

# Rational approximations of values of the Gamma function at rational points

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Computation of a given a real number  $\alpha$ .

1) **Numerical** point of view: find a sequence of rational numbers  $(\frac{u_n}{v_n})_{n \geq 0}$  such that

$$\lim_{n \rightarrow +\infty} \frac{u_n}{v_n} = \alpha$$

with fast convergence and  $u_n/v_n$  reasonably easy to compute.

2) **Diophantine** point of view: find two sequences of integers  $(u_n)_{n \geq 0}$  and  $(v_n)_{n \geq 0}$  such that

$$\lim_{n \rightarrow +\infty} (v_n \alpha - u_n) = 0.$$

If furthermore  $\forall n \ v_n \alpha - u_n \neq 0$ , then  $\alpha \notin \mathbb{Q}$ .

**Aim of this talk:** to present rational approximations of the numbers  $\Gamma(a/b)$ , where  $a/b \in \mathbb{Q}$  and  $\Gamma$  denotes the usual Gamma function.

A similar method enables us to get rational approximations of the numbers  $\gamma + \log(x)$ , where  $\gamma$  is Euler's constant and  $x > 0, x \in \mathbb{Q}$ .

None of these approximations are good enough to satisfy 2).

From the point of view of 1), they are new. The rational approximations are solutions of a **linear recurrence of finite order with polynomial coefficients**.

$\forall x > 0$  and  $\forall \alpha \in \mathbb{C}$ , consider the linear recurrence of order 3:

$$C_3(n, \alpha, x)U_{n+3} + C_2(n, \alpha, x)U_{n+2} + C_1(n, \alpha, x)U_{n+1} + C_0(n, \alpha, x)U_n = 0 \quad (1)$$

where the coefficients  $C_j(n, \alpha, x)$ ,  $j = 0, 1, 2, 3$ , are polynomials in  $n, \alpha, x$ , of degree 16 in  $n$ , whose expressions are

$$C_3(n, \alpha, x) = -(n+3)^5(n+4)^2(8n^2+4\alpha n-3xn+38n-6x-\alpha x+10\alpha+44) \\ (n+2)(8n^2+22n+4\alpha n-3xn+6\alpha-3x-\alpha x+14)^2 \\ (8n^2+38n+4\alpha n-3xn+10\alpha-6x-\alpha x+44)$$

$$C_2(n, \alpha, x) = \\ (24n^5+7xn^4+28\alpha n^4+330n^4-6x^2n^3+91xn^3+1794n^3+310\alpha n^3+7\alpha xn^3 \\ +70x\alpha n^2+13x\alpha^2n^2-5x^2\alpha n^2-4\alpha^3n^2+4824n^2-6\alpha^2n^2-45x^2n^2+418xn^2 \\ +1272\alpha n^2+576x-25x^2\alpha n+218\alpha xn+79x\alpha^2n-26\alpha^3n+5\alpha^3xn-4x^2\alpha^2n \\ +816xn+2296\alpha n+6420n-111x^2n-30\alpha^2n+3384+1540\alpha+576x+216\alpha x \\ -\alpha^3x^2+16\alpha^3x-10x^2\alpha^2-31x^2\alpha+116x\alpha^2-40\alpha^3-90x^2-36\alpha^2) \\ (n+3)^4(n+2)(-8n^2\alpha x-4\alpha n-38n+3xn6x-44-10\alpha) \\ (-8n^2-22n-4\alpha n+3xn-14-6\alpha+3x+\alpha x)^2$$

$$\begin{aligned}
C_1(n, \alpha, x) = & \\
& -(24n^4 - 57xn^3 + 20\alpha n^3 + 186n^3 - 38x\alpha n^2 + 518n^2 + 4\alpha^2 n^2 + 26x^2 n^2 - 315xn^2 \\
& + 120\alpha n^2 + 13x^2 \alpha n - 148\alpha xn - 5x\alpha^2 n - 543xn + 232\alpha n + 610n + 85x^2 n + 14\alpha^2 n \\
& - 3x^3 n + 254 + 144\alpha - 285x - 138\alpha x - x^3 \alpha + x^2 \alpha^2 + 24x^2 \alpha - 9x\alpha^2 + 59x^2 + 10\alpha^2 \\
& - 3x^3)(n + 3)^2(8n^2 + 22n + 4\alpha n - 3xn + 14 + 6\alpha - 3x - \alpha x) \\
& (8n^2 + 4\alpha n + 54n - 3xn - \alpha x + 14\alpha - 9x + 90)(n - \alpha + 2)(n + 2 + \alpha)^2 \\
& (8n^2 - 3xn + 38\alpha xn + 4\alpha n + 10\alpha - 6x + 44)(n + 2)
\end{aligned}$$

$$\begin{aligned}
C_0(n, \alpha, x) = & (n - \alpha + 1)(n + 1 + \alpha)^2(8n^2 - 3xn + 4\alpha n + 38n\alpha x + 10\alpha - 6x + 44)^2 \\
& (n + 3)^2(8n^2 + 22n + 4n\alpha - 3xn + 14 + 6\alpha - 3x - \alpha x)(n - \alpha + 2)(n + 2 + \alpha)^2 \\
& (8n^2 - 3xn + 54n + 4\alpha n + 14\alpha - \alpha x + 90 - 9x).
\end{aligned}$$

$(P_n(x, \alpha))_{n \geq 0}$  and  $(Q_n(x, \alpha))_{n \geq 0}$  solutions of (1), with initial values:

$$P_0(\alpha, x) = x - \alpha - 2, \quad P_1(\alpha, x) = \frac{1}{4} \left( (1 - \alpha)x^2 + (6\alpha + 2\alpha^2 - 4)x - \alpha^3 - 4 - 9\alpha - 6\alpha^2 \right)$$

$$P_2(\alpha, x) = \frac{1}{36} \left( (-3\alpha + \alpha^2 + 2)x^3 + (-3\alpha^3 + 30 - 9\alpha^2)x^2 \right. \\ \left. + (-108 + 33\alpha^2 + 24\alpha^3 + 3\alpha^4 - 60\alpha)x - 12\alpha^4 - \alpha^5 - 24 - 90\alpha^2 - 49\alpha^3 - 76\alpha \right)$$

$$Q_0(\alpha, x) = x - 2, \quad Q_1(\alpha, x) = \frac{1}{4} \left( (1 + \alpha^2 + 2\alpha)x^2 + (-10\alpha - 6\alpha^2 - 4)x - 4 + 4\alpha^2 \right)$$

$$Q_2(\alpha, x) = \frac{1}{72} \left( (\alpha^4 + 12\alpha + 13\alpha^2 + 4 + 6\alpha^3)x^3 + (-60\alpha^3 - 12\alpha^4 - 96\alpha^2 - 48\alpha)x^2 \right. \\ \left. + (-336\alpha - 216 + 84\alpha^3 + 30\alpha^4 - 66\alpha^2)x + 60\alpha^2 - 12\alpha^4 - 48 \right)$$

**Theorem 1** (R, 2009). (i)  $P_n(\alpha, x), Q_n(\alpha, x) \in \mathbb{Q}[\alpha, x]$ .

(ii)  $\forall \alpha = u/v \in \mathbb{Q}$  and  $\forall x = a/b \in \mathbb{Q}$ ,

$$n!^2(n+1)!^2v^{2n+1}b^{4n-1}P_n(\alpha, x) \in \mathbb{Z}, \quad n!^2(n+1)!^2v^{3n+2}b^{4n-1}Q_n(\alpha, x) \in \mathbb{Z}.$$

(iii)  $\forall x > 0$  and  $\forall \alpha \in \mathbb{C} \setminus \{0\}$ ,  $\text{Re}(\alpha) > -1$ ,  $\exists s(\alpha, x) \neq 0$  and  $q(\alpha, x) \neq 0$  such that

$$\left| Q_n(\alpha, x)\Gamma(\alpha+1) - P_n(\alpha, x)x^\alpha \right| \leq \frac{s(\alpha, x)}{n^{2-2\text{Re}(\alpha)/3}} \exp\left(-\frac{3}{2}x^{1/3}n^{2/3} + \frac{1}{2}x^{2/3}n^{1/3}\right)$$

and

$$Q_n(\alpha, x) \sim \frac{q(\alpha, x)}{n^{2-2\alpha/3}} \exp\left(3x^{1/3}n^{2/3} - x^{2/3}n^{1/3}\right).$$



**Example.** Define  $(p_n)_{n \geq 0}$  and  $(q_n)_{n \geq 0}$  to be the solutions of

$$\begin{aligned}
& 64(8n + 17)(n + 4)^2(8n + 9)(n + 3)^3(n + 2)U_{n+3} \\
& - 16(8n + 9)(n + 2)(24n^2 + 123n + 155)(2n + 7)^2(n + 3)^2U_{n+2} \\
& + 4(2n + 7)(n + 2)(8n + 25)(48n^3 + 158n^2 + 147n + 32)(2n + 5)^2U_{n+1} \\
& - (8n + 25)(8n + 17)(2n + 1)(2n + 7)(2n + 5)^2(2n + 3)^2U_n = 0
\end{aligned}$$

with  $p_0 = \frac{3}{2}$ ,  $p_1 = \frac{81}{32}$ ,  $p_2 = \frac{2185}{384}$ ,  $q_0 = 1$ ,  $q_1 = \frac{45}{16}$ ,  $q_2 = \frac{825}{128}$ . Then

$$\lim_{n \rightarrow +\infty} \frac{p_n}{q_n} = \Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}.$$

## Idea of the proof.

- Euler's functions: For  $z \notin [0, +\infty)$  and  $\operatorname{Re}(\beta) > -1$ , set

$$\mathcal{F}_\beta(z) := \int_0^\infty \frac{t^\beta e^{-t}}{z-t} dt \sim \sum_{k=1}^\infty \frac{\Gamma(\beta+k)}{z^k}.$$

- Laguerre type polynomials of degree  $2n$  in  $x$ :

$$A_{n,\alpha}(x) := \frac{1}{n!2} e^x \left( x^{n-\alpha} (e^{-x} x^{n+\alpha})^{(n)} \right)^{(n)} \in \mathbb{Q}[\alpha, x].$$

They are orthogonal on  $(0, +\infty)$  for the **two** weights  $e^{-x}$  and  $x^\alpha e^{-x}$ .  
( $\alpha \neq 0$ )

**Simultaneous type II Padé approximants at  $z = \infty$  to  $\mathcal{F}_0(z)$  and  $\mathcal{F}_\alpha(z)$ .**

**Lemma 1.** For  $\beta \in \{0, \alpha\}$ ,  $\alpha \in \mathbb{C} \setminus \{0\}$ ,  $\operatorname{Re}(\alpha) > -1$  and  $z \in \mathbb{C} \setminus [0, +\infty)$ ,

$$R_{n,\alpha,\beta}(z) := \int_0^\infty \frac{A_{n,\alpha}(t)}{z-t} t^\beta e^{-t} dt = A_{n,\alpha}(z) \mathcal{F}_\beta(z) - B_{n,\alpha,\beta}(z)$$

$$\sim \sum_{k=1}^{\infty} \frac{(k-\beta-n)_n (k-\alpha+\beta-n)_n}{n!^2} \cdot \frac{\Gamma(\beta+k)}{z^k} = \mathcal{O}\left(\frac{1}{z^{n+1}}\right).$$

Here,

$$B_{n,\alpha,\beta}(z) := \int_0^\infty \frac{A_{n,\alpha}(z) - A_{n,\alpha}(t)}{z-t} t^\beta e^{-t} dt \in \Gamma(1+\beta) \mathbb{Q}[\alpha, z]$$

is of degree at most  $2n-1$  in  $z$ .

**Crucial fact:**  $B_{n,\alpha,0}(z) \in \mathbb{Q}[\alpha, z]$  and  $B_{n,\alpha,\alpha}(z) = \Gamma(1 + \alpha)C_{n,\alpha}(z)$  for some  $C_{n,\alpha}(z) \in \mathbb{Q}[\alpha, z]$ .

For  $\beta \in \mathbb{C} \setminus \{0\}$ , take  $z^\beta$  to be given by its principal value whenever  $-\pi < \arg(z) < \pi$ .

Set

$$\mathcal{F}(z) := z^\alpha \mathcal{F}_0(z) - \mathcal{F}_\alpha(z) = \int_0^\infty \frac{z^\alpha - t^\alpha}{z - t} e^{-t} dt,$$

$$R_{n,\alpha}(z) := z^\alpha R_{n,\alpha,0}(z) - R_{n,\alpha,\alpha}(z) = \int_0^\infty \frac{z^\alpha - t^\alpha}{z - t} A_{n,\alpha}(t) e^{-t} dt,$$

which are both analytic on  $\mathbb{C} \setminus (-\infty, 0]$ .

Obviously,

$$R_{n,\alpha}(z) = A_{n,\alpha}(z)\mathcal{F}(z) + \Gamma(1 + \alpha)C_{n,\alpha}(z) - z^\alpha B_{n,\alpha,0}(z).$$

To “remove” the term  $A_{n,\alpha}(z)\mathcal{F}(z)$ , set

$$P_n(\alpha, z) = \begin{vmatrix} A_{n,\alpha}(z) & B_{n,\alpha,0}(z) \\ A_{n+1,\alpha}(z) & B_{n+1,\alpha,0}(z) \end{vmatrix} \in \mathbb{Q}[\alpha, z]$$

$$Q_n(\alpha, z) = \begin{vmatrix} A_{n,\alpha}(z) & C_{n,\alpha}(z) \\ A_{n+1,\alpha}(z) & C_{n+1,\alpha}(z) \end{vmatrix} \in \mathbb{Q}[\alpha, z],$$

$$S_n(\alpha, z) = \begin{vmatrix} A_{n,\alpha}(z) & R_{n,\alpha}(z) \\ A_{n+1,\alpha}(z) & R_{n+1,\alpha}(z) \end{vmatrix}.$$

**Lemma 2.** (i) For any  $z \in \mathbb{C} \setminus (-\infty, 0]$ , we have

$$S_n(\alpha, z) = Q_n(\alpha, z)\Gamma(1 + \alpha) - z^\alpha P_n(\alpha, z)$$

where

$$n!^2(n + 1)!^2 P_n(\alpha, z) \in \mathbb{Z}[\alpha, z], \quad n!^2(n + 1)!^2 Q_n(\alpha, z) \in \mathbb{Z}[\alpha, z].$$

(ii)  $P_n(\alpha, z)$ ,  $Q_n(\alpha, z)$  and  $S_n(\alpha, z)$  are solutions of the linear recurrence (1).

(iii) The degrees of  $Q_n$  and  $P_n$  in  $z$  are at most  $4n - 1$ , those in  $\alpha$  at most  $3n + 2$  and  $2n + 1$  respectively.

(i) and (iii) are “easy”. (ii) is consequence of the facts

– that the sequences  $A_{n,\alpha}$ ,  $B_{n,\alpha,0}$ ,  $B_{n,\alpha,\alpha}$ ,  $R_{n,\alpha,0}$ ,  $R_{n,\alpha,\alpha}$  (hence  $R_{n,\alpha}$ ) are all solutions of the same linear recurrence of order 3, say  $\mathcal{R}$ , obtained explicitly using Zeilberger’s algorithm EKHAD.

– that if  $a_n$  and  $b_n$  are solutions of a recurrence of order 3:

$$U_{n+3} = p_n U_{n+2} + q_n U_{n+1} + r_n U_n,$$

then the sequence of determinants  $a_n b_{n+1} - a_{n+1} b_n$  is solution of the linear recurrence also of order 3:

$$U_{n+3} = -q_{n+1} U_{n+2} - p_n r_{n+1} U_{n+1} + r_n r_{n+1} U_n.$$

## Final steps.

**Lemma 3.** (i)  $\forall x > 0$ , the modulus of  $A_{n,\alpha}$ ,  $B_{n,\alpha,0}$ ,  $B_{n,\alpha,\alpha}$  are bounded by

$$\frac{u(x, \alpha)}{n^{1-\operatorname{Re}(\alpha)/3}} \exp(3/2 \cdot x^{1/3} n^{2/3} - 1/2 \cdot x^{2/3} n^{1/3}),$$

where  $u(x, \alpha)$  depends on the sequence.

(ii)  $\forall x > 0$ ,  $\exists r(x, \alpha) \neq 0$  s.t.

$$R_{n,\alpha}(x) \sim \frac{r(x, \alpha)}{n^{1-\alpha/3}} \exp(-3x^{1/3} n^{2/3} + x^{2/3} n^{1/3}).$$



(i) follows from Birkhoff-Trjitzinsky theory applied to the recurrence  $\mathcal{R}$ .

(ii) is a consequence of the identity

$$R_{n,\alpha}(x) = \alpha x^\alpha \frac{(1-\alpha)_n (1+\alpha)_n}{n!^2} \int_0^\infty \int_0^\infty \frac{u^n t^{n+\alpha} e^{-t}}{(1+u)^{n+1-\alpha} (x+ut)^{n+1+\alpha}} dt du$$

which implies that  $R_{n,\alpha}(x) \rightarrow 0$  when  $n \rightarrow 0$ , and then we apply Birkhoff-Trjitzinsky theory again to  $\mathcal{R}$ .

Lemma 3 (both (i) and (ii)) implies that  $S_n(\alpha, x) \rightarrow 0$  when  $n \rightarrow +\infty$ .

Apply again Birkhoff-Trjitzinsky theory to recurrence (1) to get the bounds in Theorem 1, ie.,  $\forall x > 0$ ,

$$\left| Q_n(\alpha, x)\Gamma(\alpha+1) - P_n(\alpha, x)x^\alpha \right| \leq \frac{s(\alpha, x)}{n^{2-2\text{Re}(\alpha)/3}} \exp\left(-\frac{3}{2}x^{1/3}n^{2/3} + \frac{1}{2}x^{2/3}n^{1/3}\right),$$

$$Q_n(\alpha, x) \sim \frac{q(\alpha, x)}{n^{2-2\alpha/3}} \exp\left(3x^{1/3}n^{2/3} - x^{2/3}n^{1/3}\right).$$

## Results related to Euler's constant $\gamma$ .

**Theorem 2** (R, 2008, Transactions AMS). *There exists two polynomial solutions  $P_n(z)$  and  $Q_n(z)$  of a linear recurrence of order 3, of degree at most  $n + 1$  such that*

(i)  $n!^2 P_n(z)$  and  $n!^2 Q_n(z)$  belong to  $\mathbb{Z}[z]$ .

(ii)  $\forall x > 0, \exists s(x) \neq 0$  and  $q(x) \neq 0$  s.t.

$$|Q_n(x)(\ln(x) + \gamma) - P_n(x)| \leq \frac{s(x)}{n^2} \exp\left(-\frac{3}{2}x^{1/3}n^{2/3} + \frac{1}{2}x^{2/3}n^{1/3}\right),$$

$$Q_n(x) \sim \frac{q(x)}{n^2} \exp(3x^{1/3}n^{2/3} - x^{2/3}n^{1/3}).$$

## Examples

The recurrence

$$\begin{aligned} & (n + 3)^2(8n + 11)(8n + 19)U_{n+3} \\ &= (24n^2 + 145n + 215)(8n + 11)U_{n+2} \\ & \quad - (24n^3 + 105n^2 + 124n + 25)(8n + 27)U_{n+1} \\ & \quad \quad \quad + (n + 2)^2(8n + 19)(8n + 27)U_n \end{aligned}$$

provides two sequences of rational numbers  $(p_n)_{n \geq 0}$  and  $(q_n)_{n \geq 0}$  with  $p_0 = -1$ ,  $p_1 = 4$ ,  $p_2 = 77/4$  and  $q_0 = 1$ ,  $q_1 = 7$ ,  $q_2 = 65/2$  such that

$$\frac{p_n}{q_n} \rightarrow \gamma.$$

The first such recurrence for  $\gamma$  was obtained by Aptekarev in 2007.

The recurrence

$$(n + 1)(n + 2)(n + 3)U_{n+3} = (3n^2 + 19n + 29)(n + 1)U_{n+2} \\ - (3n^3 + 6n^2 - 7n - 13)U_{n+1} + (n + 2)^3U_n$$

provides two sequences of rational numbers  $(p_n)_{n \geq 0}$  and  $(q_n)_{n \geq 0}$  with  $p_0 = -1, p_1 = 11, p_2 = 71$  and  $q_0 = 0, q_1 = 8, q_2 = 56$  such that

$$\frac{p_n}{q_n} \rightarrow \ln(2) + \gamma.$$

The construction is based on

- Euler type functions

$$\int_0^\infty \frac{\log(t)^s e^{-t}}{z-t} dt \sim \sum_{k=1}^\infty \frac{\Gamma^{(s)}(k)}{z^k}.$$

- Laguerre type polynomials with  $\alpha = 0$ :

$$A_{n,0}(x) = \frac{1}{n!2} e^x \left( x^n (e^{-x} x^n)^{(n)} \right)^{(n)} \in \mathbb{Q}[x],$$

orthogonal on  $(0, +\infty)$  for the weights  $e^{-x}$  and  $\log(x)e^{-x}$ . (**Remember that the case  $\alpha = 0$  was excluded.**)

This construction can be viewed as a limiting case of our results for  $\Gamma(1 + \alpha)$  because

$$\lim_{\alpha \rightarrow 0} \frac{\Gamma(1 + \alpha) - 1}{\alpha} = \Gamma'(1) = -\gamma.$$

How to construct good simultaneous approximations for

$$\Gamma(\alpha) \quad \text{and} \quad \int_0^{\infty} (t+1)^{\alpha-1} e^{-t} dt,$$

and for

$$\gamma \quad \text{and} \quad \int_0^{\infty} \frac{e^{-t}}{1+t} dt?$$

Set

$$G_\alpha(z) = z^{-\alpha} \int_0^\infty (t+z)^{\alpha-1} e^{-t} dt, \quad E_\alpha(z) = \sum_{m=0}^{\infty} \frac{z^m}{m!(m+\alpha+1)}$$

and

$$E(z) = \sum_{m=1}^{\infty} \frac{z^m}{m!m}.$$

We have the identities

$$\frac{\Gamma(\alpha)}{z^\alpha} = E_{\alpha-1}(-z) + e^{-z} G_\alpha(z)$$

$$\gamma + \log(z) = E(-z) - e^{-z}(G_0(z) - \log(z)).$$



Construction of Hermite-Padé approximants.

**Proposition 1.**  $\Re(\alpha) > -1$  and  $\alpha \notin \mathbb{Z}$ .  $\exists$  polynomials  $A_n, C_n$  (degree  $\leq n$ ) and  $B_n$  (degree  $\leq n + 1$ ) s.t.

$$\begin{aligned} R_{n,\alpha}(z) &:= \frac{z^{3n+1}}{n!^2} \int_0^1 \int_0^1 e^{zuv} u^{2n+\alpha} (1-u)^n v^{2n} (1-v)^n du dv \\ &= A_n(z)e^z + B_n(z)E_\alpha(z) + C_n(z). \end{aligned}$$

Order at  $z = 0$  of  $R_n(z)$  is  $3n + 1$ .

A similar proposition holds when  $\alpha = -1$  with  $E(z)$  instead of  $E_\alpha(z)$ .

We deduce that for any integers  $p, q, r$  not all zero and any  $\varepsilon > 0$ , we have

$$\left| p + qe^{-z} + rE_{\alpha-1}(-z) \right| \geq \frac{c_1}{H^{2+\varepsilon}},$$

where  $H = \max(|p|, |q|, |r|)$  and  $c_1$  depends on  $\alpha, \varepsilon, z$ .

Using the relation between  $\Gamma(\alpha)$ ,  $G_\alpha(z)$ ,  $E_{\alpha-1}(z)$  and  $e^{-z}$ , it follows for any integers  $p, q, r$  not all zero and any  $\varepsilon > 0$ ,

$$\left| \frac{\Gamma(\alpha)}{z^\alpha} - \frac{p}{q} \right| + \left| G_\alpha(z) - \frac{r}{q} \right| \geq \frac{c_2(\alpha, \varepsilon, z)}{H^{3+\varepsilon}},$$

Similarly, we get

$$\left| \gamma + \log(z) - \frac{p}{q} \right| + \left| G_0(z) - \log(z) - \frac{r}{q} \right| \geq \frac{d(\varepsilon, z)}{H^{3+\varepsilon}}.$$

**Theorem 3.** • For any  $\alpha \in \mathbb{Q}$ ,  $\alpha > 0$ ,  $\alpha \notin \mathbb{Z}$ , at least one of

$$\Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1} e^{-t} dt$$

and

$$G_{\alpha}(1) = \int_0^{\infty} (t+1)^{\alpha-1} e^{-t} dt$$

is irrational.

• At least one of

$$\gamma = - \int_0^{\infty} \log(t) e^{-t} dt$$

and

$$G_0(1) = \int_0^{\infty} \frac{e^{-t}}{1+t} dt = \int_0^{\infty} \log(t+1) e^{-t} dt$$

is irrational.