Rational approximations of values of the Gamma function at rational points

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Computation of a given a real number α.

1) **Numerical** point of view: find a sequence of rational numbers $(\frac{u_n}{v_n})_{n\geq 0}$ such that

$$\lim_{n\to+\infty} \frac{u_n}{v_n} = \alpha$$

with fast convergence and u_n/v_n reasonably easy to compute.

2) **Diophantine** point of view: find two sequences of integers $(u_n)_{n\geq 0}$ and $(v_n)_{n\geq 0}$ such that

$$\lim_{n\to+\infty} (v_n\alpha - u_n) = 0.$$

If furthermore $\forall n \; v_n\alpha - u_n \neq 0$, then $\alpha \notin \mathbb{Q}$.

Aim of this talk: to present rational approximations of the numbers $\Gamma(a/b)$, where $a/b \in \mathbb{Q}$ and Γ denotes the usual Gamma function.

A similar method enables us to get rational approximations of the numbers $\gamma + \log(x)$, where γ is Euler's constant and $x > 0, x \in \mathbb{Q}$.

None of these approximations are good enough to satisfy 2).

From the point of view of 1), they are new. The rational approximations are solutions of a **linear recurrence of finite order with polynomial coefficients**.

$\forall x > 0$ and $\forall \alpha \in \mathbb{C}$, consider the linear recurrence of order 3:

$$C_3(n,\alpha,x)\mathbf{U}_{n+3}+C_2(n,\alpha,x)\mathbf{U}_{n+2}+C_1(n,\alpha,x)\mathbf{U}_{n+1}+C_0(n,\alpha,x)\mathbf{U}_n = 0 \quad (1)$$

where the coefficients $C_j(n,\alpha,x)$, $j = 0, 1, 2, 3$, are polynomials in n, α, x, of degree 16 in n, whose expressions are

$$C_3(n,\alpha,x) = -(n+3)^5(n+4)^2(8n^2+4\alpha n-3xn+38n-6x-\alpha x+10\alpha+44)$$
$$(n+2)(8n^2+22n+4\alpha n-3xn+6\alpha-3x-\alpha x+14)^2$$
$$(8n^2+38n+4\alpha n-3xn+10\alpha-6x-\alpha x+44)$$

$$C_2(n,\alpha,x) =$$
$$(24n^5+7xn^4+28\alpha n^4+330n^4-6x^2n^3+91xn^3+1794n^3+310\alpha n^3+7\alpha xn^3$$
$$+70x\alpha n^2+13x\alpha^2n^2-5x^2\alpha n^2-4\alpha^3n^2+4824n^2-6\alpha^2n^2-45x^2n^2+418xn^2$$
$$+1272\alpha n^2+576x-25x^2\alpha n+218\alpha xn+79x\alpha^2n-26\alpha^3n+5\alpha^3xn-4x^2\alpha^2n$$
$$+816xn+2296\alpha n+6420n-111x^2n-30\alpha^2n+3384+1540\alpha+576x+216\alpha x$$
$$-\alpha^3x^2+16\alpha^3x-10x^2\alpha^2-31x^2\alpha+116x\alpha^2-40\alpha^3-90x^2-36\alpha^2)$$
$$(n+3)^4(n+2)(-8n^2\alpha x-4\alpha n-38n+3xn6x-44-10\alpha)$$
$$(-8n^2-22n-4\alpha n+3xn-14-6\alpha+3x+\alpha x)^2$$

$$C_1(n,\alpha,x) =$$
$$-(24n^4-57xn^3+20\alpha n^3+186n^3-38x\alpha n^2+518n^2+4\alpha^2n^2+26x^2n^2-315xn^2$$
$$+120\alpha n^2+13x^2\alpha n-148\alpha xn-5x\alpha^2n-543xn+232\alpha n+610n+85x^2n+14\alpha^2n$$
$$-3x^3n+254+144\alpha-285x-138\alpha x-x^3\alpha+x^2\alpha^2+24x^2\alpha-9x\alpha^2+59x^2+10\alpha^2$$
$$-3x^3)(n+3)^2(8n^2+22n+4\alpha n-3xn+14+6\alpha-3x-\alpha x)$$
$$(8n^2+4\alpha n+54n-3xn-\alpha x+14\alpha-9x+90)(n-\alpha+2)(n+2+\alpha)^2$$
$$(8n^2-3xn+38\alpha xn+4\alpha n+10\alpha-6x+44)(n+2)$$

$$C_0(n,\alpha,x) = (n-\alpha+1)(n+1+\alpha)^2(8n^2-3xn+4\alpha n+38n\alpha x+10\alpha-6x+44)^2$$
$$(n+3)^2(8n^2+22n+4n\alpha-3xn+14+6\alpha-3x-\alpha x)(n-\alpha+2)(n+2+\alpha)^2$$
$$(8n^2-3xn+54n+4\alpha n+14\alpha-\alpha x+90-9x).$$

$(P_n(x,\alpha))_{n\geq 0}$ and $(Q_n(x,\alpha))_{n\geq 0}$ solutions of (1), with initial values:

$$P_0(\alpha,x) = x-\alpha-2,\ P_1(\alpha,x) = \frac{1}{4}\Big((1-\alpha)x^2+(6\alpha+2\alpha^2-4)x-\alpha^3-4-9\alpha-6\alpha^2\Big)$$

$$P_2(\alpha,x) = \frac{1}{36}\Big((-3\alpha+\alpha^2+2)x^3+(-3\alpha^3+30-9\alpha^2)x^2$$
$$+(-108+33\alpha^2+24\alpha^3+3\alpha^4-60\alpha)x-12\alpha^4-\alpha^5-24-90\alpha^2-49\alpha^3-76\alpha\Big)$$

$$Q_0(\alpha,x) = x-2, \quad Q_1(\alpha,x) = \frac{1}{4}\Big((1+\alpha^2+2\alpha)x^2+(-10\alpha-6\alpha^2-4)x-4+4\alpha^2\Big)$$

$$Q_2(\alpha,x) = \frac{1}{72}\Big((\alpha^4+12\alpha+13\alpha^2+4+6\alpha^3)x^3+(-60\alpha^3-12\alpha^4-96\alpha^2-48\alpha)x^2$$
$$+(-336\alpha-216+84\alpha^3+30\alpha^4-66\alpha^2)x+60\alpha^2-12\alpha^4-48\Big)$$

Theorem 1 (R, 2009). *(i)* $P_n(\alpha,x), Q_n(\alpha,x) \in \mathbb{Q}[\alpha,x]$.

(ii) $\forall \alpha = u/v \in \mathbb{Q}$ *and* $\forall x = a/b \in \mathbb{Q}$,

$$n!^2(n+1)!^2 v^{2n+1} b^{4n-1} P_n(\alpha,x) \in \mathbb{Z}, \quad n!^2(n+1)!^2 v^{3n+2} b^{4n-1} Q_n(\alpha,x) \in \mathbb{Z}.$$

(iii) $\forall x > 0$ *and* $\forall \alpha \in \mathbb{C}\setminus\{0\}$, $\mathrm{Re}(\alpha) > -1$, $\exists\, s(\alpha,x) \neq 0$ *and* $q(\alpha,x) \neq 0$ *such that*

$$\Big|Q_n(\alpha,x)\Gamma(\alpha+1) - P_n(\alpha,x)x^\alpha\Big| \leq \frac{s(\alpha,x)}{n^{2-2\mathrm{Re}(\alpha)/3}} \exp\Big(-\frac{3}{2}x^{1/3}n^{2/3} + \frac{1}{2}x^{2/3}n^{1/3}\Big)$$

and

$$Q_n(\alpha,x) \sim \frac{q(\alpha,x)}{n^{2-2\alpha/3}} \exp\Big(3x^{1/3}n^{2/3} - x^{2/3}n^{1/3}\Big).$$

Example. Define $(p_n)_{n\geq 0}$ and $(q_n)_{n\geq 0}$ to be the solutions of

$$64(8n+17)(n+4)^2(8n+9)(n+3)^3(n+2)\mathbf{U}_{n+3}$$

$$-16(8n+9)(n+2)(24n^2+123n+155)(2n+7)^2(n+3)^2\mathbf{U}_{n+2}$$

$$+4(2n+7)(n+2)(8n+25)(48n^3+158n^2+147n+32)(2n+5)^2\mathbf{U}_{n+1}$$

$$-(8n+25)(8n+17)(2n+1)(2n+7)(2n+5)^2(2n+3)^2\mathbf{U}_n=0$$

with $p_0=\frac{3}{2}$, $p_1=\frac{81}{32}$, $p_2=\frac{2185}{384}$, $q_0=1$, $q_1=\frac{45}{16}$, $q_2=\frac{825}{128}$. Then

$$\lim_{n\to+\infty}\frac{p_n}{q_n}=\Gamma\left(\frac{3}{2}\right)=\frac{\sqrt{\pi}}{2}.$$

Idea of the proof.

• Euler's functions: For $z \notin [0, +\infty)$ and $\mathrm{Re}(\beta) > -1$, set

$$\mathscr{F}_\beta(z) := \int_0^\infty \frac{t^\beta e^{-t}}{z-t}\,\mathrm{d}t \sim \sum_{k=1}^\infty \frac{\Gamma(\beta+k)}{z^k}.$$

• Laguerre type polynomials of degree $2n$ in x:

$$A_{n,\alpha}(x) := \frac{1}{n!^2}\,e^x\Big(x^{n-\alpha}(e^{-x}x^{n+\alpha})^{(n)}\Big)^{(n)} \in \mathbb{Q}[\alpha, x].$$

They are orthogonal on $(0, +\infty)$ for the **two** weights e^{-x} and $x^\alpha e^{-x}$.
$(\alpha \neq 0)$

Simultaneous type II Padé approximants at $z=\infty$ to $\mathscr{F}_0(z)$ and $\mathscr{F}_\alpha(z)$.

Lemma 1. *For* $\beta\in\{0,\alpha\}$, $\alpha\in\mathbb{C}\setminus\{0\}$, $\mathrm{Re}(\alpha)>-1$ *and* $z\in\mathbb{C}\setminus[0,+\infty)$,

$$R_{n,\alpha,\beta}(z):=\int_0^\infty\frac{A_{n,\alpha}(t)}{z-t}\,t^\beta e^{-t}\,\mathrm{d}t=A_{n,\alpha}(z)\mathscr{F}_\beta(z)-B_{n,\alpha,\beta}(z)$$

$$\sim\sum_{k=1}^\infty\frac{(k-\beta-n)_n(k-\alpha+\beta-n)_n}{n!^2}\cdot\frac{\Gamma(\beta+k)}{z^k}=\mathcal{O}\Big(\frac{1}{z^{n+1}}\Big).$$

Here,

$$B_{n,\alpha,\beta}(z):=\int_0^\infty\frac{A_{n,\alpha}(z)-A_{n,\alpha}(t)}{z-t}\,t^\beta e^{-t}\,\mathrm{d}t\in\Gamma(1+\beta)\mathbb{Q}[\alpha,z]$$

is of degree at most $2n-1$ in z.

Crucial fact: $B_{n,\alpha,0}(z) \in \mathbb{Q}[\alpha, z]$ and $B_{n,\alpha,\alpha}(z) = \Gamma(1+\alpha)C_{n,\alpha}(z)$ for some $C_{n,\alpha}(z) \in \mathbb{Q}[\alpha, z]$.

For $\beta \in \mathbb{C} \setminus \{0\}$, take z^β to be given by its principal value whenever $-\pi < \arg(z) < \pi$.

Set

$$\mathscr{F}(z) := z^\alpha \mathscr{F}_0(z) - \mathscr{F}_\alpha(z) = \int_0^\infty \frac{z^\alpha - t^\alpha}{z - t} e^{-t} \, dt,$$

$$R_{n,\alpha}(z) := z^\alpha R_{n,\alpha,0}(z) - R_{n,\alpha,\alpha}(z) = \int_0^\infty \frac{z^\alpha - t^\alpha}{z - t} A_{n,\alpha}(t) e^{-t} \, dt,$$

which are both analytic on $\mathbb{C} \setminus (-\infty, 0]$.

Obviously,

$$R_{n,\alpha}(z) = A_{n,\alpha}(z)\mathscr{F}(z) + \Gamma(1+\alpha)C_{n,\alpha}(z) - z^{\alpha}B_{n,\alpha,0}(z).$$

To "remove" the term $A_{n,\alpha}(z)\mathscr{F}(z)$, set

$$P_n(\alpha, z) = \begin{vmatrix} A_{n,\alpha}(z) & B_{n,\alpha,0}(z) \\ A_{n+1,\alpha}(z) & B_{n+1,\alpha,0}(z) \end{vmatrix} \in \mathbb{Q}[\alpha, z]$$

$$Q_n(\alpha, z) = \begin{vmatrix} A_{n,\alpha}(z) & C_{n,\alpha}(z) \\ A_{n+1,\alpha}(z) & C_{n+1,\alpha}(z) \end{vmatrix} \in \mathbb{Q}[\alpha, z],$$

$$S_n(\alpha, z) = \begin{vmatrix} A_{n,\alpha}(z) & R_{n,\alpha}(z) \\ A_{n+1,\alpha}(z) & R_{n+1,\alpha}(z) \end{vmatrix}.$$

Lemma 2. (i) *For any* $z \in \mathbb{C} \setminus (-\infty, 0]$*, we have*

$$S_n(\alpha, z) = Q_n(\alpha, z)\Gamma(1+\alpha) - z^{\alpha} P_n(\alpha, z)$$

where

$$n!^2(n+1)!^2 P_n(\alpha, z) \in \mathbb{Z}[\alpha, z], \qquad n!^2(n+1)!^2 Q_n(\alpha, z) \in \mathbb{Z}[\alpha, z].$$

(ii) $P_n(\alpha, z)$*,* $Q_n(\alpha, z)$ *and* $S_n(\alpha, z)$ *are solutions of the linear recurrence* (1)*.*

(iii) *The degrees of* Q_n *and* P_n *in* z *are at most* $4n-1$*, those in* α *at most* $3n+2$ *and* $2n+1$ *respectively.*

(i) and (iii) are "easy". (ii) is consequence of the facts

– that the sequences $A_{n,\alpha}$, $B_{n,\alpha,0}$, $B_{n,\alpha,\alpha}$, $R_{n,\alpha,0}$, $R_{n,\alpha,\alpha}$ (hence $R_{n,\alpha}$) are all solutions of the same linear recurrence of order 3, say $\mathcal{R}$, obtained explicitly using Zeilberger's algorithm EKHAD.

– that if a_n and b_n are solutions of a recurrence of order 3:

$$U_{n+3} = p_n U_{n+2} + q_n U_{n+1} + r_n U_n,$$

then the sequence of determinants $a_n b_{n+1} - a_{n+1} b_n$ is solution of the linear recurrence also of order 3:

$$U_{n+3} = -q_{n+1} U_{n+2} - p_n r_{n+1} U_{n+1} + r_n r_{n+1} U_n.$$

Final steps.

Lemma 3. *(i)* $\forall x > 0$, *the modulus of* $A_{n,\alpha}$, $B_{n,\alpha,0}$, $B_{n,\alpha,\alpha}$ *are bounded by*

$$\frac{u(x,\alpha)}{n^{1-\mathrm{Re}(\alpha)/3}} \exp(3/2 \cdot x^{1/3} n^{2/3} - 1/2 \cdot x^{2/3} n^{1/3}),$$

where $u(x,\alpha)$ *depends on the sequence.*

(ii) $\forall x > 0$, $\exists r(x,\alpha) \neq 0$ *s.t.*

$$R_{n,\alpha}(x) \sim \frac{r(x,\alpha)}{n^{1-\alpha/3}} \exp(-3x^{1/3} n^{2/3} + x^{2/3} n^{1/3}).$$

(i) follows from Birkhoff-Trjitzinsky theory applied to the recurrence $\mathcal{R}$.

(ii) is a consequence of the identity

$$R_{n,\alpha}(x) = \alpha x^{\alpha} \frac{(1-\alpha)_n(1+\alpha)_n}{n!^2} \int_0^{\infty}\int_0^{\infty} \frac{u^n t^{n+\alpha} e^{-t}}{(1+u)^{n+1-\alpha}(x+ut)^{n+1+\alpha}} \mathrm{d}t\mathrm{d}u$$

which implies that $R_{n,\alpha}(x) \to 0$ when $n \to 0$, and then we apply Birkhoff-Trjitzinsky theory again to $\mathcal{R}$.

Lemma 3 (both (i) and (ii)) implies that $S_n(\alpha, x) \to 0$ when $n \to +\infty$.

Apply again Birkhoff-Trjitzinsky theory to recurrence (1) to get the bounds in Theorem 1, ie., $\forall x > 0$,

$$\left| Q_n(\alpha, x)\Gamma(\alpha+1) - P_n(\alpha, x)x^\alpha \right| \leq \frac{s(\alpha, x)}{n^{2-2\mathrm{Re}(\alpha)/3}} \exp\left(-\frac{3}{2}x^{1/3}n^{2/3} + \frac{1}{2}x^{2/3}n^{1/3}\right),$$

$$Q_n(\alpha, x) \sim \frac{q(\alpha, x)}{n^{2-2\alpha/3}} \exp\left(3x^{1/3}n^{2/3} - x^{2/3}n^{1/3}\right).$$

Results related to Euler's constant γ.

Theorem 2 (R, 2008, Transactions AMS). *There exists two polynomial solutions $P_n(z)$ and $Q_n(z)$ of a linear recurrence of order* 3, *of degree at most $n+1$ such that*

(i) $n!^2 P_n(z)$ *and* $n!^2 Q_n(z)$ *belong to* $\mathbb{Z}[z]$.

(ii) $\forall x > 0$, $\exists s(x) \neq 0$ *and* $q(x) \neq 0$ *s.t.*

$$|Q_n(x)\Big(\ln(x) + \gamma\Big) - P_n(x)| \leq \frac{s(x)}{n^2} \exp(-\frac{3}{2}x^{1/3}n^{2/3} + \frac{1}{2}x^{2/3}n^{1/3}),$$

$$Q_n(x) \sim \frac{q(x)}{n^2} \exp(3x^{1/3}n^{2/3} - x^{2/3}n^{1/3}).$$

Examples

The recurrence

$$(n+3)^2(8n+11)(8n+19)U_{n+3}$$

$$=(24n^2+145n+215)(8n+11)U_{n+2}$$

$$-(24n^3+105n^2+124n+25)(8n+27)U_{n+1}$$

$$+(n+2)^2(8n+19)(8n+27)U_n$$

provides two sequences of rational numbers $(p_n)_{n\geq 0}$ and $(q_n)_{n\geq 0}$ with $p_0=-1$, $p_1=4$, $p_2=77/4$ and $q_0=1, q_1=7, q_2=65/2$ such that

$$\frac{p_n}{q_n}\to\gamma.$$

The first such recurrence for γ was obtained by Aptekarev in 2007.

The recurrence

$$(n+1)(n+2)(n+3)U_{n+3} = (3n^2 + 19n + 29)(n+1)U_{n+2}$$

$$- (3n^3 + 6n^2 - 7n - 13)U_{n+1} + (n+2)^3 U_n$$

provides two sequences of rational numbers $(p_n)_{n\geq 0}$ and $(q_n)_{n\geq 0}$ with $p_0 = -1$, $p_1 = 11$, $p_2 = 71$ and $q_0 = 0, q_1 = 8, q_2 = 56$ such that

$$\frac{p_n}{q_n} \to \ln(2) + \gamma.$$

The construction is based on

- Euler type functions

$$\int_0^\infty \frac{\log(t)^s e^{-t}}{z-t}\,\mathrm{d}t \sim \sum_{k=1}^{\infty} \frac{\Gamma^{(s)}(k)}{z^k}.$$

- Laguerre type polynomials with $\alpha = 0$:

$$A_{n,0}(x) = \frac{1}{n!^2}\, e^x \Big(x^n (e^{-x} x^n)^{(n)}\Big)^{(n)} \in \mathbb{Q}[x],$$

orthogonal on $(0, +\infty)$ for the weigths e^{-x} and $\log(x)e^{-x}$. (**Remember that the case $\alpha = 0$ was excluded.**)

This construction can be viewed as a limiting case of our results for $\Gamma(1+\alpha)$ because

$$\lim_{\alpha\to 0} \frac{\Gamma(1+\alpha) - 1}{\alpha} = \Gamma'(1) = -\gamma.$$

How to construct good simultaneous approximations for

$$\Gamma(\alpha) \quad \text{and} \quad \int_0^\infty (t+1)^{\alpha-1} e^{-t} \mathrm{d}t,$$

and for

$$\gamma \quad \text{and} \quad \int_0^\infty \frac{e^{-t}}{1+t} \mathrm{d}t?$$

Set

$$G_\alpha(z) = z^{-\alpha}\int_0^\infty (t+z)^{\alpha-1}e^{-t}\mathrm{d}t, \qquad E_\alpha(z) = \sum_{m=0}^{\infty} \frac{z^m}{m!(m+\alpha+1)}$$

and

$$E(z) = \sum_{m=1}^{\infty} \frac{z^m}{m!m}.$$

We have the identities

$$\frac{\Gamma(\alpha)}{z^\alpha} = E_{\alpha-1}(-z) + e^{-z}G_\alpha(z)$$

$$\gamma + \log(z) = E(-z) - e^{-z}(G_0(z) - \log(z)).$$

Construction of Hermite-Padé approximants.

Proposition 1. $\Re(\alpha) > -1$ *and* $\alpha \notin \mathbb{Z}$. $\exists$ *polynomials* A_n, C_n *(degree* $\leq n$*) and* B_n *(degree* $\leq n+1$*) s.t.*

$$R_{n,\alpha}(z) := \frac{z^{3n+1}}{n!^2} \int_0^1 \int_0^1 e^{zuv} u^{2n+\alpha}(1-u)^n v^{2n}(1-v)^n \mathrm{d}u \mathrm{d}v$$

$$= A_n(z)e^z + B_n(z)E_\alpha(z) + C_n(z).$$

Order at $z = 0$ *of* $R_n(z)$ *is* $3n+1$.

A similar proposition holds when $\alpha = -1$ with $E(z)$ instead of $E_\alpha(z)$.

We deduce that for any integers p, q, r not all zero and any $\varepsilon > 0$, we have

$$\left|p + qe^{-z} + rE_{\alpha-1}(-z)\right| \geq \frac{c_1}{H^{2+\varepsilon}},$$

where $H = \max(|p|, |q|, |r|)$ and c_1 depends on α, ε, z.

Using the relation between $\Gamma(\alpha)$, $G_\alpha(z)$, $E_{\alpha-1}(z)$ and e^{-z}, it follows for any integers p, q, r not all zero and any $\varepsilon > 0$,

$$\left|\frac{\Gamma(\alpha)}{z^\alpha} - \frac{p}{q}\right| + \left|G_\alpha(z) - \frac{r}{q}\right| \geq \frac{c_2(\alpha, \varepsilon, z)}{H^{3+\varepsilon}},$$

Similarly, we get

$$\left|\gamma + \log(z) - \frac{p}{q}\right| + \left|G_0(z) - \log(z) - \frac{r}{q}\right| \geq \frac{d(\varepsilon, z)}{H^{3+\varepsilon}}.$$

Theorem 3. • *For any* $\alpha \in \mathbb{Q}$, $\alpha > 0$, $\alpha \notin \mathbb{Z}$, *at least one of*

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} \mathrm{d}t$$

and

$$G_\alpha(1) = \int_0^\infty (t+1)^{\alpha-1} e^{-t} \mathrm{d}t$$

is irrational.

• *At least one of*

$$\gamma = -\int_0^\infty \log(t) e^{-t} \mathrm{d}t$$

and

$$G_0(1) = \int_0^\infty \frac{e^{-t}}{1+t} \mathrm{d}t = \int_0^\infty \log(t+1) e^{-t} \mathrm{d}t$$

is irrational.