# FREE PROBABILITY AND RANDOM MATRICES

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Free probability, invented by D. Voiculescu, is a tool for understanding spectral properties of sets of large random matrices
$X$=hermitan $N \times N$ matrix.

$$X = UDU^*$$

$U$=unitary (eigenvectors of $X$);
$D$=real diagonal (eigenvalues)

$$D = \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & \lambda_N \end{pmatrix}$$

The geometry of $X$ is specified, up to conjugation by a unitary matrix, by its spectrum.
$\mathrm{Spec}(X) \Longleftrightarrow \frac{1}{N} Tr(X^n) = \frac{1}{N} \sum_{i=1}^{N} \lambda_i^n; n = 1, 2, \ldots$

$X_1, \ldots, X_n$ $N \times N$ hermitian matrices.
In general they do not commute: no joint spectrum.
Up to conjugation by a unitary

$$X_1, \ldots, X_n \mapsto UX_1U^*, \ldots, UX_nU^*$$

the $n$-tuple of matrices $X_1, \ldots, X_n$
can be recovered from their moments

$$\frac{1}{N} Tr(X_{i_1} \ldots X_{i_k}); \qquad i_1, \ldots, i_k \in \{1, \ldots, n\}$$

# A MODEL FOR INDEPENDENT MATRICES

Take $X_i = U_i D_i U_i^*$ where $D_i$ are fixed real diagonal, and $U_i$ are random unitaries (taken with Haar measure on $U(N)$).

Haar measure on $U(N)$:
$U = (V_1 \ V_2 \ \dots V_N)$ column vectors. Choose $V_1$ at random with norm 1. Then choose $V_2 \perp V_1$ at random with norm 1, then $V_3 \perp V_1, V_2$, etc...

**Theorem**

(Voiculescu, 1990) *When* $N \to \infty$ *with probability almost one,*

$$\frac{1}{N} Tr(X_{i_1} \dots X_{i_k})$$

*can be expressed asymptotically, as polynomial functions, in terms of the moments* $\frac{1}{N} Tr(D_i^k) = \frac{1}{N} Tr(X_i^k)$

Examples: $(\frac{1}{N} Tr = tr)$

$$tr(X_1 X_2) \sim tr(X_1) tr(X_2)$$

$$tr(X_1 X_2 X_1 X_2) \sim$$
$$tr(X_1^2) tr(X_2)^2 + tr(X_1)^2 tr(X_2^2) - tr(X_1)^2 tr(X_2)^2$$

**Corollary**

If we know the spectra of $X_1, \ldots, X_n$
then we can compute, with good approximation,
and high probability, the spectrum of any combination
of $X_i$'s (e.g. sum, product etc...).
e.g.

$$tr((X_1 + X_2)^n) = \sum_{i_1 \ldots i_n} tr(X_{i_1} \ldots X_{i_n})$$

can be computed from the values $tr(X_1^k), tr(X_2^k), k = 1, 2, ....$

# FREENESS

$A$=algebra (of noncommutative random variables);
$1 \in A, \quad a+b, \quad ab, \quad \lambda a \quad \in A$ if $a, b \in A$
$\tau : A \to \mathbf{C}$=linear functional (=expectation). $\tau(1) = 1$

**Definition** (Voiculescu, 1983)
$\{A_i; i \in I\}$=family of algebras are *free* in $(A, \tau)$ iff
*for all* $a_1, \ldots, a_n \in A$ *such that*
*i)* $\tau(a_j) = 0$ *for all* $j$,
*ii)* $a_j \in A_{i_j}$, $i_1 \neq i_2$, $i_2 \neq i_3, \ldots, i_{n-1} \neq i_n$,
*one has*

$$\tau(a_1 \ldots a_n) = 0$$

Example: $a_1 \in A_1, a_2 \in A_2$, free in $(A, \tau)$

$$a_1 = \bar{a}_1 + \tau(a_1)1; \quad a_2 = \bar{a}_2 + \tau(a_2)1; \quad \tau(\bar{a}_1) = \tau(\bar{a}_2) = 0$$

$$\tau(a_1 a_2) = \tau((\bar{a}_1 + \tau(a_1))(\bar{a}_2 + \tau(a_2)))$$

by freeness assumption

$$\tau(\bar{a}_1 \bar{a}_2) = 0$$

finally

$$\tau(a_1 a_2) = \tau(a_1)\tau(a_2)$$

Similarly

$$\tau(a_1 a_2 a_1 a_2) = \tau(a_1^2)\tau(a_2)^2 + \tau(a_1)^2\tau(a_2^2) - \tau(a_1)^2\tau(a_2)^2$$

In general

$$\tau(a_1 \dots a_n)$$

for $a_j \in A_{i_j}$ can be computed by a polynomial in moments

$$\tau(a_{i_1} \dots a_{j_r})$$

with $a_{j_1} \dots a_{j_r}$ in the same algebra.

# FREENESS AND RANDOM MATRICES

Take $X_i = U_i D_i U_i^*$ where $D_i$ are fixed real diagonal, and $U_i$ are random unitaries. Let $a_1, \ldots, a_n \in (A, \tau)$ be free and such that

$$\tau(a_i^k) = tr(X_i^k) \qquad k = 1, 2, \ldots$$

then for $N$ large

$$tr(X_{i_1} \ldots X_{i_k}) \sim \tau(a_{i_1} \ldots a_{i_k})$$

with probability close to 1. As we saw $\tau(a_{i_1} \ldots a_{ik})$ can be written as a polynomial in the moments $\tau(a_i^k) = tr(X_i^k)$.

This solves the problem at the beginning.

# COMBINATORICS OF FREENESS

A combinatorial way of dealing with freeness has been devised by R. Speicher,
using noncrossing partitions.
A partition of $\{1, \ldots, n\}$ is noncrossing if there is no crossing. A crossing is a quadruple $(i, j, k, l)$ with

$$i < j < k < l$$

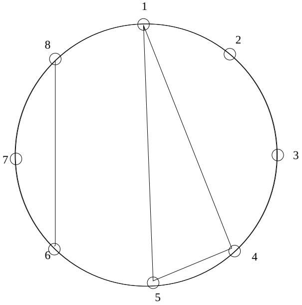
and

$$i \sim k, \qquad k \sim l$$

and $i, j$ not in the same part.

$$\{1,4,5\}\cup\{2\}\cup\{3\}\cup\{6,8\}\cup\{7\}$$

has no crossing



## NONCROSSING CUMULANTS

On $(A, \tau)$ define $R_n$, multilinear functionals by

$$\tau(a_1 \dots a_n) = \sum_{\pi \in NC(n)} R_\pi(a_1, \dots, a_n)$$

$$R_\pi(a_1, \dots, a_n) = \prod_{\text{parts of}\pi} R_{|p|}(a_{i_1}, \dots, a_{i_{|p|}})$$

where $p = \{i_1, \dots, i_{|p|}\}$ is a part of $\pi$.

Example:

$$\begin{array}{rcc} \tau(a_1a_2a_3) = & R_3(a_1,a_2,a_3) & \{1,2,3\} \\ & +R_1(a_1)R_2(a_2,a_3) & \{1\}\cup\{2,3\} \\ & +R_2(a_1,a_3)R_1(a_2) & \{1,3\}\cup\{2\} \\ & +R_2(a_1,a_2)R_1(a_3) & \{1,2\}\cup\{3\} \\ & +R_1(a_1)R_1(a_2)R_1(a_3) & \{1\}\cup\{2\}\cup\{2\} \end{array}$$

$$\begin{array}{rcl} R_1(a) & = & \tau(a) \\ R_2(a_1,a_2) & = & \tau(a_1a_2)-\tau(a_1)\tau(a_2) \\ R_3(a_1,a_2,a_3) & = & \tau(a_1a_2a_3)-\tau(a_1a_2)\tau(a_3) \\ & & -\tau(a_1a_3)\tau(a_2) \\ & & -\tau(a_1)\tau(a_2a_3) \\ & +2\tau(a_1)\tau(a_2)\tau(a_3) & \end{array}$$

# FREENESS AND FREE CUMULANTS

**Theorem** (Speicher). If $A_i \subset A; i \in I$ are free, and $a_1 \in A_{i_1}, \dots, a_n \in A_{i_n}$, then

$$R_n(a_1, \dots, a_n) = 0$$

if there exists $j, k$ such that $i_j \neq i_k$.

**Remark** If one uses all partitions instead of noncrossing partitions, this is Rota's combinatorial approach to independence.
Example:
$a, b$ free in $(A, \tau)$ then

$$R_n(a+b, \dots, a+b) = R_n(a, \dots, a) + R_n(b, \dots, b)$$

## FREE CONVOLUTION

$A$ =*-algebra; $\tau$=tracial state on $A$. Let $X_1, X_2$ be free, selfadjoint in $A$.

$$\tau(X_1^n) = \int_{\mathbb{R}} x^n \mu_1(dx); \qquad \tau(X_2^n) = \int_{\mathbb{R}} x^n \mu_2(dx)$$

$$\tau((X_1 + X_2)^n) = \int_{\mathbb{R}} x^n \mu_1 \boxplus \mu_2(dx)$$

$$G_\mu(z) = \int \frac{1}{z - x}\mu(dx) = \frac{1}{z} + \sum_{n=1}^{\infty} z^{-n-1} \int x^n \mu(dx)$$

$$K_\mu(G_\mu(z)) = G_\mu(K_\mu(z)); \qquad K_\mu(z) = \frac{1}{z} + \sum_{n=0}^{\infty} R_n(\mu) z^n$$

**Theorem** (Voiculescu, 1986)

$$R_n(\mu_1 \boxplus \mu_2) = R_n(\mu_1) + R_n(\mu_2)$$

$R_n(\mu)$ are called the *free cumulants* of $\mu$. Compare with

$$\log \int e^{itx} \mu(dx) = \sum_n (it)^n C_n(\mu)/n!$$

where $C_n$ are the *cumulants* of $\mu$.

$$C_n(\mu_1 * \mu_2) = C_n(\mu_1) + C_n(\mu_2).$$

Examples:

$$\frac{1}{2}(\delta_0 + \delta_1) \boxplus \frac{1}{2}(\delta_0 + \delta_1)$$

Random matrix model

$$\Pi_1 + \Pi_2$$

where $\Pi_1, \Pi_2 =$ orthogonal projections on a random subspaces of dimension $N/2$.

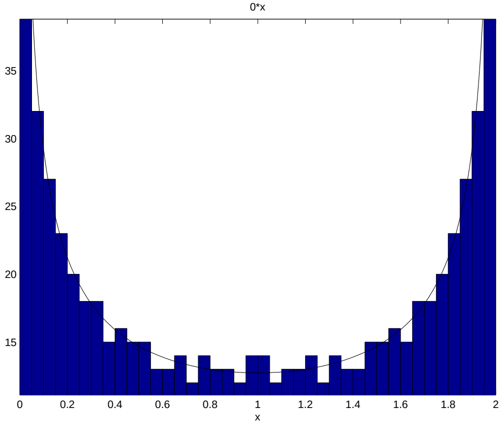
$$y = \frac{1}{\pi\sqrt{x(2-x)}}$$

$$\frac{1}{2}(\delta_0 + \delta_1) \boxplus \frac{1}{2}(\delta_0 + \delta_1) \boxplus \frac{1}{2}(\delta_0 + \delta_1)$$

Random matrix model

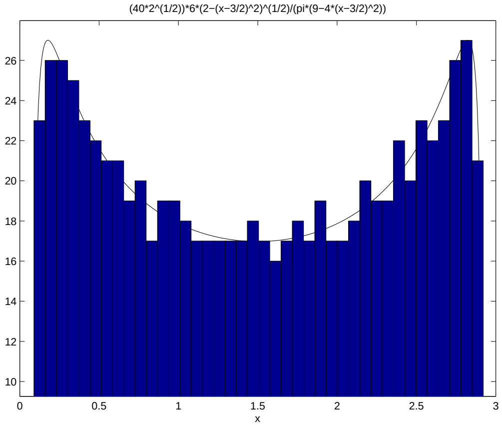
$$\Pi_1 + \Pi_2 + \Pi_3$$

$$y = \frac{\sqrt{8 - (2x-3)^2}}{\pi\sqrt{x(3-x)}}$$




$$\Pi_1 + \Pi_2$$

$$y = \frac{1}{\pi\sqrt{x(2-x)}}$$




$$\Pi_1 + \Pi_2 + \Pi_3$$

$$y = \frac{\sqrt{8-(2x-3)^2}}{\pi\sqrt{x(3-x)}}$$

# FREE CENTRAL LIMIT THEOREM

Let $X_1, \ldots, X_n \in (A, \tau)$ be free random variables, identically distributed.

$$\tau(X_i) = 0 \qquad \tau(X_i^2) = \sigma^2$$

**Theorem** (Voiculescu, 1983) As $n \to \infty$ the distribution of $\frac{X_1 + \ldots + X_n}{\sqrt{n}}$ converges to the semi-circular distribution with density

$$\frac{1}{\pi\sigma}\sqrt{4\sigma^2 - x^2} \qquad x \in [-2\sigma, 2\sigma]$$

This should be compared with Wigner's theorem

Let $M$ be a random hermitian gaussian matrix such that

$$E[Tr(M^2)] = N$$

then the distribution of eigenvalues of $M$ converges to the semi-circular distribution as $N \to \infty$.Indeed one has

$$M = \frac{M_1 + M_2 + \ldots + M_n}{\sqrt{n}}$$

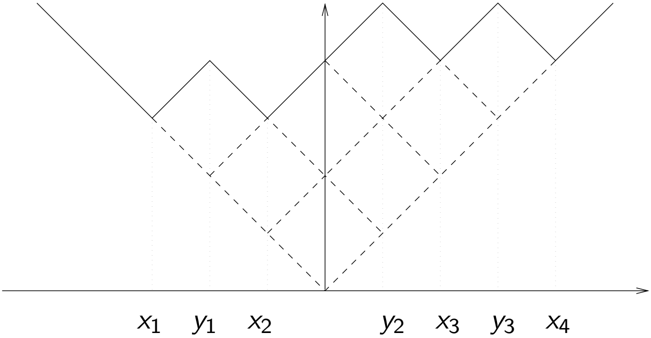
with independent random matrices $M_1, \ldots, M_n$, which are

# YOUNG DIAGRAMS

A Young diagram is a sequence of integers

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \leq 0$$

Young diagrams label *irreducible representations of symmetric groups.*



A diagram may be identified with a function $\omega(x)$ such that

$$|\omega(x)| = |x| \text{ for } x >> 1$$

$$|\omega(x) - \omega(y)| \leq |x - y|.$$

# TRANSITION MEASURES

Take $\omega$ as above, put

$$\sigma(u) = (\omega(u) - |u|)/2$$

then (S.Kerov) there exists a unique probability measure

$$m_\omega$$

such that

$$\begin{aligned} G_\omega(z) &= \tfrac{1}{z} \exp \int \tfrac{1}{x-z} \sigma'(x)\, dx \\ &= \int \tfrac{1}{z-x} m_\omega(dx) \\ &= \tfrac{\prod_{i=1}^{n-1}(z-y_k)}{\prod_{i=1}^{n}(z-x_k)} \end{aligned}$$

$$m_\omega = \sum_{k=1}^{n} \mu_k \delta_{x_k} \qquad \mu_k = \frac{\prod_{i=1}^{n-1}(x_k - y_i)}{\prod_{i \neq k}(x_k - x_i)}$$

$$K_\omega = G_\omega^{\langle -1 \rangle}$$

## ASYMPTOTIC EVALUATION OF CHARACTERS

$\lambda =$ Young diagram with $q$ boxes, $\lambda \sim \sqrt{q}\omega$. Number of rows and columns $= O(\sqrt(q))$. $\chi_\lambda =$ normalized character of $\lambda$.

$$\chi_\lambda(\sigma) = q^{-|\sigma|/2}(\prod_{c|\sigma} R_{|c|+2}(\omega) + O(q^{-1}))$$

$|\sigma|=$ length of $\sigma$ w.r.t generating set of all transpositions, the product is over cycles of $\sigma$.

# ASYMPTOTIC OF RESTRICTION

For a continuous diagram $\omega$, and $0 < t < 1$, define $\omega_t$ by

$$R_n(\omega_t) = t^{n-1} R_n(\omega)$$

The restriction of $\lambda$ to $S_p \times S_{q-p} \subset S_q$ splits into irreducible

$$\bigoplus c^{\lambda}_{\mu\nu} [\mu] \otimes [\nu] \quad \text{(Littlewood-Richarson rule).}$$

Give a weight $c^{\lambda}_{\mu\nu}\, dim(\mu)\, dim(\nu)$ to the pair $(\mu, \nu)$.
Then as $q \to \infty$ and $p/q \to t$, almost all pairs $(\mu, \nu)$ (rescaled by $\sqrt{q}$),
become close to $(\omega_t, \omega_{1-t})$.

# ASYMPTOTIC OF INDUCTION

For continuous diagrams $\omega, \omega'$, define $\omega \boxplus \omega'$ by

$$R_n(\omega \boxplus \omega') = R_n(\omega) + R_n(\omega')$$

The induction of $[\mu] \otimes [\nu]$ from $S_p \times S_{q-p}$ to $S_q$ splits into

irreducible

$$\bigoplus c^{\lambda}_{\mu\nu} [\lambda]$$

Frobenius duality: the coefficients are given by Littlewood-Richardson rule.Give a weight $c^{\lambda}_{\mu\nu}\, dim(\lambda)$ to

$\lambda$.Rescaling by a common factor $\mu \to \omega$ and $\nu \to \omega'$,

then almost all $\lambda$ become close to the shape $\omega \boxplus \omega'$.

# REFERENCES

P. Biane Free probability for probabilists
available at
http://www.dma.ens.fr/ biane/

Hiai, Fumio; Petz, Dénes The semicircle law, free random variables and entropy.
Mathematical Surveys and Monographs, 77. American Mathematical Society, Providence, RI, 2000. Speicher, Roland

Combinatorial theory of the free product with amalgamation and operator-valued free probability theory. Mem. Amer. Math. Soc. 132 (1998), no. 627 Voiculescu, Dan Lectures on free probability

theory.
Lectures on probability theory and statistics
(Saint-Flour, 1998), 279–349, Lecture Notes in Math., 1738, Springer, Berlin, 2000.