Random 2-XORSAT/MAX-2-XORSAT and their phase transitions

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Outline of the talk

- Introduction & motivations.
- The 2-XORSAT phase transition.
- MAX-2-XORSAT.
- Conclusion and perspectives.

Introduction & Motivations

Decision and optimization problems

- Decision and optimization problems play central key rôle in CS (cf. [GAREY, JOHNSON 79] , [AUSIELLO *et al.* 03])
  1. A **decision problem** is a question in some formal system with a **yes**/**no** answer :

$$\begin{cases} \text{INPUT : an instance } \mathcal{I} \text{ and a property } \mathcal{P}. \\ \\ \text{OUTPUT : yes or no } \mathcal{I} \text{ satisfies } \mathcal{P}. \end{cases}$$

  2. An **optimization problem** is the problem of finding the **best solution** from all feasible solutions.
- In this talk, we consider two such problems : **2-XORSAT** and **MAX-2-XORSAT**.

SAT-like problems

- Random k-SAT formulas ($k > 2$) are subject to phase transition phenomena [FRIEDGUT, BOURGAIN 1999] .
- Main research tasks include
  1. **Localization** of the threshold (ex. **3-SAT** 4.2. . . ? **3-XORSAT** 0.91. . . [DUBOIS, MANDLER 03)])
  2. Nature of the phenomena : **sharp/coarse**. [CREIGNOU, DAUDÉ 2000++] .
  3. Details inside the **window of transition** (ex. **2-SAT** [BOLLOBÀS, BORGS, KIM, WILSON 01])
  4. **Space** of solutions (ex. [ACHLIOPTAS, NAOR, PERES 07] or [MONASSON *et al.* 07])

- An instance : $(\mathbf{v_1} \vee \mathbf{v_2}) \wedge (\neg \mathbf{v_1} \vee \mathbf{v_3}) \wedge (\neg \mathbf{v_1} \vee \neg \mathbf{v_2})$
- A solution : SAT with $(\mathbf{v_1} = \mathbf{1}, \mathbf{v_2} = \mathbf{0}, \mathbf{v_3} = \mathbf{1})$.
- Localization of the threshold : n variables, $m = c \times n$ clauses randomly picked from the set of $4\binom{n}{2}$ clauses.
  $c < 1$ Proba SAT $\rightarrow 1$, $c > 1$ Proba SAT $\rightarrow 0$.
  Underlying combinatorial structures : directed graphs.

$$\texttt{Write} \quad x \vee y \quad \texttt{as} \quad \begin{cases} \neg x = 1 \Longrightarrow y = 1 \\ \neg y = 1 \Longrightarrow x = 1 \end{cases}$$

Characterization : SAT iff no directed path between x and $\neg x$ (and vice-versa).
Proof. First and second moments method [GOERDT 92, DE LA VEGA 92, CHVÀTAL, REED 92] .

Main motivations

- Since the empirical results ([KIRKPATRICK, SELMAN 90] about k-SAT, rigorous results are quite **limited**!
- What are the contributions of **ENUMERATIVE/ANALYTIC COMBINATORICS** to SAT/CSP-like problems?
- MONASSON (2007) inferred that (statistical physics) :

$$\lim_{n\to+\infty} n^{\text{critical exponent}} \times \mathtt{Proba}\left[2\text{XORSAT}(n, \frac{n}{2})\right] = O(1),$$

  where "critical exponent" $= \mathbf{1/12}$.
- We will show that "critical exponent" $= \mathbf{1/12}$ and will explicit the hidden constant behind the $O(1)$.
- We will quantify the MAXIMUM number of satisfiable clauses in random formula.

The 2-XORSAT phase transition

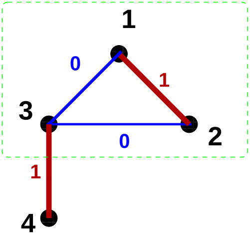
- Ex :
  $x_1 \oplus x_2 = 1, x_2 \oplus x_3 = 0, x_1 \oplus x_3 = 0, x_3 \oplus x_4 = 1, \cdots.$
- General form : $AX = C$ where A has m rows and 2 columns and C is a m-dimensional $0/1$ vector.
- Distribution : uniform. We pick m clauses of the form $x_i \oplus x_j = \varepsilon \in \{0, 1\}$ from the set of $n(n-1)$ clauses.
- Underlying structures : **graphs with weighted edges**
  $x \oplus y = \varepsilon \Longleftrightarrow$ edges of weight $\varepsilon \in \{0, 1\}$.

Characterisation :

SAT iff no elementary cycle of odd weight.

SAT iff no elementary cycle of odd weight

$$\begin{cases} x_1 \oplus x_2 = 1 \\ x_2 \oplus x_3 = 0 \\ x_1 \oplus x_3 = 0 \\ x_3 \oplus x_4 = 1 \end{cases}$$



- UNSAT $\Longleftarrow$ Fix a cycle of odd weight ...
- SAT $\Longleftarrow$ No cycles of odd weight. DFS affectation based proof.

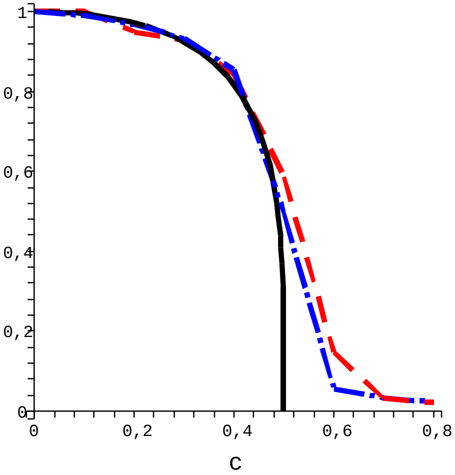
A basic scheme

1. Enumeration of "SAT"-graphs (graphs without cycles of odd weight) by means of generating functions.

2. Use the obtained results with analytic combinatorics to compute :

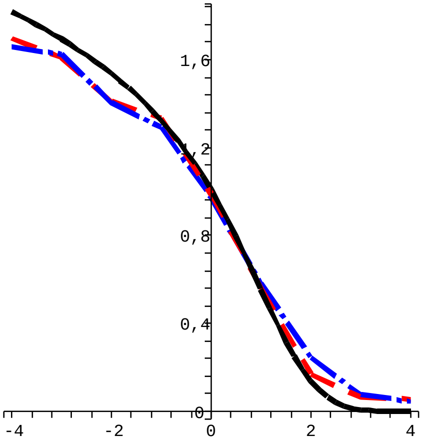
$$\textbf{Prob. SAT} = \frac{\textbf{Nbr of configurations without cycles of odd weight}}{\textbf{Nbr total of configurations}} .$$

Taste of our results : the whole window



$\mathbf{p(n, cn)} \stackrel{\text{def}}{=} \texttt{Proba}\,[\mathbf{2} - \text{XOR with } \mathbf{n} \text{ variables }, \mathbf{cn} \text{ clauses }]$ is **SAT**
for $n = 1000$, $n = 2000$ and the **theoretical** function : $\mathbf{e^{c/2}(1 - 2c)^{1/4}}$.

Taste of our results: rescaling the critical window



Rescaling at the point "zero", i.e $c = 1/2$: $n = 1000$, $n = 2000$ and $\lim_{n\to\infty} \underbrace{n^{1/12}\times}\ p(n, n/2 + \mu n^{2/3})$ as a **function of** μ.

Enumerating graphs of 2-XORSAT.

We will **enumerate** the `connected graphs without cycles of odd weight` according to two parameters: **number of vertices n** and **number of edges n** $+ \ell$. $\ell \stackrel{\text{def}}{=}$ **excess.**

Let

$$C_\ell(z) = \sum_{n>0} c_{n,n+\ell} \frac{z^n}{n!} .$$

What are the series C_ℓ?

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What are the series C_ℓ?

Th.

$$C_\ell(z) = \frac{1}{2} W_\ell(2z)$$

with W_ℓ = Exponential generating functions of connected graphs WRIGHT (1977).

Enumerations: trees and unicyclic components

- Rooted and unrooted trees (excess $= -1$)

$$T(z) = ze^{2T(z)} = \sum_{n>0} (2n)^{n-1} \frac{z^n}{n!}, \qquad C_{-1}(z) = T - T^2 .$$

- Unicyclic components (excess $= 0$)
  1. Number of labellings of a *smooth* cycle (i.e. without vertices of degree 1) using $n > 2$ vertices :

$$\frac{2^n n!}{2n} .$$

  2. Thus, the EGF of smooth unicyclic components

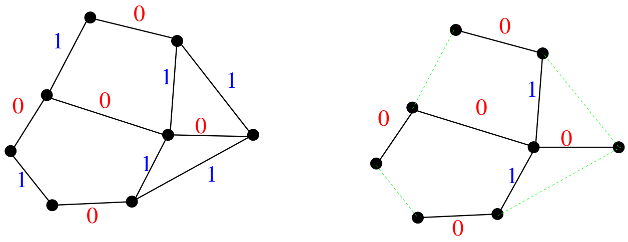
$$\tilde{C}_0(z) = -\frac{1}{4} \log(1 - 2z) - z/2 - z^2/2 .$$

  3. Substituting each vertex with a full rooted tree, we get

$$C_0(z) = -\frac{1}{4} \log(1 - 2T) - T/2 - T^2/2 .$$

- What about multicyclic components? (excess > 0)

Enumerations: connected multicyclic components



On a connected "SAT"-graph with n vertices and $n+\ell$ edges, the edges of a spanning tree can be colored in 2^{n-1} ways. The colors of the other edges are "determined".

Let $F_r(z)$ be the EGF of all complex weighted labelled graphs (**connected** or **not**), with a positive *total excess*[1] r and without cycles of odd weight ("SAT-graph").

$$\sum_{r\geq 0} F_r(z) = \exp\left(\sum_{k\geq 1} \frac{W_k(2z)}{2}\right)$$

and for any $r \geq 1$

$$rF_r(z) = \sum_{k=1}^{r} k \frac{W_k(2z)}{2} F_{r-k}(z)\,, \quad F_0(z) = 1\,.$$

Since $W_k(x) \asymp \frac{w_k}{(1-T(x))^{3r}}$ [WRIGHT 80] , we also have $F_k(x) \asymp \frac{f_k}{(1-T(2x))^{3r}}$ with $2rf_r = \sum_{k=1}^{r} kb_k f_{r-k}\,, \quad r > 0$.

[1] total excess of the random graphs $\stackrel{\text{def}}{=}$ **nbr of edges** $+$ **number of trees** $-$ **number of vertices**

Th.

The probability that a random formula with n variables and m clauses is SAT satisfies the following :

(i) Sub-critical phase : As $0 < n - 2m \ll n^{2/3}$,

$$\mathbf{Pr}(\mathbf{n},\mathbf{m}) = \mathbf{e}^{\mathbf{m}/2\mathbf{n}}\left(\mathbf{1} - \mathbf{2}\frac{\mathbf{m}}{\mathbf{n}}\right)^{\mathbf{1/4}} + \mathbf{O}\left(\frac{\mathbf{n}^2}{(\mathbf{n}-\mathbf{2m})^3}\right).$$

(ii) Critical phase : As $m = \frac{n}{2} + \mu n^{2/3}$, $\mu \in \mathbb{R}$ fixed

$$\lim_{\mathbf{n}\to\infty} \mathbf{n}^{\mathbf{1/12}}\,\mathbf{Pr}\left(\mathbf{n}, \frac{\mathbf{n}}{\mathbf{2}}(\mathbf{1} + \mu\mathbf{n}^{-\mathbf{1/3}})\right) = \Psi(\mu)\,,$$

where Ψ can be expressed in terms of the Airy function.

(iii) Super-critical phase : As $m = \frac{n}{2} + \mu n^{2/3}$ with $\mu = o(n^{1/12})$

$$\mathbf{Pr}\left(\mathbf{n}, \frac{\mathbf{n}}{\mathbf{2}}(\mathbf{1} + \mu\mathbf{n}^{-\mathbf{1/3}})\right) = \mathbf{Poly}(\mathbf{n},\,\mu)\,\mathbf{e}^{-\frac{\mu^3}{6}}\,.$$

Proof of (i) : the sub-critical phase

1. As $0 < n - 2m \ll n^{2/3}$, the probability that a Erdős-Rényi random graph $\mathbb{G}(n, m)$ has NO MULTICYCLIC COMPONENTS is

$$\mathbf{1} - \mathbf{O}\left(\frac{\mathbf{n^2}}{(\mathbf{n} - \mathbf{2m})^3}\right) \begin{cases} \text{if } m = cn \text{ with } \limsup c < 1/2, \ \text{BigOh} = O(1/n) \\ \text{if } m = \frac{n}{2} - \mu(n)n^{2/3}, \ \text{BigOh} = O(1/\mu^3) \end{cases}$$

2. Then, the probability that the graph associated to random 2-XORSAT formula is SAT (conditionally that there is no multicyclic components) is given by

$$\frac{\mathbf{n}!}{\binom{\mathbf{n(n-1)}}{\mathbf{m}}} [\mathbf{z^n}] \quad \underbrace{\frac{C_{-1}(z)^{n-m}}{(n-m)!}}_{\text{unrooted trees}} \times \underbrace{e^{C_0(z)}}_{\substack{\text{set of even weighted} \\ \text{unicyclic components}}}$$

Saddle-point method for random 2-XORSAT sub-critical phase

$$\mathbf{m} \leq \frac{\mathbf{n}}{\mathbf{2}} - \mu \mathbf{n}^{2/3}, \quad \mathbf{1} \ll \mu$$

1. Cauchy integral formula leads to

$$\text{coeff}(n,m) \times \frac{1}{2\pi i} \oint \frac{e^{-T(2z)/4 - T(2z)^2/8}}{(1-T(2z))^{1/4}} \left(\frac{T(2z)}{2} - \frac{T(2z)^2}{4} \right)^{n-m} \frac{dz}{z^{n+1}}$$

2. "Lagrangian" substitution $u = T(2z)$.
3. $$\text{coeff}(n,m) \times \frac{1}{2\pi i} \oint g(u) \exp\left(nh(u)\right) du$$
4. $\mathbf{h(u) = u - \frac{m}{n} \log u + \left(1 - \frac{m}{n}\right) \log (2 - u)}$.
   Saddle-points at $\mathbf{u_0 = 2m/n < 1}$ and $\mathbf{u_1 = 1}$.
   $h''(1) = 2m/n - 1 < 0$ and $h''(2m/n) = \frac{n(n-2m)}{4m(n-m)} > 0$.
   Saddle-point method applies on circular path $|z| = 2m/n \cdots$

Proof of (ii) : Inside the critical phase (1/2)

$$\mathbf{m} = \frac{\mathbf{n}}{\mathbf{2}} \pm \mu \mathbf{n}^{\mathbf{2/3}}, \quad |\mu| = \mathbf{O}(\mathbf{n}^{\mathbf{1/12}})$$

Some MULTICYCLIC COMPONENTS (can) appear and the general formula for the integral becomes

1. $$\text{coeff}(n, m, r) \times \frac{1}{2\pi i} \oint \frac{e^{-T(2z)/4 - T(2z)^2/8}}{(1 - T(2z))^{1/4+3r}} \left(\frac{T(2z)}{2} - \frac{T(2z)^2}{4} \right)^{n-m+r} \frac{dz}{z^{n+1}}$$

2. $$\text{coeff}(n, m, r) e^n \times \frac{1}{2\pi i} \oint g_r(u) \exp(nh(u)) du$$

3. $\mathbf{h(u) = u - 1 - \frac{m}{n} \log u + \left(1 - \frac{m}{n}\right) \log (2 - u)}$.
   Saddle-points at $\mathbf{u_0 = 2m/n = 1 + 2\mu n^{-1/3}}$ and $\mathbf{u_1 = 1}$.
   BUT at the critical point $m = 2n$ ($\mu = 0$), we have $\mathbf{u_0 = u_1 = 1}$ with **triple zero** $\mathbf{h(1) = h'(1) = h''(1) = 0}$.

Integral representation on the complex plane

The Airy function is given by

$$\mathrm{Ai}(z) = \frac{1}{2\pi i} \int_C \exp\left(\frac{t^3}{3} - zt\right) dt \ ,$$

where the integral is over a path C starting at the point at infinity with argument $-\pi/3$ and ending at the point at infinity with argument $\pi/3$.

Airy function and the critical window of transition

Integral representation on the complex plane

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Well suited for our purpose (see also [FLAJOLET, KNUTH, PITTEL 89] , [JANSON, KNUTH, ŁUZCAK, PITTEL 93] , [FLAJOLET, SALVY, SCHAEFFER 02] , [BANDERIER, FLAJOLET, SCHAEFFER, SORIA 01])!

Integrating on a path $\mathbf{z = e^{-(\alpha+it)n^{-1/3}}}$, we get

$$\frac{e^{-\mu^3/6-n}}{2^{2m-n-2r}} \times \frac{1}{2\pi i} \oint \frac{e^{-T(2z)/4-T(2z)^2/8}}{(1-T(2z))^{1/4+3r}} \left(\frac{T(2z)}{2} - \frac{T(2z)^2}{4}\right)^{n-m+r} \frac{dz}{z^{n+1}}$$
$$\sim e^{-3/8} A(1/4+3r,\mu)\, n^{r-7/12} \,,$$

where $A(y,\mu) = \dfrac{e^{-\mu^3/6}}{3^{(y+1)/3}} \displaystyle\sum_{k\geq 0} \frac{\left(\frac{1}{2}3^{2/3}\mu\right)^k}{k!\,\Gamma((y+1-2k)/3)}$

Define $p_r(n, m) =$ Proba to have SAT-graph of excess r. The proba. that a random formula is given by $p(n, m) = \sum_{r \geq 0} p_r(n, m)$.
The proof of part (ii) can now be completed by means of the following facts

1. Using the Airy stuff, we compute for fixed r

$$n^{1/12} \times p_r(n, m) \sim \frac{\sqrt{2\pi}\, e^{1/4} f_r}{2^r} A(3r + 1/4, \mu)\,.$$

2. Bounding the magnitude of the integral, it can be proved that there exist R, C, $\epsilon > 0$ such that for all $r \geq R$ and all n:

$$n^{1/12}\, p_r(n, m) \leq C\, e^{-\epsilon\, r}\,.$$

(dominated convergence theorem applies).

Remark

- On the first hand, writing $m = \frac{n}{2} - \mu n^{2/3}$ the probability is about :

$$e^{m/2n}\left(1 - \frac{2m}{n}\right)^{1/4} \sim e^{1/4}\,\mu^{1/4}\,n^{-1/12}\,.$$

- On the other hand, the Airy stuff are valid for $m = \frac{n}{2} + \mu\, n^{2/3}$, $|\mu| = O(n^{1/12})$. Using $A(r,\mu) = \frac{1}{\sqrt{2\pi}\,|\mu|^{y-1/2}}\left(1 - \frac{3y^2+3y-1}{6|\mu|^3} + O(|\mu|^{-6})\right)$ as $\mu \to -\infty$ we get

$$\sum_r p_r\,(n,\ m) \sim n^{-1/12}\left(\sum_{r=0}^{\infty} \frac{\sqrt{2\pi}\,e^{1/4} f_r}{2^r}\,A(3r+1/4,\mu)\right) \sim e^{1/4}\,\mu^{1/4}\,n^{-1/12}\,.$$

Proof of (iii) : the supercritical phase

For the case (iii) of the theorem, we use

$$A(y,\mu) = \frac{e^{-\mu^3/6}}{2^{y/2}\mu^{1-y/2}}\left(\frac{1}{\Gamma(y/2)} + \frac{4\mu^{-3/2}}{3\sqrt{2}\,\Gamma(y/2-3/2)} + O(\mu^{-2})\right).$$

Random MAX-2-XORSAT

Context

- **MAX-2-XORSAT** is an NP-optimization problem (NPO). The corresponding decision problem is in NP (deciding if the size of the MAX is k ...).

- **MAX/MIN** problems are interesting (and difficult) in randomness context.

- **PREVIOUS WORKS :** [COPPERSMITH, GAMARNIK, HAJIAGHAYI, SORKIN 04] **Expectations** of the **Maximum** number of satisfiable clauses in MAX-2-SAT and MAX-CUT for the subcritical phases. **Bounds** of these expectations for some cases (namely for the critical and supercritical phases of random graphs)!

- **OUR WORK :**
  Quantification of the **Minimum** number of clauses to remove in order to get satisfiable formula.

Th.

Let $\mathbf{X_{n,m}}$ be the minimum number of clauses UNSAT in a random 2-XOR formula with n variables and m clauses. We have :

(i) Sub-critical phase : If $\limsup \frac{m}{n} < 1/2$ then

$$\mathbf{X_{n,m}} \xrightarrow{\text{dist.}} \text{Poisson}\left(\frac{\log \mathbf{n} - \mathbf{3} \log\left(\frac{\mathbf{n-2m}}{\mathbf{n^{2/3}}}\right) - \mathbf{3}\left(\mathbf{1} - \frac{\mathbf{2m}}{\mathbf{n}}\right)}{\mathbf{12}} \right).$$

(ii) Critical phase : If $m = \frac{n}{2}(1 - \mu n^{-1/3})$, $1 \ll \mu \ll n^{1/3}$ then

$$\mathbb{P}\left(\mathbf{X_{n,m}} - \frac{\mathbf{1}}{\mathbf{4}} \log(\mu \mathbf{n}^{-\mathbf{1/3}}) \leq \mathbf{x} \sqrt{\frac{\mathbf{1}}{\mathbf{4}} \log(\mu \mathbf{n}^{-\mathbf{1/3}})} \right) \to \frac{\mathbf{1}}{\sqrt{\mathbf{2}\pi}} \int_{-\infty}^{\mathbf{x}} \mathbf{e}^{-\mathbf{u^2/2}} \mathbf{du}$$

(iii) Supercritical phase : If $m = \frac{n}{2} + \mu n^{2/3}$ with $\mu = o(n^{1/3})$ (resp. $m = \frac{n}{2}(1 + \varepsilon)$)

$$\frac{\mathbf{6\,X_{n,m}}}{\frac{\mathbf{(2m-n)^3}}{\mathbf{n^2}} + \mathbf{O}(\log \mathbf{n})} \xrightarrow{\text{dist.}} \mathbf{1}. \qquad \left(\text{resp. } \frac{\mathbf{8(1+\varepsilon)}}{\mathbf{n}(\varepsilon^{\mathbf{2}} - \sigma^{\mathbf{2}})} \mathbf{X_{n,m}} \xrightarrow{\text{dist.}} \mathbf{1}, \right)$$

where σ is the solution of $(1+\varepsilon)e^{-\varepsilon} = (1-\sigma)e^{\sigma}$.

Notations

- $\mathbf{X_{n,m}}$: minimum number of UNSAT clauses in random formula with *n* variables and *m* clauses.

- $\mathbf{Y_{n,m}}$: minimum number of clauses to suppress in unicyclic components.

- $\mathbf{Z_{n,m}}$: minimum number of clauses to suppress in multicyclic components.

$$X_{n,m} = Y_{n,m} + Z_{n,m} \,.$$

Proof of the sub-critical phase

In the sub-critical random graphs, we know that $Z_{n,m} = O_p(1)$.

- if $m = cn$, $c \in]0, \frac{1}{2}[$ $\forall R$ fixed, we have
$$\mathbf{Pr}\left(\mathbf{Y_{n,m}} = \mathbf{R}\right) = \mathbf{e}^{-\alpha(\mathbf{c})} \frac{\alpha(\mathbf{c})^{\mathbf{R}}}{\mathbf{R}!} \left(\mathbf{1} + \mathbf{O}\left(\frac{\mathbf{1}}{\mathbf{n}}\right)\right).$$
- If $m = \frac{n}{2}(1 - \mu n^{-1/3})$ with $\mu \to \infty$ but $\mu = o(n^{1/3})$, we get $\forall\ R \leq 4\beta(n)$
$$\mathbf{Pr}\left(\mathbf{Y_{n,m}} = \mathbf{R}\right) = \mathbf{e}^{-\beta(\mathbf{n})} \frac{\beta(\mathbf{n})^{\mathbf{R}}}{\mathbf{R}!} \left(\mathbf{1} + \mathbf{O}\left(\frac{\mathbf{1}}{\mu^{\mathbf{3}}}\right)\right).$$
- There are $R_0, C, \varepsilon > 0$, s. t. $\forall R > R_0$
$$\mathbf{Pr}\left(\mathbf{Y_{n,m}} = \mathbf{R}\right) \leq \mathbf{C}\mathbf{e}^{-\varepsilon \mathbf{R}}.$$

with
$$\beta(\mathbf{n}) = \frac{\mathbf{1}}{\mathbf{12}} \log(\mathbf{n}) - \frac{\mathbf{1}}{\mathbf{4}} \log(\mu) - \frac{\mathbf{1}}{\mathbf{4}} + \frac{\mathbf{1}}{\mathbf{4}} \mu \mathbf{n}^{-\mathbf{1/3}},\ \alpha(\mathbf{c}) = -\frac{\mathbf{1}}{\mathbf{4}} \log(\mathbf{1} - \mathbf{2c}) - \frac{\mathbf{c}}{\mathbf{2}}$$

Corollary. As $n \to \infty$ and $m = cn, c \in]0, 1/2[$, for all $k \geq 1$,

$$\mathbf{E}\,(\mathbf{X_{n,m}})_{\mathbf{k}} \sim \alpha(\mathbf{c})^{\mathbf{k}}\,.$$

If $m = \frac{n}{2}(1 - \mu n^{-1/3})$ but $\mu = o(n^{1/3})$

$$\mathbf{E}\,(\mathbf{X_{n,m}})_{\mathbf{k}} \sim \beta(\mathbf{n})^{\mathbf{k}}\,.$$

with

$$\beta(\mathbf{n}) = \frac{\mathbf{1}}{\mathbf{12}}\log(\mathbf{n}) - \frac{\mathbf{1}}{\mathbf{4}}\log(\mu) - \frac{\mathbf{1}}{\mathbf{4}} + \frac{\mathbf{1}}{\mathbf{4}}\mu\mathbf{n}^{-\mathbf{1/3}}$$

and

$$\alpha(\mathbf{c}) = -\frac{\mathbf{1}}{\mathbf{4}}\log(\mathbf{1} - \mathbf{2c}) - \frac{\mathbf{c}}{\mathbf{2}}$$

Theorem. As $n, m \to \infty$ and $m = \frac{n}{2} \pm O(1)n^{2/3}$, then for all k

$$\mathbf{E}(\mathbf{X}_{\mathbf{n},\mathbf{m}})_{\mathbf{k}} \sim \frac{\mathbf{1}}{\mathbf{12}^{\mathbf{k}}} \log(\mathbf{n})^{\mathbf{k}} .$$

- Here again, by [JANSON, KNUTH, ŁUCZAK, PITTEL 93] $\mathbf{Z}_{\mathbf{n},\mathbf{m}} = O_p(1)$.
- The k-th factorial moment of $\mathbf{Y}_{\mathbf{n},\mathbf{m}}$ is $\sum_{\mathbf{r}} \mathbf{s}_{\mathbf{r},\mathbf{k}}$ where

$$s_{r,k} = \frac{1}{\binom{n(n-1)}{m}} \frac{n!}{2\pi i} \oint \frac{\partial^k}{\partial u^k} S_r(u,z)_{|u=1} \frac{dz}{z^{n+1}}$$

$$S_r(u,z) = \frac{(T(z) - T(z)^2)^{n-m+r}}{(n-m+r)!} \exp\left(C_{0,0}(z) + uC_{1,0}(z)\right) F_r(z) ,$$

$F_r(z)$: EGF of multicyclic components.

Lemma. As $\ell \to \infty$, the probability that the number of edges to suppress in order to obtain a (weighted) connected graph without cycles of odd weight from a (weighted) connected graph of excess ℓ is larger than

$$\frac{\ell}{4} - o(\ell)$$

is at least

$$1 - e^{-O(\ell)} - e^{-4c(\ell)^2 + \frac{1}{2}\log(\ell)}$$

where $c(\ell)^2 \gg \log(\ell)$

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is at least

$$\mathbf{1} - \mathbf{e}^{-\mathbf{O}(\ell)} - \mathbf{e}^{-\mathbf{4c}(\ell)^{\mathbf{2}} + \frac{\mathbf{1}}{\mathbf{2}} \log(\ell)}$$

where $c(\ell)^2 \gg \log(\ell)$

To prove this lemma, we need another one!

Let $C_{s,\ell}$ be the EGFs of connected components of EXCESS ℓ and where **at LEAST** s edges have to be suppressed to obtain components without cycles of odd weight.

Lemma. For all $s \geq 0$, we have

$$\mathbf{C_{s,\ell}(z)} \prec \left(\sum_{i=s}^{2s} \binom{\ell + 1}{i} \right) \mathbf{C_{0,\ell}(z)}$$

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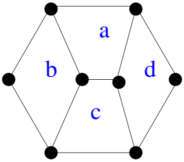
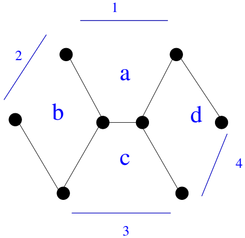
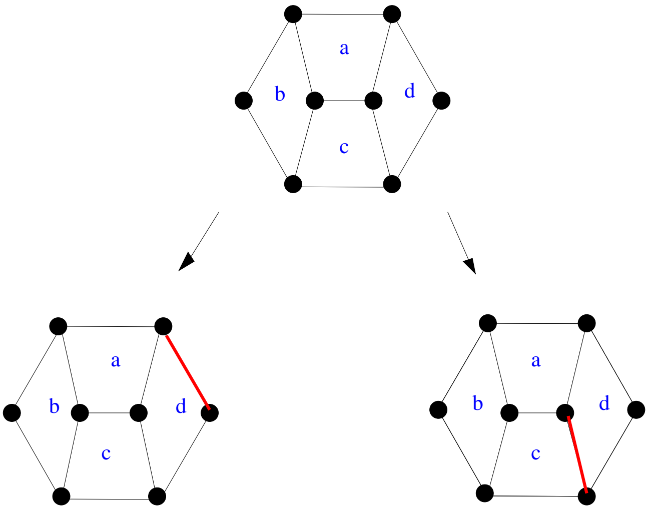
$$\mathbf{C_{s,\ell}(z)} \prec \left(\sum_{\mathbf{i=s}}^{\mathbf{2s}} \binom{\ell + \mathbf{1}}{\mathbf{i}} \right) \mathbf{C_{0,\ell}(z)}$$

Idea of the proof.





SAT=>UNSAT



Upper-bound of the probability (super-critical phase)

Lemma. If in a connected component of excess ℓ we have to suppress at least s edges to obtain a SAT-graph then this component has at most s fundamental and distinct cycles of **odd weight**.

Idea of the proof. Immediate.

As a crucial **consequence**, such a connected component has a cactus (as a subgraph) with at most s cycles of odd weight.

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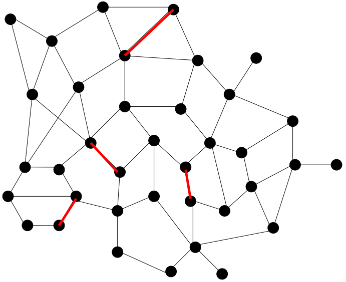
Example.



Lemma. Let $\tilde{\Xi}_s(z)$ be the EGF of smooth cactii (Husimi trees) with s cycles, we have :

$$\partial_z \tilde{\Xi}_s + (s-1)\tilde{\Xi}_s = \frac{1}{2}\sum_{i=1}^{s-1}(\partial_z \tilde{\Xi}_i)(\partial_z \tilde{\Xi}_{s-i})\ (\partial(P) - P) + \sum_{k=1}^{s-1} z^k \frac{\partial^k}{\partial z^k}\partial_z \tilde{\Xi}_1$$

$$\times \sum_{\substack{\ell_1+2\ell_2+\cdots+(s-1)\ell_{s-1}=s-1\\ \ell_1+\ell_2+\cdots+\ell_{s-1}=k,\ell_i\in\mathbb{N}}} \frac{\left(\partial_z \tilde{\Xi}_1\right)^{\ell_1}}{\ell_1!}\cdots\frac{\left(\partial_z \tilde{\Xi}_{s-1}\right)^{\ell_{s-1}}}{\ell_{s-1}!}\left(\frac{1}{z}+\frac{P}{z^2}\right)^k$$

, with $P \equiv P(z) = \frac{z^2}{1-z}$.

Counting cactii (...)

Lemma. We have

$$\Xi_s(z) \preceq \frac{\xi_s}{(1 - t(z))^{3s-3}}, \qquad s > 1$$

where $(\xi_s)_{s>1}$ satisfies $\xi_2 = \frac{1}{8}$, $\xi_3 = \frac{1}{12}$ and for $s \geq 3$, we have :

$$\mathbf{3(s-1)\xi_s = \frac{3}{2}(s-2)\xi_{s-1} + \frac{9}{2}\sum_{i=2}^{s-2}(i-1)(s-i-1)\xi_i\xi_{s-i} +}$$

$$\mathbf{\frac{1}{2}\sum_{k=1}^{s-1} k!} \left(\sum_{\substack{\ell_1+2\ell_2+\cdots+(s-1)\ell_{s-1}=s-1 \\ \ell_1+\ell_2+\cdots+\ell_{s-1}=k}} \frac{\left(\frac{1}{2}\right)^{\ell_1}}{\ell_1!} \frac{(3\xi_2)^{\ell_2}}{\ell_2!} \cdots \frac{(3(s-2)\xi_{s-1})^{\ell_{s-1}}}{\ell_{s-1}!} \right)$$

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Lemma. As $s \to \infty$,

$$\xi_s = \frac{1}{6}\left(\frac{3}{2}\right)^{s-1} \frac{3^{s/2}}{\sqrt{2\pi s^3}(s-1)} \left(1 + O\left(\frac{1}{s}\right)\right).$$

Corollary. The number of connected component of excess ℓ obtained by adding edges from cactii with s cycles can be neglected if $s > \frac{\ell}{2} + O\left(\frac{\ell}{\log(\ell)}\right)$.

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Idea of the proof.

- Pick a cactus with s cycles.
- Add $(\ell - s)$ edges to obtain a connected component of excess ℓ. The number of such constructions can be bounded by **pointing/depointing** the last added edge.
- The ratio of the number these objects over the number of all connected components of excess ℓ is exponentially small as $s > \frac{\ell}{2} + O(\ell/\log \ell)$.

Proof for the super-critical phase

1. On connected components of excess ℓ the number of edges to suppress lies w.h.p. between

$$\frac{\ell}{4} - O(\ell^{1/2}) \leq |\text{suppressions}| \leq \frac{\ell}{4} + O\left(\frac{\ell}{\log \ell}\right).$$

2. Now, we can use the result on random graphs from [PITTEL, WORMALD 05] to complete the proof of the theorem.

Conclusion and perspectives

Enumerative/Analytic approaches of

1. a `decision problem` and its `phase transition`
2. an `NP-optimization` problem.

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Enumerative/Analytic approaches of

1. a **`decision problem`** and its **`phase transition`**
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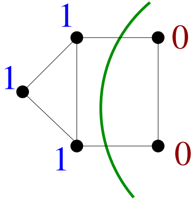
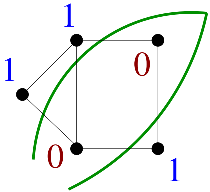
Similar **methods** on other problems such as

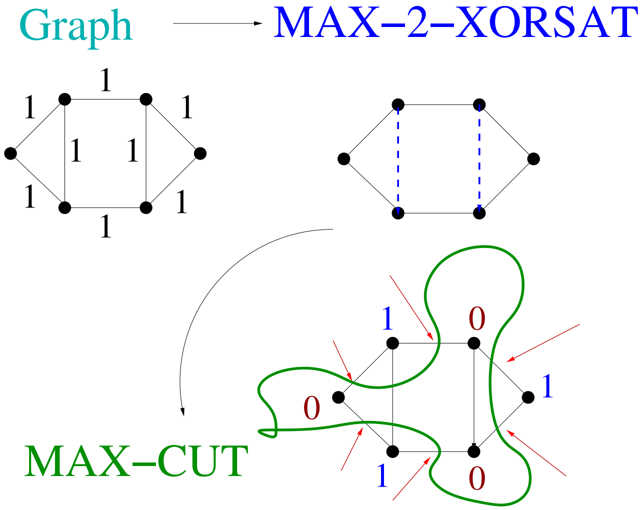
1. bipartiness (or 2-COL),
2. MAX-2-COL, MAX-CUT, MIN-VERTEX-COVER, MIN-BISECTION (all are hard optimization problems related to bipartiteness/2-COL).
3. 2-QXORSAT (quantified formula).



CUT



MAX–CUT


Graph
MAX–2–XORSAT
1
1
1
1
1
1
1
1
1
0
1
0
1
0
MAX–CUT