$n^3 A_n - (34n^3 - 51n^2 + 27n - 5)A_{n-1} + (n-1)^3 A_{n-2} = 0.$$

Proof by Zagier and Cohen: Let $a(n, m) = \binom{n}{m}^2 \binom{n+m}{m}^2$ and

$$L(n, E_n) := n^3 - (34n^3 - 51n^2 + 27n - 5)E_n^{-1} + (n-1)^3 E_n^{-2}.$$

1. There exists $b(n, m)$ such that $b/a \in \mathbb{Q}(n, m)$ and
$$L(n, E_n) \cdot a(n, m) = \Delta_m \cdot b(n, m), \quad \text{where} \quad \Delta_m = E_m - 1.$$
2. Note that $\sum_m$ commutes with $L(n, E_n)$, then
$$L(n, E_n) \cdot A_n = \sum_m \Delta_m \cdot b(n, m) = 0.$$

Telescoping problems: the bivariate case

Discrete case:

$$\sum_m h(n, m) \quad \rightsquigarrow \quad \underbrace{L(n, E_n)}_{\text{Telescoper}}(h) = \Delta_m(g)$$

Continuous case:

$$\int_a^b h(x, y)\, dy \quad \rightsquigarrow \quad L(x, D_x)(h) = D_y(g)$$

Mixed case:

$$\sum_n h(x, n) \quad \rightsquigarrow \quad L(x, D_x)(h) = \Delta_n(g)$$

$$\int_a^b h(x, n)\, dx \quad \rightsquigarrow \quad L(n, E_n)(h) = D_x(g)$$

From Gosper's algorithm to Zeilberger's algorithm

Gosper's algorithm: (Gosper 1978) For hypergeometric $h(n)$, decide

$$\text{whether} \quad h(n) = \Delta_n(g(n)), \quad \text{for some hypergeometric } g(n).$$

Zeilberger's algorithm $\mathcal{Z}$: (Zeilberger 1990) For hypergeometric $h(n)$, find (L, g), such that

$$L(n, E_n)(h) = \Delta_m(g), \quad \text{for some hypergeometric } g(n).$$

0. Initialize $\rho := 0$;
1. Set $L_\rho := \sum_{i=0}^{\rho} \ell_i(n) E_n^i$;
2. Solve $L_\rho(h) = \Delta_m(g_\rho)$ via ParaGosper;
3. If find a nontrivial solution in Step 2, return; otherwise increase ρ by 1 and go to Step 1.

Termination of Zeilberger's algorithm

Problem: When does Zeilberger's algorithm terminate?

Termination $\Leftrightarrow$ Existence of telescopers

Sufficient conditions: h is holonomic $\Rightarrow$ $\mathcal{Z}$ terminates on h

Holonomicity:

1. $\mathcal{W}_s := k\langle x_1, \ldots, x_s, D_1, \ldots, D_s\rangle$, in which

$$D_i x_i = x_i D_i + 1, \quad \text{for all } i = 1, \ldots, s.$$

2. $h(\mathbf{x})$ is holonomic if ann(h) $\subseteq \mathcal{W}_s$ has Hilbert dimension s.
3. $h(\mathbf{x}, \mathbf{n})$ is holonomic if the generating function

$$H(\mathbf{x}, \mathbf{y}) := \sum_{n_1, \ldots, n_t \geq 0} h(\mathbf{x}, \mathbf{n})\, y_1^{n_1} \cdots y_t^{n_t} \quad \text{is holonomic in } \mathcal{W}_{s+t}.$$

Detecting holonomicity and the WZ conjecture

During the discussion of structure theorems, k is algebraically closed.

Wilf and Zeilberger's conjecture: Let $h(\mathbf{x}, \mathbf{n})$ be hyper-hyper, then

$$h(\mathbf{x}, \mathbf{n}) \text{ is holonomic} \quad \Leftrightarrow \quad h(\mathbf{x}, \mathbf{n}) \text{ is proper.}$$

Properness: A hyper-hyper function is proper if it has the form

$$P \exp(g_0) \prod_{\ell=1}^{L} g_\ell^{c_\ell} \prod_{j=1}^{t} \beta_j^{n_j} \prod_{p=1}^{P} \underbrace{(\phi_{p,1} n_1 + \cdots + \phi_{p,t} n_t + \varphi_p)}_{\text{Integer-linear polynomial}}!^{e_p}$$

where $P \in k[\mathbf{x}, \mathbf{n}]$, $g_0 \in k(\mathbf{x})$, $g_\ell, \beta_j \in k(\mathbf{x})$, $c_\ell, \varphi_p \in k$, and $\phi_{p,j}, e_p \in \mathbb{Z}$.

The discrete case of WZ conjecture is solved!

Discrete case of the WZ conjecture: If $h(\mathbf{x})$ is hypergeometric,

$$h(\mathbf{x}) \text{ is holonomic} \quad \Leftrightarrow \quad h(\mathbf{x}) \text{ is proper}.$$

Ore-Sato theorem (Ore1930, Sato1990?): Hypergeometric term $h(\mathbf{n})$ can be written as

$$\lambda \cdot f(\mathbf{n}) \cdot \prod_{i=1}^{I} u_i^{n_i} \cdot \prod_{p=1}^{P} (\phi_{p,1} n_1 + \cdots + \phi_{p,t} n_t + \varphi_p)!^{e_p},$$

where $\lambda \in k$, $f \in k(\mathbf{n})$, $u_i, \varphi_p \in k$, and $\phi_{j,p} \in \mathbb{Z}$.

Historical note:

- Bivariate case: Abramov–Petkovšek 2001, Hou 2001.
- Multivariate case: Payne 1997, Abramov–Petkovšek 2002.

Existence criteria: the bivariate hyper-hyper case

Simple situation:

- Differential case (Bernstein 1971, Lipshitz 1988):

Hyperexponential ⇒ Holonomic ⇒ Telescoper exists

Involved situation:

- Shift case (Abramov 2002), q-Shift case (Chen–Hou–Mu 2005):

$$L(n, E_n)(h) = \Delta_m(g) \quad \Leftrightarrow \quad h = \Delta_m(h_1) + \text{ proper term}$$

- Differential-shift case (Chen–Chyzak–Feng–Li 2010): new!

$$L(x, D_x)(h) = \Delta_n(g) \quad \Leftrightarrow \quad h = \Delta_n(h_1) + \text{ proper term}$$
$$L(n, E_n)(h) = D_x(g) \quad \Leftrightarrow \quad h = D_x(h_1) + \text{ proper term}$$

Our contributions

1. Extension of the Ore–Sato theorem to the multivariate continuous-discrete case.

2. A proof that all hyperexponential functions are proper.

3. Criteria for the existence of telescopers in the bivariate continuous-discrete case.

First-order fully integrable system

First-order systems (FOS):

$$D_1(z) = a_1 z, \ldots, D_s(z) = b_s z, E_1(z) = b_1 z, \ldots, E_t(z) = b_t z$$

where $a_i, b_j \in k(\mathbf{x}, \mathbf{n})$. The FOS is fully integrable if it satisfies the

integrability conditions (ICs):

$$\begin{array}{llcl}
\text{D-D:} & D_i(a_j) = D_j(a_i) & \rightsquigarrow & D_i D_j(h) = D_j D_i(h) \\
\text{S-S:} & \dfrac{E_i(b_j)}{b_j} = \dfrac{E_j(b_i)}{b_i} & \rightsquigarrow & E_i E_j(h) = E_j E_i(h) \\
\text{D-S:} & \dfrac{D_i(b_j)}{b_j} = E_j(a_i) - a_i & \rightsquigarrow & D_i E_j(h) = E_j D_i(h)
\end{array}$$

For brevity, denote the fully integrable FOS by $(a_1, \ldots, a_s, b_1, \ldots, b_t)$.

Correspondence between functions and systems

Theorem (Bronstein–Li–Wu 2005): If k is algebraically closed, the solution space of a fully integrable FOS is one-dimensional over k.

Dictionary

$$
\begin{aligned}
h(\mathbf{x}, \mathbf{n}) \quad &\leftrightarrow \quad (a_1, \ldots, a_s, b_1, \ldots, b_t) \\
\exp(f(\mathbf{x})) \quad &\leftrightarrow \quad (D_1(f), \ldots, D_s(f), 1, \ldots, 1) \\
\beta(\mathbf{x})^\lambda \quad &\leftrightarrow \quad (\lambda \frac{D_1(\beta)}{\beta}, \ldots, \lambda \frac{D_s(\beta)}{\beta}, 1, \ldots, 1) \\
\beta(\mathbf{x})^{n_j} \quad &\leftrightarrow \quad (n_j \frac{D_1(\beta)}{\beta}, \ldots, n_j \frac{D_s(\beta)}{\beta}, 1, \ldots, 1, \beta, 1, \ldots, 1) \\
L(\mathbf{n})!^\star \quad &\leftrightarrow \quad (0, \ldots, 0, \prod_{\ell=1}^{\phi_1} L(\mathbf{n}) + \ell, \ldots, \prod_{\ell=1}^{\phi_t} L(\mathbf{n}) + \ell)
\end{aligned}
$$

$\star\, L(\mathbf{x})$ is an integer-linear polynomial $\phi_1 n_1 + \ldots + \phi_t n_t + \varphi$.

Multiplicative structure

Correspondence:

$$h(\mathbf{x},\mathbf{n}) \cdot h'(\mathbf{x},\mathbf{n}) \quad \leftrightarrow \quad (a_1 + a_1', \ldots, a_s + a_s', b_1 b_1', \ldots, b_t b_t')$$

Theorem (new): $(a_1, \ldots, a_s, b_1, \ldots, b_t)$ has a solution of the form

$$f \cdot \exp(g_0) \prod_{\ell=1}^{L} g_\ell^{c_\ell} \cdot \prod_{j=1}^{t} \beta_j^{n_j} \cdot \prod_{p=1}^{P} (\phi_{p,1} n_1 + \cdots + \phi_{p,t} n_t + \varphi_p)!^{e_p},$$

where $f \in k(\mathbf{x},\mathbf{n})$, $g_0, g_\ell, \beta_j \in k(\mathbf{x})$, $c_\ell, \varphi_p \in k$, and $\phi_{p,j}, e_p \in \mathbb{Z}$.

$$\updownarrow$$

$$a_i = \frac{D_i(f)}{f} + D_i(g_0) + \sum_{\ell=1}^{L} c_\ell \frac{D_i(g_\ell)}{g_\ell} + \sum_{j=1}^{t} n_j \frac{D_i(\beta_j)}{\beta_j}, \qquad 1 \le i \le s,$$

$$b_j = \frac{E_j(f)}{f} \cdot \beta_j \cdot \alpha_j, \qquad 1 \le j \le t,\ \alpha_j \in k(\mathbf{n}) \text{ factor into integer-linear polys.}$$

Separating continuous and discrete parts

Lemma (Feng–Singer–Wu 2010): For $\frac{D(b)}{b} = E(a) - a$ in $k(x, n)$, there exist $f \in k(x, n)$, $\beta, \gamma \in k(x)$ and $\alpha \in k(n)$ s.t.

$$a = \frac{D_x(f)}{f} + n\frac{D_x(\beta)}{\beta} + \gamma \quad \text{and} \quad b = \frac{E_n(f)}{f}\beta\alpha.$$

Theorem (new): Certificates of a hyper-hyper function can be written as

$$a_i = \frac{D_i(f)}{f} + \sum_{j=1}^{t} n_j \frac{D_i(\beta_j)}{\beta_j} + A_i(\mathbf{x}), \qquad 1 \leq i \leq s,$$

$$b_j = \frac{E_j(f)}{f} \cdot \beta_j \cdot B_j(\mathbf{n}), \qquad 1 \leq j \leq t,$$

where $f \in k(\mathbf{x}, \mathbf{n})$, $g_j, A_i \in k(\mathbf{x})$, $B_j \in k(\mathbf{n})$, and A_i, B_j satisfy ICs.

The Ore–Sato theorem and its differential analogue

Ore–SatoTheorem (shift case, certificate version):

$$\frac{E_i(B_j)}{B_j} = \frac{E_j(B_i)}{B_i} \quad \Rightarrow \quad B_j = \frac{E_j(f)}{f}\,\alpha_j, \quad 1 \leq j \leq t,$$

where $f \in k(\mathbf{n})$ and $\alpha_j \in k(\mathbf{n})$'s factor into integer-linear polys.

Theorem (differential case, new):

$$D_i(A_j) = D_j(A_i) \quad \Rightarrow \quad a_i = D_i(f) + \frac{D_i(g_0)}{g_0} + \sum_{\ell=1}^{L} c_\ell \frac{D_i(g_\ell)}{g_\ell}, \quad 1 \leq i \leq s,$$

where $f, g_0 \in k(\mathbf{x})$, $c_1, \ldots, c_L \in k$, and $g_1, \ldots, g_L \in k(\mathbf{x})$.

Continuous case of the WZ conjecture

Cor: Multivariate hyperexponential functions are proper.

$$\frac{P}{Q}\exp(g_0)\prod_{\ell=1}^{L} g_\ell^{c_\ell} \quad \leftrightarrow \quad P\exp(g_0)\prod_{\ell=1}^{L+1} g_\ell^{c_\ell}, \quad c_{L+1}=-1, g_{L+1}=Q$$

Theorem (Kashiwara1978, Takayama1992):

$$h(\mathbf{x}) \text{ is } D\text{-finite} \quad \Rightarrow \quad h(\mathbf{x}) \text{ is holonomic.}$$

Cor: Multivariate hyperexponential functions are holonomic.

This makes the WZ conjecture self-evident in the continuous setting!

Towards a proof of the general case of WZ conjecture

Recall: $h(\mathbf{x}, \mathbf{n})$ is holonomic iff

$$H(\mathbf{x}, \mathbf{y}) = \sum_{n_1, \ldots, n_s \geq 0} h(\mathbf{x}, \mathbf{n}) y_1^{n_1} \cdots y_n^{n_t} \quad \text{is holonomic over } k[\mathbf{x}, \mathbf{y}].$$

Need to show or clarify the literature:

1. Closure properties: product, antiderivative;
2. Holonomic $\Rightarrow$ Telescoper exists;
3. If $f \in k(\mathbf{x}, \mathbf{n})$ is holonomic,

$$f = \frac{P(\mathbf{x}, \mathbf{n})}{Q_1(\mathbf{x}) Q_2(\mathbf{n})}, \quad \text{where } Q_2 \text{ factors into integer-linear polys.}$$

Termination of Creative telescoping: the mixed case

Problem: Given a bivariate hyper-hyper function $h(x, n)$, decide:

$$\text{whether} \,\exists\, L \in k(x)\langle D_x\rangle \setminus \{0\} \text{ s.t. } L(x, D_x)(h) = \Delta_n(g)?$$
$$\text{whether} \,\exists\, L \in k(n)\langle E_n\rangle \setminus \{0\} \text{ s.t. } L(n, E_n)(h) = D_x(g)?$$

where g is hyper-hyper over $k(x, n)$.

$\mathcal{H}(a, b)$: The solution space of the fully integrable system

$$D_x(h) = ah, \quad E_n(h) = bh.$$

Split polynomials

Splitness: A polynomial $p \in k[x, n]$ is split if

$$p = p_1(x) \cdot p_2(n) \text{ with } p_1 \in k[x] \text{ and } p_2 \in k[n].$$

By the structure theorem, $h(x, n)$ can be written as

$$f(x, n) \cdot \beta(x)^n \cdot h'(x, n), \qquad \text{(multiplicative form)}$$

where $f \in k(x, n)$, $\beta \in k(x)$, and $h' \in \mathcal{H}(\gamma, \alpha)$ with $\gamma \in k(x), \alpha \in k(n)$.

Proposition:

$$h(x, n) \text{ is proper} \quad \Leftrightarrow \quad \text{denom. of } f \text{ is split}$$

Proof:

$$\frac{P}{A(x)B(n)} \cdot \beta(x)^n \cdot \mathcal{H}(\gamma, \alpha) \quad \leftrightarrow \quad P \cdot \beta(x)^n \cdot \mathcal{H}\left(\gamma - \frac{D_x A}{A}, \alpha \frac{B}{E_n B}\right)$$

Introductory example

$$f(x, n) = \frac{1}{x + n}.$$

Claim 1: f has no telescoper w.r.t. x.

Proof. For any nonzero $L = \sum_{i=0}^{r} e_i(n) E_n^i$,

$$L(f) = \sum_{i=0}^{r} \frac{e_i(n)}{x + n + i} = \frac{P}{(x + n + r + 1) \cdots (x + n)}.$$

1. $L(f)$ is proper w.r.t. x $+$ its denom. is square free w.r.t. x.
2. If $L(f) = D_x(g)$, then $L(f) = 0$.
3. The denominator of a rational solution of $L(y) = 0$ is split, but $x + n$ is not split, contradiction!

Introductory example (continued)

Claim 2: f has no telescoper w.r.t. n.

Proof. For any nonzero $L = \sum_{i=0}^{r} d_i(x) D_x^i$,

$$L(f) = \sum_{i=0}^{r} \frac{p_i}{(x+n)^{i+1}} = \frac{P}{(x+n)^{r+1}}.$$

1. $L(f)$ is proper w.r.t. n $+$ its denom. is shift-free w.r.t. n.
2. If $L(f) = \Delta_n(g)$, then $L(f) = 0$.
3. The denominator of a rational solution of $L(y) = 0$ is split, but $x + n$ is not split, contradiction!

Additive decomposition (discrete case)

$$h(x,n) = f(x,n)\cdot \beta(x)^n \cdot \mathcal{H}(\gamma(x), \alpha(n))$$

Abramov-Petkovšek decomposition: $\exists\, h_1$ with $h/h_1 \in k(x,n)$, s.t.,

$$h = \Delta_n(h_1) + \frac{v}{u}\cdot \beta(x)^n \cdot \mathcal{H}(\tilde{\gamma}(x), \tilde{\alpha}(n)),$$

1. $v, u \in k(x)[n]$ and $\gcd(v,u) = 1$;
2. u is shift-free w.r.t. n;
3. $\beta(x)\tilde{\alpha}(n)$ is shift-reduced w.r.t. n;
4. $\gcd(u, E_n^{-\ell}(\text{num}(\tilde{\alpha}))) = \gcd(u, E_n^{\ell}(\text{den}(\tilde{\alpha}))) = 1$ for all $\ell \in \mathbb{N}$.

Lemma (Abramov-Petkovšek, 2002):

$$h = \Delta_n(g) \;\Rightarrow\; u \in k(x).$$

Additive decomposition (continuous case)

$$h(x,n) = f(x,n) \cdot \beta(x)^n \cdot \mathcal{H}(\gamma(x), \alpha(n))$$

Geddes-Le-Li decomposition: $\exists\, h_1$ with $h/h_1 \in k(x,n)$, s.t.,

$$h = D_x(h_1) + \frac{v}{u} \cdot \beta(x)^n \cdot \mathcal{H}(\tilde{\gamma}(x), \alpha(n)),$$

1. $v, u \in k(n)[x]$ and $\gcd(v,u) = 1$;
2. u is square-free w.r.t. x;
3. $\tilde{\gamma} + n\frac{D_x(\beta)}{\beta}$ is differential-reduced w.r.t. x;
4. $\gcd(u, \mathrm{den}(\tilde{\gamma}(x) + n\frac{D_x(\beta)}{\beta})) = 1$.

Lemma (Geddes-Le-Li, 2004):

$$h = D_x(g) \;\Rightarrow\; u \in k(n).$$

Applying operators to additive decompositions

Applying differential operators: For all $L(x, D_x) \in k(x)\langle D_x \rangle$,

$$h = \Delta_n(h_1) + \frac{v}{u} \cdot \beta(x)^n \cdot \mathcal{H}(\tilde{\gamma}(x), \tilde{\alpha}(n)) \quad \text{add. decomp.}$$

$$\Downarrow$$

$$L(h) = \Delta_n(L(h_1)) + \frac{V}{U} \cdot \beta(x)^n \cdot \mathcal{H}(\tilde{\gamma}(x), \tilde{\alpha}(n)) \quad \text{add. decomp.}$$

Applying recurrence operators: For all $L(n, E_n) \in k(n)\langle E_n \rangle$,

$$h = D_x(h_1) + \frac{v}{u} \cdot \beta(x)^n \cdot \mathcal{H}(\tilde{\gamma}(x), \alpha(n)) \quad \text{add. decomp.}$$

$$\Downarrow$$

$$L(h) = D_x(L(h_1)) + \frac{V}{U\mathrm{den}(\beta)^{\rho}} \cdot \beta(x)^n \cdot \mathcal{H}(\tilde{\gamma}(x), \alpha(n)) \quad \text{add. decomp.}$$

Two criteria for existence

$$h(x,n) = f(x,n) \cdot \beta(x)^n \cdot \mathcal{H}(\gamma(x), \alpha(n))$$

Discrete case: $h = \Delta_n(h_1) + h_2$ with $h_2 = \frac{v}{u} \cdot \beta(x)^n \cdot \mathcal{H}(\tilde{\gamma}(x), \tilde{\alpha}(n))$,

h has a telescoper w.r.t. n $\Leftrightarrow$ u is split $\Leftrightarrow$ h_2 is proper.

Continuous case: $h = D_x(h_1) + h_2$ with $h_2 = \frac{v}{u} \cdot \beta(x)^n \cdot \mathcal{H}(\tilde{\gamma}(x), \alpha(n))$,

h has a telescoper w.r.t. x $\Leftrightarrow$ u is split $\Leftrightarrow$ h_2 is proper.

Sufficient part: h has a telescoper if it is proper.

Wilf and Zeilberger's fundamental theorem $+$ Wegschaider's trick.

Two observations

Continuous case: Let p be a polynomial in $k[x, n]$ and

$$L(x, n, D_x) = e_n(x, n) D_x^{\rho} + e_n(x, n) D_x^{\rho-1} + \cdots + e_0(x, n)$$

$$f \in k(x, n) \text{ s.t. } L(f) = p \text{ and } e_n \text{ is split} \quad \Rightarrow \quad \text{denom. of } f \text{ is split.}$$

Discrete case: Let p be a polynomial in $k[x, n]$ and

$$L(x, n, E_n) = e_n(x, n) E_n^{\rho} + e_n(x, n) E_n^{\rho-1} + \cdots + e_0(x, n).$$

$$f \in k(x, n) \text{ s.t. } L(f) = p \text{ and } e_n \text{ is split} \quad \Rightarrow \quad \text{denom. of } f \text{ is split.}$$

Proof of necessary part (the discrete case)

Assumption:
A hypergeometric $h(x, n)$ has a telescoper L in $k(x)\langle D_x \rangle$.

A sketch:

1. Additive decomposition:

$$h = \Delta_n(h_1) + h_2$$

   where

$$h_2 = \frac{v}{u} \cdot \tilde{h}_2 \quad \text{with} \quad \tilde{h}_2 = \beta(x)^n \cdot \mathcal{H}(\tilde{\gamma}(x), \alpha(n)).$$

2. Applying L:

$$L(n, E_n)(h) = D_x(g) \quad \Rightarrow \quad L(h_2) = D_x(g - L(h_1)).$$

Goal: Prove u split.

3 Rewriting $L(h_2)$:
$\forall\ i \in \mathbb{N},\ \exists\ M_i \in k(x)[n]\langle D_x \rangle$ with leading coefficient 1 s.t.

$$D_x^i(h_2) = M_i\left(\frac{v}{u}\right) \cdot \tilde{h}_2$$

$$\Downarrow$$

$$L(h_2) = M\left(\frac{v}{u}\right) \cdot \tilde{h}_2 = \frac{V}{U} \cdot \tilde{h}_2 \ \Rightarrow\ M\left(\frac{v}{u}\right) = \frac{V}{U}.$$

where $M \in k(x)[n]\langle D_x \rangle$ with leading coefficient in $k(x)$, and $U, V \in k[x, n]$.

4 Applying Lemma (Abramov-Petkovšek, 2002):
Since $L(h_2)$ is hypergeometric summable w.r.t. n,

$$L(h_2) = \frac{V}{U} \cdot \tilde{h}_2 \quad \Rightarrow \quad U \in k[x].$$

5. Applying the first observation:
Since v/u is a rational solution of $(U \cdot M)(y) = \tilde{V}$, u is split.

Two criteria are not redundant!

Example:

$$h(x, n) = \frac{1}{(x+n)^s},$$

- $s = 1$: h has no telescoper w.r.t. n or x;
- $s > 1$: h has no telescoper w.r.t. n, but it has a telescoper w.r.t. x,

$$h = D_x \left(\frac{-1}{(s-1)(x+n)^{s-1}} \right).$$

Non-proper example

Example:

$$h(x, n) = \frac{-x + 2\,nx + 2\,n^2}{(x+n)^2\,x} \cdot x^n \cdot e^{-x}.$$

Though h is not proper, it still has a telescoper w.r.t. x since

$$h = D_x\left(\frac{1}{x+n} x^n e^{-x}\right) + \frac{1}{x} \cdot x^n \cdot e^{-x}.$$

Summary

Present:

1. Structure theorem for hyper-hyper functions;
2. The continuous case of WZ conjecture is true;
3. Two criteria for the existence of telescopers:

$$h \text{ has a telescoper w.r.t. } n \quad \Leftrightarrow \quad h = \Delta_n(h_1) + \text{proper term.}$$

$$h \text{ has a telescoper w.r.t. } x \quad \Leftrightarrow \quad h = D_x(h_1) + \text{proper term.}$$

Future:

- The general case of WZ conjecture;
- The existence criterion for multivariate hyper-hyper functions;
- Implement Zeilberger's algorithm with termination criteria;
- The mixed differential and q-shift cases.