

FREE PROBABILITY AND RANDOM MATRICES

Philippe Biane
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Free probability, invented by D. Voiculescu, is a tool for understanding spectral properties of sets of large random matrices X =hermitian $N \times N$ matrix.

$$X = UDU^*$$

U =unitary (eigenvectors of X);

D =real diagonal (eigenvalues)

$$D = \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & \lambda_N \end{pmatrix}$$

The geometry of X is specified, up to conjugation by a unitary matrix, by its spectrum.

$$\text{Spec}(X) \iff \frac{1}{N} \text{Tr}(X^n) = \frac{1}{N} \sum_{i=1}^N \lambda_i^n; n = 1, 2, \dots$$

X_1, \dots, X_n $N \times N$ hermitian matrices.

In general they do not commute: no joint spectrum.

Up to conjugation by a unitary

$$X_1, \dots, X_n \mapsto UX_1U^*, \dots, UX_nU^*$$

the n -tuple of matrices X_1, \dots, X_n

can be recovered from their moments

$$\frac{1}{N} \text{Tr}(X_{i_1} \dots X_{i_k}); \quad i_1, \dots, i_k \in \{1, \dots, n\}$$

A MODEL FOR INDEPENDENT MATRICES

Take $X_i = U_i D_i U_i^*$ where D_i are fixed real diagonal, and U_i are random unitaries (taken with Haar measure on $U(N)$).

Haar measure on $U(N)$:

$U = (V_1 \ V_2 \ \dots \ V_N)$ column vectors. Choose V_1 at random with norm 1. Then choose $V_2 \perp V_1$ at random with norm 1, then $V_3 \perp V_1, V_2$, etc...

Theorem

(Voiculescu, 1990) *When $N \rightarrow \infty$ with probability almost one,*

$$\frac{1}{N} \text{Tr}(X_{i_1} \dots X_{i_k})$$

can be expressed asymptotically, as polynomial functions, in terms of the moments $\frac{1}{N} \text{Tr}(D_i^k) = \frac{1}{N} \text{Tr}(X_i^k)$

Examples: $(\frac{1}{N} \text{Tr} = \text{tr})$

$$\text{tr}(X_1 X_2) \sim \text{tr}(X_1) \text{tr}(X_2)$$

$$\text{tr}(X_1 X_2 X_1 X_2) \sim$$

$$\text{tr}(X_1^2) \text{tr}(X_2)^2 + \text{tr}(X_1)^2 \text{tr}(X_2^2) - \text{tr}(X_1)^2 \text{tr}(X_2)^2$$

Corollary

If we know the spectra of X_1, \dots, X_n
then we can compute, with good approximation,
and high probability, the spectrum of any combination
of X_i 's (e.g. sum, product etc...).

e.g.

$$\operatorname{tr}((X_1 + X_2)^n) = \sum_{i_1 \dots i_n} \operatorname{tr}(X_{i_1} \dots X_{i_n})$$

can be computed from the values $\operatorname{tr}(X_1^k), \operatorname{tr}(X_2^k), k = 1, 2, \dots$

FREENESS

A =algebra (of noncommutative random variables);
 $1 \in A$, $a + b$, ab , $\lambda a \in A$ if $a, b \in A$
 $\tau : A \rightarrow \mathbf{C}$ =linear functional (=expectation). $\tau(1) = 1$

Definition (Voiculescu, 1983)

$\{A_i; i \in I\}$ =family of algebras are *free* in (A, τ) iff

for all $a_1, \dots, a_n \in A$ such that

i) $\tau(a_j) = 0$ for all j ,

ii) $a_j \in A_{i_j}$, $i_1 \neq i_2, i_2 \neq i_3, \dots, i_{n-1} \neq i_n$,

one has

$$\tau(a_1 \dots a_n) = 0$$

Example: $a_1 \in A_1, a_2 \in A_2$, free in (A, τ)

$$a_1 = \bar{a}_1 + \tau(a_1)1; \quad a_2 = \bar{a}_2 + \tau(a_2)1; \quad \tau(\bar{a}_1) = \tau(\bar{a}_2) = 0$$

$$\tau(a_1 a_2) = \tau((\bar{a}_1 + \tau(a_1))(\bar{a}_2 + \tau(a_2)))$$

by freeness assumption

$$\tau(\bar{a}_1 \bar{a}_2) = 0$$

finally

$$\tau(a_1 a_2) = \tau(a_1)\tau(a_2)$$

Similarly

$$\tau(a_1 a_2 a_1 a_2) = \tau(a_1^2)\tau(a_2)^2 + \tau(a_1)^2\tau(a_2^2) - \tau(a_1)^2\tau(a_2)^2$$

In general

$$\tau(a_1 \dots a_n)$$

for $a_j \in A_{j_i}$ can be computed by a polynomial in moments

$$\tau(a_{i_1} \dots a_{j_r})$$

with $a_{j_1} \dots a_{j_r}$ in the same algebra.

FREENESS AND RANDOM MATRICES

Take $X_i = U_i D_i U_i^*$ where D_i are fixed real diagonal, and U_i are random unitaries. Let $a_1, \dots, a_n \in (A, \tau)$ be free and such that

$$\tau(a_i^k) = \text{tr}(X_i^k) \quad k = 1, 2, \dots$$

then for N large

$$\text{tr}(X_{i_1} \dots X_{i_k}) \sim \tau(a_{i_1} \dots a_{i_k})$$

with probability close to 1. As we saw $\tau(a_{i_1} \dots a_{i_k})$ can be written as a polynomial in the moments $\tau(a_i^k) = \text{tr}(X_i^k)$.

This solves the problem at the beginning.

COMBINATORICS OF FREENESS

A combinatorial way of dealing with freeness has been devised by R. Speicher, using noncrossing partitions.

A partition of $\{1, \dots, n\}$ is noncrossing if there is no crossing. A crossing is a quadruple (i, j, k, l) with

$$i < j < k < l$$

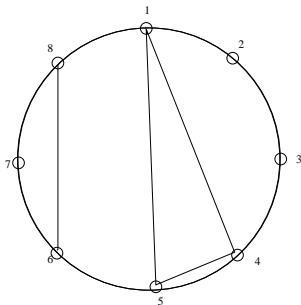
and

$$i \sim k, \quad k \sim l$$

and i, j not in the same part.

$$\{1, 4, 5\} \cup \{2\} \cup \{3\} \cup \{6, 8\} \cup \{7\}$$

has no crossing



NONCROSSING CUMULANTS

On (A, τ) define R_n , multilinear functionals by

$$\tau(a_1 \dots a_n) = \sum_{\pi \in NC(n)} R_{\pi}(a_1, \dots, a_n)$$

$$R_{\pi}(a_1, \dots, a_n) = \prod_{\text{parts of } \pi} R_{|p|}(a_{i_1}, \dots, a_{i_{|p|}})$$

where $p = \{i_1, \dots, i_{|p|}\}$ is a part of π .

Example:

$$\begin{aligned}\tau(a_1 a_2 a_3) = & R_3(a_1, a_2, a_3) && \{1, 2, 3\} \\ & + R_1(a_1) R_2(a_2, a_3) && \{1\} \cup \{2, 3\} \\ & + R_2(a_1, a_3) R_1(a_2) && \{1, 3\} \cup \{2\} \\ & + R_2(a_1, a_2) R_1(a_3) && \{1, 2\} \cup \{3\} \\ & + R_1(a_1) R_1(a_2) R_1(a_3) && \{1\} \cup \{2\} \cup \{3\}\end{aligned}$$

$$\begin{aligned}R_1(a) &= \tau(a) \\ R_2(a_1, a_2) &= \tau(a_1 a_2) - \tau(a_1)\tau(a_2) \\ R_3(a_1, a_2, a_3) &= \tau(a_1 a_2 a_3) - \tau(a_1 a_2)\tau(a_3) \\ &\quad - \tau(a_1 a_3)\tau(a_2) \\ &\quad - \tau(a_1)\tau(a_2 a_3) \\ &\quad + 2\tau(a_1)\tau(a_2)\tau(a_3)\end{aligned}$$

FREENESS AND FREE CUMULANTS

Theorem (Speicher). If $A_i \subset A; i \in I$ are free, and $a_1 \in A_{i_1}, \dots, a_n \in A_{i_n}$, then

$$R_n(a_1, \dots, a_n) = 0$$

if there exists j, k such that $i_j \neq i_k$.

Remark If one uses all partitions instead of noncrossing partitions, this is Rota's combinatorial approach to independence.

Example:

a, b free in (A, τ) then

$$R_n(a + b, \dots, a + b) = R_n(a, \dots, a) + R_n(b, \dots, b)$$

FREE CONVOLUTION

A =*-algebra; τ =tracial state on A . Let X_1, X_2 be free, selfadjoint in A .

$$\tau(X_1^n) = \int_{\mathbb{R}} x^n \mu_1(dx); \quad \tau(X_2^n) = \int_{\mathbb{R}} x^n \mu_2(dx)$$

$$\tau((X_1 + X_2)^n) = \int_{\mathbb{R}} x^n \mu_1 \boxplus \mu_2(dx)$$

$$G_\mu(z) = \int \frac{1}{z-x} \mu(dx) = \frac{1}{z} + \sum_{n=1}^{\infty} z^{-n-1} \int x^n \mu(dx)$$

$$K_\mu(G_\mu(z)) = G_\mu(K_\mu(z)); \quad K_\mu(z) = \frac{1}{z} + \sum_{n=0}^{\infty} R_n(\mu) z^n$$

Theorem (Voiculescu, 1986)

$$R_n(\mu_1 \boxplus \mu_2) = R_n(\mu_1) + R_n(\mu_2)$$

$R_n(\mu)$ are called the *free cumulants* of μ . Compare with

$$\log \int e^{itx} \mu(dx) = \sum_n (it)^n C_n(\mu)/n!$$

where C_n are the *cumulants* of μ .

$$C_n(\mu_1 * \mu_2) = C_n(\mu_1) + C_n(\mu_2).$$

Examples:

$$\frac{1}{2}(\delta_0 + \delta_1) \boxplus \frac{1}{2}(\delta_0 + \delta_1)$$

Random matrix model

$$\Pi_1 + \Pi_2$$

where $\Pi_1, \Pi_2 =$ orthogonal projections on a random subspaces of dimension $N/2$.

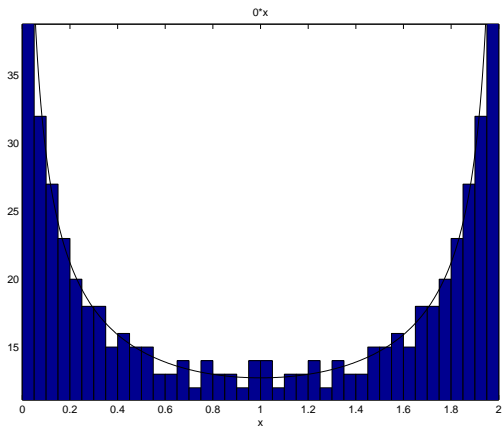
$$y = \frac{1}{\pi \sqrt{x(2-x)}}$$

$$\frac{1}{2}(\delta_0 + \delta_1) \boxplus \frac{1}{2}(\delta_0 + \delta_1) \boxplus \frac{1}{2}(\delta_0 + \delta_1)$$

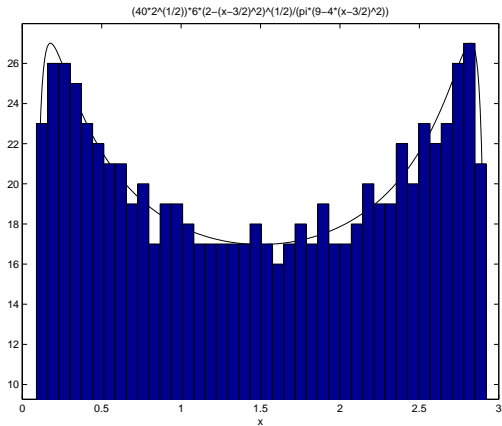
Random matrix model

$$\Pi_1 + \Pi_2 + \Pi_3$$

$$y = \frac{\sqrt{8 - (2x - 3)^2}}{\pi \sqrt{x(3-x)}}$$



$$y = \frac{\Pi_1 + \Pi_2}{\pi \sqrt{x(2-x)}}$$



$$\begin{aligned}
 & \Pi_1 + \Pi_2 + \Pi_3 \\
 y = & \frac{\sqrt{8 - (2x - 3)^2}}{\pi \sqrt{x(3 - x)}}
 \end{aligned}$$

FREE CENTRAL LIMIT THEOREM

Let $X_1, \dots, X_n \in (A, \tau)$ be free random variables, identically distributed.

$$\tau(X_j) = 0 \quad \tau(X_j^2) = \sigma^2$$

Theorem (Voiculescu, 1983) As $n \rightarrow \infty$ the distribution of $\frac{X_1 + \dots + X_n}{\sqrt{n}}$ converges to the semi-circular distribution with density

$$\frac{1}{\pi\sigma} \sqrt{4\sigma^2 - x^2} \quad x \in [-2\sigma, 2\sigma]$$

This should be compared with Wigner's theorem

Let M be a random hermitian gaussian matrix such that

$$E[\text{Tr}(M^2)] = N$$

then the distribution of eigenvalues of M converges to the semi-circular distribution as $N \rightarrow \infty$. Indeed one has

$$M = \frac{M_1 + M_2 + \dots + M_n}{\sqrt{n}}$$

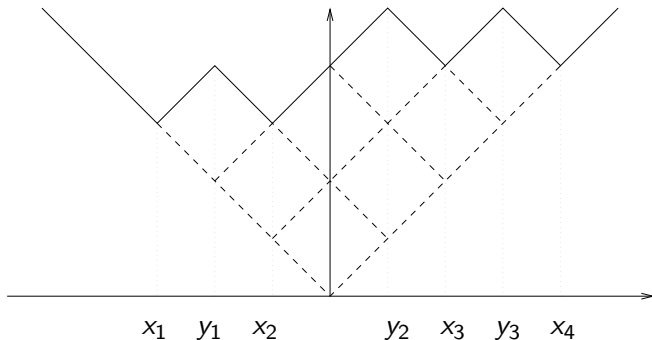
with independent random matrices M_1, \dots, M_n , which are

YOUNG DIAGRAMS

A Young diagram is a sequence of integers

$$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq 0$$

Young diagrams label *irreducible representations of symmetric groups*.



A diagram may be identified with a function $\omega(x)$ such that

$$|\omega(x)| = |x| \text{ for } x \gg 1$$

$$|\omega(x) - \omega(y)| \leq |x - y|.$$

TRANSITION MEASURES

Take ω as above, put

$$\sigma(u) = (\omega(u) - |u|)/2$$

then (S.Kerov) there exists a unique probability measure

$$m_\omega$$

such that

$$\begin{aligned} G_\omega(z) &= \frac{1}{z} \exp \int \frac{1}{x-z} \sigma'(x) dx \\ &= \int \frac{1}{z-x} m_\omega(dx) \\ &= \frac{\prod_{i=1}^{n-1} (z-y_i)}{\prod_{i=1}^n (z-x_i)} \end{aligned}$$

$$m_\omega = \sum_{k=1}^n \mu_k \delta_{x_k} \quad \mu_k = \frac{\prod_{i=1}^{n-1} (x_k - y_i)}{\prod_{i \neq k} (x_k - x_i)}$$

$$K_\omega = G_\omega^{\langle -1 \rangle}$$

ASYMPTOTIC EVALUATION OF CHARACTERS

$\lambda =$ Young diagram with q boxes, $\lambda \sim \sqrt{q}\omega$. Number of rows and columns $= O(\sqrt{q})$. $\chi_\lambda =$ normalized character of λ .

$$\chi_\lambda(\sigma) = q^{-|\sigma|/2} \left(\prod_{c|\sigma} R_{|c|+2}(\omega) + O(q^{-1}) \right)$$

$|\sigma| =$ length of σ w.r.t generating set of all transpositions, the product is over cycles of σ .

ASYMPTOTIC OF RESTRICTION

For a continuous diagram ω , and $0 < t < 1$, define ω_t by

$$R_n(\omega_t) = t^{n-1} R_n(\omega)$$

The restriction of λ to $S_p \times S_{q-p} \subset S_q$ splits into irreducible

$$\bigoplus c_{\mu\nu}^\lambda [\mu] \otimes [\nu] \quad (\text{Littlewood-Richarson rule}).$$

Give a weight $c_{\mu\nu}^\lambda \dim(\mu) \dim(\nu)$ to the pair (μ, ν) .

Then as $q \rightarrow \infty$ and $p/q \rightarrow t$, almost all pairs (μ, ν) (rescaled by \sqrt{q}),
become close to (ω_t, ω_{1-t}) .

ASYMPTOTIC OF INDUCTION

For continuous diagrams ω, ω' , define $\omega \boxplus \omega'$ by

$$R_n(\omega \boxplus \omega') = R_n(\omega) + R_n(\omega')$$

The induction of $[\mu] \otimes [\nu]$ from $S_p \times S_{q-p}$ to S_q splits into irreducible

$$\bigoplus c_{\mu\nu}^\lambda [\lambda]$$

Frobenius duality: the coefficients are given by Littlewood-Richardson rule. Give a weight $c_{\mu\nu}^\lambda \dim(\lambda)$ to

λ . Rescaling by a common factor $\mu \rightarrow \omega$ and $\nu \rightarrow \omega'$, then almost all λ become close to the shape $\omega \boxplus \omega'$.

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