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joint work with

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Outline of the talk

- Introduction & motivations.
- The 2-XORSAT phase transition.
- MAX-2-XORSAT.
- Conclusion and perspectives.
Introduction & Motivations
Decision and optimization problems play central key rôle in CS (cf. [GAREY, JOHNSON 79], [AUSIELLO et al. 03])

1. A decision problem is a question in some formal system with a yes/no answer:

   \[
   \begin{cases}
   \text{INPUT : an instance } I \text{ and a property } P. \\
   \text{OUTPUT : yes or no } I \text{ satisfies } P.
   \end{cases}
   \]

2. An optimization problem is the problem of finding the best solution from all feasible solutions.

In this talk, we consider two such problems: 2-XORSAT and MAX-2-XORSAT.
Random $k$-SAT formulas ($k > 2$) are subject to phase transition phenomena \cite{FRIEDGUT,BOURGAIN1999}.

Main research tasks include

1. **Localization** of the threshold (ex. \textbf{3-SAT} 4.2 \ldots \textbf{3-XORSAT} 0.91 \ldots \cite{DUBOIS,MANDLER03})

2. Nature of the phenomena: \textbf{sharp/coarse}. \cite{CREIGNOU,DAUDE2000++}.

3. Details inside the \textbf{window of transition} (ex. \textbf{2-SAT} \cite{BOLLOBAS,BORGSKIM,WILSON01})

4. \textbf{Space} of solutions (ex. \cite{ACHLIOPTAS,NAOR,PERES07} or \cite{MONASSONETAL07})
SAT-like problems: localization of 2-SAT’s threshold

- An instance: \((v_1 \lor v_2) \land (\neg v_1 \lor v_3) \land (\neg v_1 \lor \neg v_2)\)
- A solution: SAT with \((v_1 = 1, v_2 = 0, v_3 = 1)\).
- Localization of the threshold: \(n\) variables, \(m = c \times n\) clauses randomly picked from the set of \(4\binom{n}{2}\) clauses.
  - \(c < 1\) Proba SAT → 1, \(c > 1\) Proba SAT → 0.

Underlying combinatorial structures: directed graphs.

Write \(x \lor y\) as \(\begin{cases} \neg x = 1 \implies y = 1 \\ \neg y = 1 \implies x = 1 \end{cases}\)

Characterization: SAT iff no directed path between \(x\) and \(\neg x\) (and vice-versa).

Proof. First and second moments method [Göerdt 92, De la Vega 92, Chvátal, Reed 92].
Main motivations

- Since the empirical results (\cite{KirkpatrickSelman90} about $k$-SAT, rigorous results are quite **limited**!
- What are the contributions of **Enumerative/Analytic Combinatorics** to SAT/CSP-like problems?
- **Monasson** (2007) inferred that (statistical physics):

$$ \lim_{n \to +\infty} n^{\text{critical exponent}} \times \text{Proba} \left[ 2XORSAT(n, \frac{n}{2}) \right] = O(1), $$

where “critical exponent” = $1/12$.

- We will **show** that “critical exponent” = $1/12$ and will **explicit** the hidden constant behind the $O(1)$.
- We will **quantify** the **MAXIMUM** number of satisfiable clauses in random formula.
The 2-XORSAT phase transition
Ex:
\[ x_1 \oplus x_2 = 1, \ x_2 \oplus x_3 = 0, \ x_1 \oplus x_3 = 0, \ x_3 \oplus x_4 = 1, \ldots \]

General form: \( AX = C \) where \( A \) has \( m \) rows and 2 columns and \( C \) is a \( m \)-dimensional 0/1 vector.

Distribution: uniform. We pick \( m \) clauses of the form \( x_i \oplus x_j = \varepsilon \in \{0, 1\} \) from the set of \( n(n-1) \) clauses.

Underlying structures: graphs with weighted edges
\[ x \oplus y = \varepsilon \iff \text{edges of weight } \varepsilon \in \{0, 1\}. \]

Characterisation:
SAT iff no elementary cycle of odd weight.
SAT iff no elementary cycle of odd weight

\[
\begin{align*}
  x_1 \oplus x_2 &= 1 \\
  x_2 \oplus x_3 &= 0 \\
  x_1 \oplus x_3 &= 0 \\
  x_3 \oplus x_4 &= 1
\end{align*}
\]

- **UNSAT** $\iff$ Fix a cycle of odd weight ...

- **SAT** $\iff$ No cycles of odd weight. DFS affectionation based proof.
Main ideas of our approach

A basic scheme

1. **Enumeration** of “SAT”-graphs (graphs without cycles of odd weight) by means of generating functions.

2. Use the obtained results with **analytic combinatorics** to compute:

\[
\text{Prob. SAT} = \frac{\text{Nbr of configurations without cycles of odd weight}}{\text{Nbr total of configurations}}.
\]
$p(n, cn) \overset{\text{def}}{=} \text{Proba}[2 - \text{XOR with } n \text{ variables}, \text{cn clauses}]$ is SAT for $n = 1000$, $n = 2000$ and the theoretical function: $e^{c/2}(1 - 2c)^{1/4}$. 

Taste of our results: the whole window
Rescaling at the point “zero”, i.e \( c = 1/2 \) : \( n = 1000 \), \( n = 2000 \) and 
\[
\lim_{n \to \infty} n^{1/12} \times p(n, n/2 + \mu n^{2/3})
\]
as a function of \( \mu \).
We will **enumerate** the **connected graphs without cycles of odd weight** according to two parameters: **number of vertices** $n$ and **number of edges** $n + \ell$. $\ell \overset{\text{def}}{=} \text{excess}$.

Let

$$C_\ell(z) = \sum_{n>0} c_{n,n+\ell} \frac{z^n}{n!}.$$ 

What are the series $C_\ell$?
Enumerating graphs of 2-XORSAT.

We will enumerate the connected graphs without cycles of odd weight according to two parameters: number of vertices \( n \) and number of edges \( n + \ell \). \( \ell \) \( \text{def} \) excess.

Let

\[
C_\ell(z) = \sum_{n>0} c_{n,n+\ell} \frac{z^n}{n!}.
\]

What are the series \( C_\ell \)?

**Th.**

\[
C_\ell(z) = \frac{1}{2} W_\ell(2z)
\]

with \( W_\ell = \) Exponential generating functions of connected graphs \( \text{WRIGHT (1977)}. \)
Enumerations: trees and unicyclic components

- **Rooted and unrooted trees** (excess $= -1$)
  \[
  T(z) = ze^{2T(z)} = \sum_{n>0} (2n)^{-1} \frac{z^n}{n!}, \quad C_{-1}(z) = T - T^2.
  \]

- **Unicyclic components** (excess $= 0$)
  1. Number of labellings of a *smooth* cycle (i.e. without vertices of degree 1) using $n > 2$ vertices:
     \[
     \frac{2^n n!}{2n}.
     \]
  2. Thus, the EGF of smooth unicyclic components
     \[
     \tilde{C}_0(z) = -\frac{1}{4} \log (1 - 2z) - z/2 - z^2/2.
     \]
  3. Substituting each vertex with a full rooted tree, we get
     \[
     C_0(z) = -\frac{1}{4} \log (1 - 2T) - T/2 - T^2/2.
     \]

- **What about multicyclic components?** (excess $> 0$)
On a connected “SAT”-graph with $n$ vertices and $n + \ell$ edges, the edges of a spanning tree can be colored in $2^{n-1}$ ways. The colors of the other edges are “determined”.
Let $F_r(z)$ be the EGF of all complex weighted labelled graphs (connected or not), with a positive \textit{total excess}$^1$ $r$ and without cycles of odd weight ("SAT-graph").

$$
\sum_{r \geq 0} F_r(z) = \exp \left( \sum_{k \geq 1} \frac{W_k(2z)}{2} \right)
$$

and for any $r \geq 1$

$$
rF_r(z) = \sum_{k=1}^{r} k \frac{W_k(2z)}{2} F_{r-k}(z), \quad F_0(z) = 1.
$$

Since $W_k(x) \asymp \frac{w_k}{(1-T(x))^{3r}}$ \cite{wright80}, we also have $F_k(x) \asymp \frac{f_k}{(1-T(2x))^{3r}}$ with

$$
2rf_r = \sum_{k=1}^{r} kb_k f_{r-k}, \quad r > 0.
$$

---

$^1$total excess of the random graphs $\overset{\text{def}}{=} \text{nbr of edges} + \text{number of trees} - \text{number of vertices}$
The Random 2-XORSAT Transition

The probability that a random formula with \( n \) variables and \( m \) clauses is SAT satisfies the following:

(i) **Sub-critical phase**: As \( 0 < n - 2m < n^{2/3} \),

\[
\Pr(n, m) = e^{m/2n} \left(1 - 2\frac{m}{n}\right)^{1/4} + O\left(\frac{n^2}{(n - 2m)^3}\right).
\]

(ii) **Critical phase**: As \( m = \frac{n}{2} + \mu n^{2/3} \), \( \mu \in \mathbb{R} \) fixed

\[
\lim_{n \to \infty} n^{1/12} \Pr\left(n, \frac{n}{2} (1 + \mu n^{-1/3})\right) = \Psi(\mu),
\]

where \( \Psi \) can be expressed in terms of the Airy function.

(iii) **Super-critical phase**: As \( m = \frac{n}{2} + \mu n^{2/3} \) with \( \mu = o(n^{1/12}) \)

\[
\Pr\left(n, \frac{n}{2} (1 + \mu n^{-1/3})\right) = \text{Poly}(n, \mu) e^{-\frac{\mu^3}{6}}.
\]
Proof of (i) : the sub-critical phase

1. As $0 < n - 2m \ll n^{2/3}$, the probability that a Erdős-Rényi random graph $G(n, m)$ has NO MULTICYCLIC COMPONENTS is

$$1 - O\left(\frac{n^2}{(n-2m)^3}\right) \left\{ \begin{array}{ll}
\text{if } m = cn \text{ with } \lim \sup c < 1/2, \text{ BigOh } = O(1/n) \\
\text{if } m = \frac{n}{2} - \mu(n)n^{2/3}, \text{ BigOh } = O(1/\mu^3)
\end{array} \right. $$

2. Then, the probability that the graph associated to random 2-XORSAT formula is SAT (conditionally that there is no multicyclic components) is given by

$$\frac{n!}{(n(n-1))} \left[ z^n \right] \frac{C_{-1}(z)^{n-m}}{(n-m)!} \times \underbrace{e^{C_0(z)}}_{\text{set of even weighted unicyclic components}}$$
Saddle-point method for random 2-XORSAT sub-critical phase

\[ m \leq \frac{n}{2} - \mu n^{2/3}, \quad 1 \ll \mu \]

1. Cauchy integral formula leads to

\[ \text{coeff}(n, m) \times \frac{1}{2\pi i} \oint \frac{e^{-T(2z)/4-T(2z)^2/8}}{(1 - T(2z))^{1/4}} \left( \frac{T(2z)}{2} - \frac{T(2z)^2}{4} \right)^{n-m} \frac{dz}{z^{n+1}} \]

2. “Lagrangian” substitution \( u = T(2z) \).

3. \[ \text{coeff}(n, m) \times \frac{1}{2\pi i} \oint g(u) \exp(nh(u)) du \]

4. \( h(u) = u - \frac{m}{n} \log u + (1 - \frac{m}{n}) \log (2 - u) \).
   Saddle-points at \( u_0 = 2m/n < 1 \) and \( u_1 = 1 \).
   \( h''(1) = 2m/n - 1 < 0 \) and \( h''(2m/n) = \frac{n(n-2m)}{4m(n-m)} > 0 \).
   Saddle-point method applies on circular path \(|z| = 2m/n \cdots\)
Proof of (ii) : Inside the critical phase (1/2)

\[ m = \frac{n}{2} \pm \mu n^{2/3}, \quad |\mu| = O(n^{1/12}) \]

Some MULTICYCLIC COMPONENTS (can) appear and the general formula for the integral becomes:

1

\[
\text{coeff}(n, m, r) \times \frac{1}{2\pi i} \oint e^{-T(2z)/4 + T(2z)^2/8} \left( \frac{T(2z)}{2} - \frac{T(2z)^2}{4} \right)^{n-m+r} \frac{dz}{z^{n+1}}
\]

2

\[
\text{coeff}(n, m, r)e^n \times \frac{1}{2\pi i} \oint g_r(u) \exp(nh(u))du
\]

3

\[ h(u) = u - 1 - \frac{m}{n} \log u + (1 - \frac{m}{n}) \log (2 - u). \]

Saddle-points at \( u_0 = 2m/n = 1 + 2\mu n^{-1/3} \) and \( u_1 = 1 \).

BUT at the critical point \( m = 2n (\mu = 0) \), we have \( u_0 = u_1 = 1 \) with **triple zero**

\[ h(1) = h'(1) = h''(1) = 0. \]
Airy function and the critical window of transition

Integral representation on the complex plane

The Airy function is given by

\[
Ai(z) = \frac{1}{2\pi i} \int_C \exp \left( \frac{t^3}{3} - zt \right) dt,
\]

where the integral is over a path \( C \) starting at the point at infinity with argument \(-\pi/3\) and ending at the point at infinity with argument \( \pi/3 \).
The Airy function is given by

\[ \text{Ai}(z) = \frac{1}{2\pi i} \int_C \exp \left( \frac{t^3}{3} - zt \right) dt , \]

where the integral is over a path \( C \) starting at the point at infinity with argument \(-\pi/3\) and ending at the point at infinity with argument \(\pi/3\).

Well suited for our purpose (see also \[\text{Flajolet, Knuth, Pittel 89}\], \[\text{Janson, Knuth, Łuczak, Pittel 93}\], \[\text{Flajolet, Salvy, Schaeffer 02}\], \[\text{Banderier, Flajolet, Schaeffer, Soria 01}\])!

Integrating on a path \( z = e^{-(\alpha +\text{i}t)n^{-1/3}} \), we get

\[ e^{-\mu^3/6-n} \frac{1}{2^{2m-n-2r}} \times \frac{1}{2\pi i} \int \frac{e^{-T(2z)/4-T(2z)^2/8}}{(1 - T(2z))^{1/4+3r}} \left( \frac{T(2z)}{2} - \frac{T(2z)^2}{4} \right)^{n-m+r} \frac{dz}{z^{n+1}} \]

\[ \sim e^{-3/8} A(1/4 + 3r, \mu) n^{-7/12} , \]

where \( A(y, \mu) = \frac{e^{-\mu^3/6}}{3(y+1)/3} \sum_{k>0} \frac{ \left( \frac{1}{2} 3^{2/3} \mu \right)^k }{k! \Gamma ((y + 1 - 2k)/3)} \).
Proof of (ii) : Inside the critical phase (2/2)

Define $p_r(n, m) = \text{Proba to have SAT-graph of excess } r$. The proba.
that a random formula is given by $p(n, m) = \sum_{r \geq 0} p_r(n, m)$.
The proof of part (ii) can now be completed by means of the following
facts

1. Using the Airy stuff, we compute for fixed $r$

$$n^{1/12} \times p_r(n, m) \sim \frac{\sqrt{2\pi} e^{1/4} f_r}{2^r} A(3r + 1/4, \mu).$$

2. Bounding the magnitude of the integral, it can be proved that there
exist $R, C, \epsilon > 0$ such that for all $r \geq R$ and all $n$:

$$n^{1/12} p_r(n, m) \leq C e^{-\epsilon r}.$$

(dominated convergence theorem applies).
Remark

On the first hand, writing \( m = \frac{n}{2} - \mu n^{2/3} \) the probability is about:

\[
e^{m/2n} \left( 1 - \frac{2m}{n} \right)^{1/4} \sim e^{1/4} \mu^{1/4} n^{-1/12}.
\]

On the other hand, the Airy stuff are valid for \( m = \frac{n}{2} + \mu n^{2/3}, \quad |\mu| = O(n^{1/12}) \). Using

\[
A(r, \mu) = \frac{1}{\sqrt{2\pi} |\mu|^{y-1/2}} \left( 1 - \frac{3y^2 + 3y - 1}{6|\mu|^3} + O(|\mu|^{-6}) \right)
\]
as \( \mu \to -\infty \) we get

\[
\sum_{r} p_r (n, m) \sim n^{-1/12} \left( \sum_{r=0}^{\infty} \frac{\sqrt{2\pi} e^{1/4} f_r}{2^r} A(3r + 1/4, \mu) \right) \sim e^{1/4} \mu^{1/4} n^{-1/12}.
\]
For the case (iii) of the theorem, we use

\[ A(y, \mu) = \frac{e^{-\mu^3/6}}{2^{y/2} \mu^{1-y/2}} \left( \frac{1}{\Gamma(y/2)} + \frac{4\mu^{-3/2}}{3\sqrt{2} \Gamma(y/2 - 3/2)} + O(\mu^{-2}) \right). \]
Random MAX-2-XORSAT
**Context**

- **Max-2-XORSAT** is an NP-optimization problem (NPO). The corresponding decision problem is in NP (deciding if the size of the MAX is \( k \) ...).

- **Max/Min** problems are interesting (and difficult) in randomness context.

- **Previous works**: [Coppersmith, Gamarnik, Hajiaghayi, Sorkin 04] Expectations of the **Maximum** number of satisfiable clauses in MAX-2-SAT and MAX-CUT for the subcritical phases. **Bounds** of these expectations for some cases (namely for the critical and supercritical phases of random graphs)!

- **Our work**: Quantification of the **Minimum** number of clauses to remove in order to get satisfiable formula.
Let $X_{n,m}$ be the minimum number of clauses UNSAT in a random 2-XOR formula with $n$ variables and $m$ clauses. We have:

(i) **Sub-critical phase**: If $\limsup \frac{m}{n} < 1/2$ then

$$X_{n,m} \xrightarrow{\text{dist.}} \text{Poisson} \left( \log n - 3 \log \left( \frac{n-2m}{n^{2/3}} \right) - 3 \left( 1 - \frac{2m}{n} \right) \right).$$

(ii) **Critical phase**: If $m = \frac{n}{2}(1 - \mu n^{-1/3})$, $1 \ll \mu \ll n^{1/3}$ then

$$\mathbb{P} \left( X_{n,m} - \frac{1}{4} \log(\mu n^{-1/3}) \leq x \sqrt{\frac{1}{4} \log(\mu n^{-1/3})} \right) \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2} du$$

(iii) **Supercritical phase**: If $m = \frac{n}{2} + \mu n^{2/3}$ with $\mu = o(n^{1/3})$ (resp. $m = \frac{n}{2}(1 + \varepsilon)$)

$$\frac{6X_{n,m}}{(2m-n)^3 + O(\log n)} \xrightarrow{\text{dist.}} 1.$$ (resp. $\frac{8(1 + \varepsilon)}{n(\varepsilon^2 - \sigma^2)} X_{n,m} \xrightarrow{\text{dist.}} 1,$)

where $\sigma$ is the solution of $(1 + \varepsilon)e^{-\varepsilon} = (1 - \sigma)e^\sigma$. 
Notations

- $X_{n,m}$: minimum number of UNSAT clauses in random formula with $n$ variables and $m$ clauses.

- $Y_{n,m}$: minimum number of clauses to suppress in unicyclic components.

- $Z_{n,m}$: minimum number of clauses to suppress in multicyclic components.

$$X_{n,m} = Y_{n,m} + Z_{n,m}.$$
Proof of the sub-critical phase

In the sub-critical random graphs, we know that $Z_{n,m} = O_p(1)$.

- If $m = cn$, $c \in ]0, \frac{1}{2}[$, $\forall R$ fixed, we have
  \[
  \Pr(Y_{n,m} = R) = e^{-\alpha(c)} \frac{\alpha(c)^R}{R!} \left(1 + O\left(\frac{1}{n}\right)\right).
  \]

- If $m = \frac{n}{2}(1 - \mu n^{-1/3})$ with $\mu \to \infty$ but $\mu = o(n^{1/3})$, we get $\forall R \leq 4\beta(n)$
  \[
  \Pr(Y_{n,m} = R) = e^{-\beta(n)} \frac{\beta(n)^R}{R!} \left(1 + O\left(\frac{1}{\mu^3}\right)\right).
  \]

- There are $R_0, C, \varepsilon > 0$, s. t. $\forall R > R_0$
  \[
  \Pr(Y_{n,m} = R) \leq Ce^{-\varepsilon R}.
  \]

with

\[
\beta(n) = \frac{1}{12} \log(n) - \frac{1}{4} \log(\mu) - \frac{1}{4} + \frac{1}{4} \mu n^{-1/3},
\alpha(c) = -\frac{1}{4} \log(1 - 2c) - \frac{c}{2}.
\]
Corollary. As \( n \to \infty \) and \( m = cn, c \in ]0, 1/2[ \), for all \( k \geq 1 \),

\[
E(X_{n,m})_k \sim \alpha(c)^k.
\]

If \( m = \frac{n}{2}(1 - \mu n^{-1/3}) \) but \( \mu = o(n^{1/3}) \)

\[
E(X_{n,m})_k \sim \beta(n)^k.
\]

with

\[
\beta(n) = \frac{1}{12} \log(n) - \frac{1}{4} \log(\mu) - \frac{1}{4} + \frac{1}{4} \mu n^{-1/3}
\]

and

\[
\alpha(c) = -\frac{1}{4} \log(1 - 2c) - \frac{c}{2}
\]
Theorem. As $n, m \to \infty$ and $m = \frac{n}{2} \pm O(1)n^{2/3}$, then for all $k$

$$E(X_{n,m})_k \sim \frac{1}{12^k} \log(n)^k.$$ 

Here again, by [JANSON, KNUTH, ŁUCZAK, PITTEL 93] $Z_{n,m} = O_{p}(1)$.

The $k$-th factorial moment of $Y_{n,m}$ is $\sum_r s_{r,k}$ where

$$s_{r,k} = \frac{1}{n(n-1)} \frac{n!}{2\pi i} \int \frac{\partial^k}{\partial u^k} S_r(u, z)|_{u=1} \frac{dz}{z^{n+1}}$$

$$S_r(u, z) = \frac{\left(T(z) - T(z)^2\right)^{n-m+r}}{(n-m+r)!} \exp \left(C_{0,0}(z) + uC_{1,0}(z)\right) F_r(z),$$

$F_r(z)$ : EGF of multicyclic components.
Lemma. As $\ell \to \infty$, the probability that the number of edges to suppress in order to obtain a (weighted) connected graph without cycles of odd weight from a (weighted) connected graph of excess $\ell$ is larger than

$$\frac{\ell}{4} - o(\ell)$$

is at least

$$1 - e^{-O(\ell)} - e^{-4c(\ell)^2 + \frac{1}{2} \log(\ell)}$$

where $c(\ell)^2 \gg \log(\ell)$.
Lemma. As $\ell \to \infty$, the probability that the number of edges to suppress in order to obtain a (weighted) connected graph without cycles of odd weight from a (weighted) connected graph of excess $\ell$ is larger than

$$\frac{\ell}{4} - o(\ell)$$

is at least

$$1 - e^{-O(\ell)} - e^{-4c(\ell)^2 + \frac{1}{2} \log(\ell)}$$

where $c(\ell)^2 \gg \log(\ell)$

To prove this lemma, we need another one!
Let $C_{s,\ell}$ be the EGFs of connected components of EXCESS $\ell$ and where at \textbf{LEAST} $s$ edges have to be suppressed to obtain components without cycles of odd weight.

**Lemma.** For all $s \geq 0$, we have

$$C_{s,\ell}(z) \prec \left( \sum_{i=s}^{2s} \binom{\ell + 1}{i} \right) C_{0,\ell}(z)$$
Lower bound of the probability (super-critical phase)

Let \( C_{s,\ell} \) be the EGFs of connected components of EXCESS \( \ell \) and where at least \( s \) edges have to be suppressed to obtain components without cycles of odd weight.

**Lemma.** For all \( s \geq 0 \), we have

\[
C_{s,\ell}(z) \prec \left( \sum_{i=s}^{2s} \binom{\ell + 1}{i} \right) C_{0,\ell}(z)
\]

**Idea of the proof.**
SAT $\Rightarrow$ UNSAT
**Lemma.** If in a connected component of excess $\ell$ we have to suppress at least $s$ edges to obtain a SAT-graph then this component has at most $s$ fundamental and distinct cycles of **odd weight**.

**Idea of the proof.** Immediate.

As a crucial **consequence**, such a connected component has a **cactus** (as a subgraph) with at most $s$ cycles of odd weight.
**Lemma.** If in a connected component of excess $\ell$ we have to suppress at least $s$ edges to obtain a SAT-graph then this component has at most $s$ fundamental and distinct cycles of **odd weight**.

**Idea of the proof.** Immediate. As a crucial **consequence**, such a connected component has a cactus (as a subgraph) with at most $s$ cycles of odd weight.

**Example.**
Lemma. Let $\tilde{\Xi}_s(z)$ be the EGF of smooth cactii (Husimi trees) with $s$ cycles, we have:

$$\frac{\partial}{\partial z} \tilde{\Xi}_s + (s - 1) \tilde{\Xi}_s = \frac{1}{2} \sum_{i=1}^{s-1} (\partial z \tilde{\Xi}_i) (\partial z \tilde{\Xi}_{s-i}) (\partial (P) - P) + \sum_{k=1}^{s-1} z^k \frac{\partial^k}{\partial z^k} \partial z \tilde{\Xi}_1$$

$$\times \sum_{\ell_1 + 2\ell_2 + \cdots + (s-1)\ell_{s-1} = s-1} \frac{(\partial z \tilde{\Xi}_1)^{\ell_1}}{\ell_1!} \cdots \frac{(\partial z \tilde{\Xi}_{s-1})^{\ell_{s-1}}}{\ell_{s-1}!} \left(\frac{1}{z} + \frac{P}{z^2}\right)^k$$

with $P \equiv P(z) = \frac{z^2}{1-z}$.
Lemma. We have

$$\Xi_s(z) \leq \frac{\xi_s}{(1 - t(z))^{3s-3}}, \quad s > 1$$

where \((\xi_s)_{s>1}\) satisfies \(\xi_2 = \frac{1}{8}, \xi_3 = \frac{1}{12}\) and for \(s \geq 3\), we have:

$$3(s - 1)\xi_s = \frac{3}{2}(s - 2)\xi_{s-1} + \frac{9}{2} \sum_{i=2}^{s-2} (i - 1)(s - i - 1)\xi_i \xi_{s-i} +$$

$$\frac{1}{2} \sum_{k=1}^{s-1} k! \left( \sum_{l_1 + 2l_2 + \cdots + (s-1)l_{s-1} = s-1} \frac{(\frac{1}{2})^{l_1}}{l_1!} \frac{(3\xi_2)^{l_2}}{l_2!} \cdots \frac{(3(s-2)\xi_{s-1})^{l_{s-1}}}{l_{s-1}!} \right)$$
Lemma. We have

\[ \Xi_s(z) \leq \frac{\xi_s}{(1 - t(z))^{3s-3}}, \quad s > 1 \]

where \((\xi_s)_{s>1}\) satisfies \(\xi_2 = \frac{1}{8}, \xi_3 = \frac{1}{12}\) and for \(s \geq 3\), we have:

\[
3(s - 1)\xi_s = \frac{3}{2}(s - 2)\xi_{s-1} + \frac{9}{2} \sum_{i=2}^{s-2} (i - 1)(s - i - 1)\xi_i \xi_{s-i} +
\]

\[
\frac{1}{2} \sum_{k=1}^{s-1} \frac{1}{k!} \left( \sum_{\ell_1 + 2\ell_2 + \cdots + (s-1)\ell_{s-1} = s-1} \frac{\left(\frac{1}{2}\right)^{\ell_1} (3\xi_2)^{\ell_2} \cdots (3(s-2)\xi_{s-1})^{\ell_{s-1}}}{\ell_1! \ell_2! \cdots \ell_{s-1}!} \right)
\]

Lemma. As \(s \to \infty\),

\[
\xi_s = \frac{1}{6} \left(\frac{3}{2}\right)^{s-1} \frac{3^{s/2}}{\sqrt{2\pi s^3(s - 1)}} \left(1 + O\left(\frac{1}{s}\right)\right).
\]
Corollary. The number of connected component of excess $\ell$ obtained by adding edges from cactii with $s$ cycles can be neglected if

$$s > \frac{\ell}{2} + O\left(\frac{\ell}{\log(\ell)}\right).$$
**Corollary.** The number of connected component of excess $\ell$ obtained by adding edges from cactii with $s$ cycles can be neglected if $s > \frac{\ell}{2} + O\left(\frac{\ell}{\log(\ell)}\right)$.

**Idea of the proof.**

- Pick a cactus with $s$ cycles.
- Add $(\ell - s)$ edges to obtain a connected component of excess $\ell$. The number of such constructions can be bounded by pointing/depointing the last added edge.
- The ratio of the number of these objects over the number of all connected components of excess $\ell$ is exponentially small as $s > \frac{\ell}{2} + O(\ell/\log \ell)$. 
On connected components of excess $\ell$ the number of edges to suppress lies w.h.p. between

$$\frac{\ell}{4} - O(\ell^{1/2}) \leq |\text{suppressions}| \leq \frac{\ell}{4} + O\left(\frac{\ell}{\log \ell}\right).$$

Now, we can use the result on random graphs from [PITTEL, WORMALD 05] to complete the proof of the theorem.
Conclusion and perspectives
Enumerative/Analytic approaches of

1. a decision problem and its phase transition
2. an NP-optimization problem.
Conclusion and perspectives

Enumerative/Analytic approaches of

1. a decision problem and its phase transition
2. an NP-optimization problem.

Similar methods on other problems such as

1. bipartiness (or 2-COL),
2. MAX-2-COL, MAX-CUT, MIN-VERTEX-COVER, MIN-BISECTION (all are hard optimization problems related to bipartiteness/2-COL).
3. 2-QXORSAT (quantified formula).
MAX-CUT $\sim$ MAX-2-XORSAT (i)
MAX-CUT $\sim$ MAX-2-XORSAT (ii)

Graph $\rightarrow$ MAX-2-XORSAT

MAX-CUT