Projet Algorithmes Seminar, INRIA Rocquencourt, 19th October 2009

The scaling limit of critical random graphs

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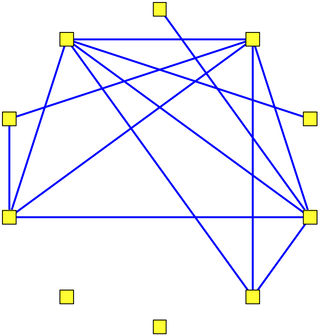
The Erdős-Rényi random graph

Take n vertices labelled by $[n] := \{1, 2, \ldots, n\}$ and put an edge between any pair independently with probability p. Call the resulting model $G(n, p)$.

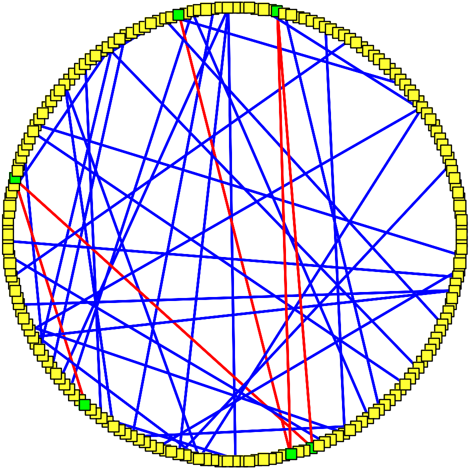
Example: $n = 10$, $p = 0.4$ (vertex labels omitted).



The phase transition

Let $p = c/n$ and consider the largest component (vertices in green, edges in red).

$n = 200$, $c = 0.4$



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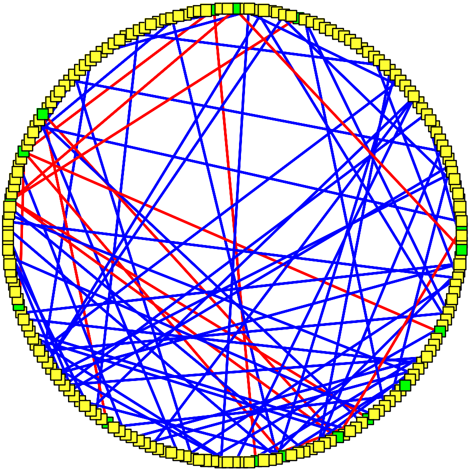
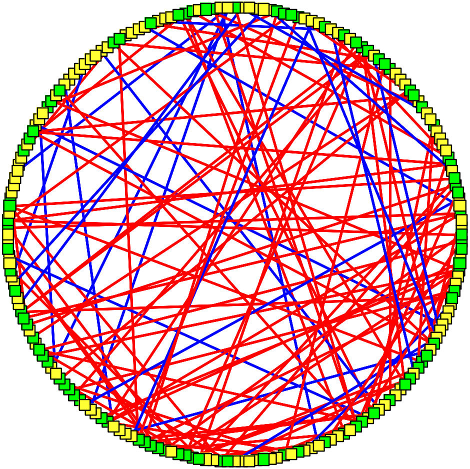
$n = 200$, $c = 0.8$



The phase transition

Let $p = c/n$ and consider the largest component (vertices in green, edges in red).

$n = 200$, $c = 1.2$



The phase transition (Erdős and Rényi (1960))

Consider $p = c/n$.

- For $c < 1$, the largest connected component has size $O(\log n)$;
- for $c > 1$, the largest connected component has size $\Theta(n)$ (and the others are all $O(\log n)$).

The critical random graph

The critical window: $p = \frac{1}{n} + \frac{\lambda}{n^{4/3}}$, where $\lambda \in \mathbb{R}$. For such p, the largest components have size $\Theta(n^{2/3})$.

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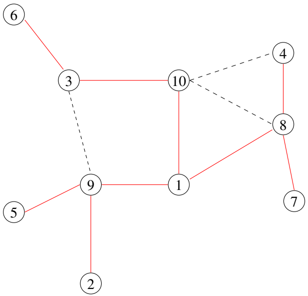
We will also be interested in the surplus of a component, the number of edges more than a tree that it has.

A component with surplus 3:



Convergence of the sizes and surpluses

Fix λ and let $C_1^n, C_2^n, \ldots$ be the sequence of component sizes in decreasing order, and let $S_1^n, S_2^n, \ldots$ be their surpluses.

Write $\mathbf{C}^n = (C_1^n, C_2^n, \ldots)$ and $\mathbf{S}^n = (S_1^n, S_2^n, \ldots)$.

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Here, convergence in the first co-ordinate takes place in

$$\ell^2_{\searrow} := \left\{ \mathbf{x} = (x_1, x_2, \ldots) : x_1 \geq x_2 \geq \ldots \geq 0, \sum_{i=1}^{\infty} x_i^2 < \infty \right\}.$$

Limiting sizes and surpluses

Let $W^\lambda(t) = W(t) + \lambda t - \frac{t^2}{2}$, $t \geq 0$, where $(W(t), t \geq 0)$ is a standard Brownian motion.

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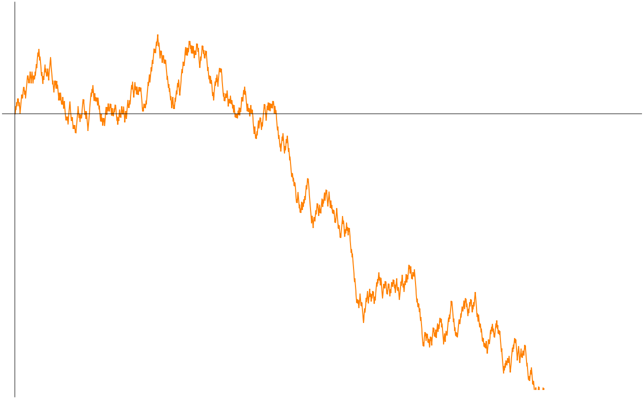

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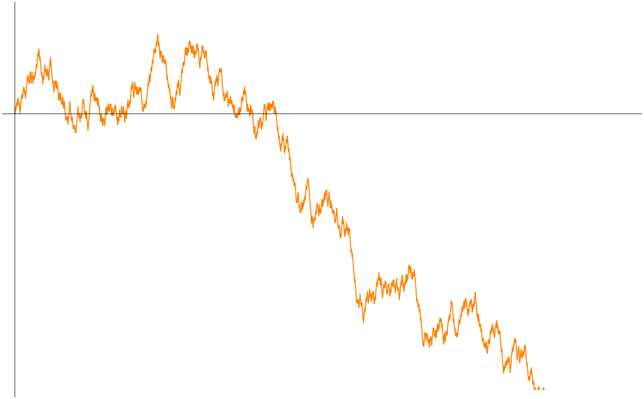
Let $B^\lambda(t) = W^\lambda(t) - \min_{0 \leq s \leq t} W^\lambda(s)$ be the process reflected at its minimum.



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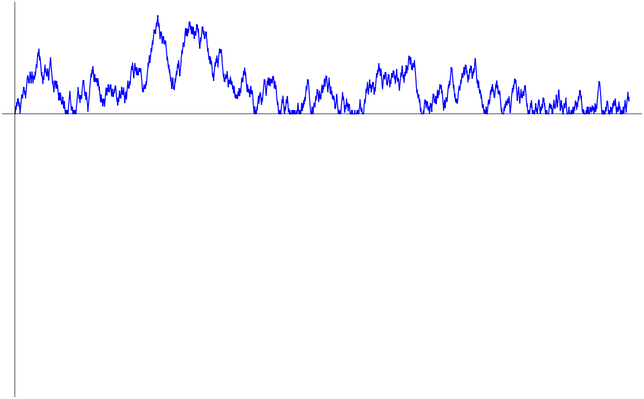

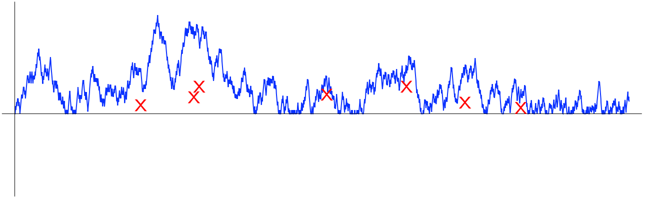

Decorate the picture with the points of a rate one Poisson process which fall above the x-axis and below the graph.

C is the sequence of excursion-lengths of this process, in decreasing order.

S is the sequence of numbers of points falling in the corresponding excursions.

Question

What do the limiting components look like?

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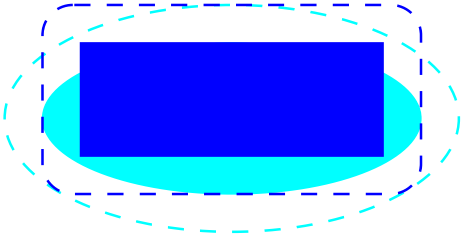
The vertex-labels are irrelevant: we are really interested in what distances look like in the limit. So we will give a metric space answer.

Measuring the distance between metric spaces

The Hausdorff distance between two compact subsets K and K' of a metric space (M, δ) is

$$d_H(K, K') = \inf\{\epsilon > 0 : K \subseteq F_\epsilon(K'), K' \subseteq F_\epsilon(K)\},$$

where $F_\epsilon(K) := \{x \in M : \delta(x, K) \leq \epsilon\}$ is the ϵ-fattening of K.



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So define the Gromov-Hausdorff distance

$$d_{GH}(X, X') = \inf\{d_H(\phi(X), \phi'(X'))\},$$

where the infimum is taken over all choices of metric space (M, δ) and all isometric embeddings $\phi : X \to M$, $\phi' : X' \to M$.

Our approach

Simple but important fact: a component of $G(n, p)$ conditioned to have m vertices and s surplus edges is a uniform connected graph on those m vertices with $m + s - 1$ edges.

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Our general approach is to pick out a (well-chosen) spanning tree, and then to put in the surplus edges.

There is one case which we already understand very well: when the surplus of a component is 0 and so we have a uniform random tree.

Warm-up: the tree case

Take a uniform random tree T_m on vertices labelled by $[m] = \{1, 2, \ldots, m\}$ and, using the ordinary graph distance, think of it as a metric space.

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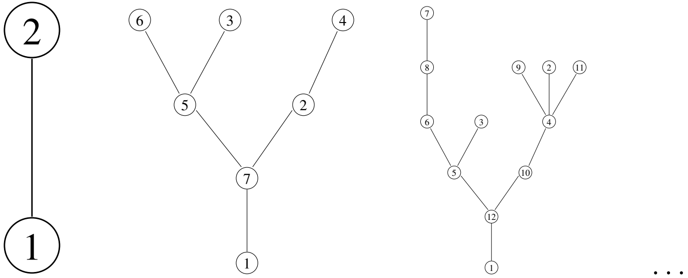
Take a uniform random tree T_m on vertices labelled by $[m] = \{1, 2, \ldots, m\}$ and, using the ordinary graph distance, think of it as a metric space.

Now rescale the edge-lengths by $1/\sqrt{m}$:



Warm-up: the tree case

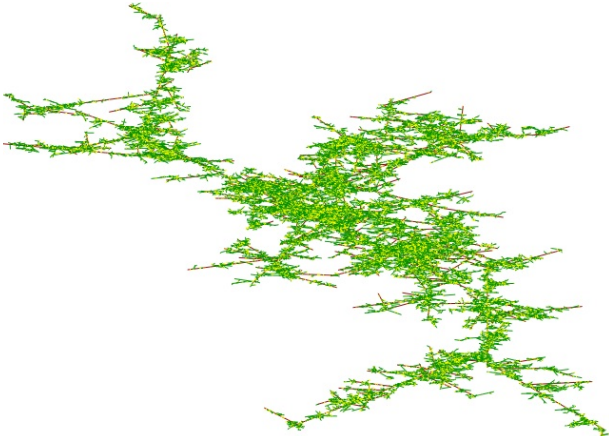
Theorem. (Aldous (1993), Le Gall (2005)) As $m \to \infty$,

$$\frac{1}{\sqrt{m}} T_m \xrightarrow{d} \mathcal{T},$$

where the convergence is in the Gromov-Hausdorff distance.

The limit $\mathcal{T}$ is the Brownian continuum random tree.

The Brownian continuum random tree



[Picture by Grégory Miermont]

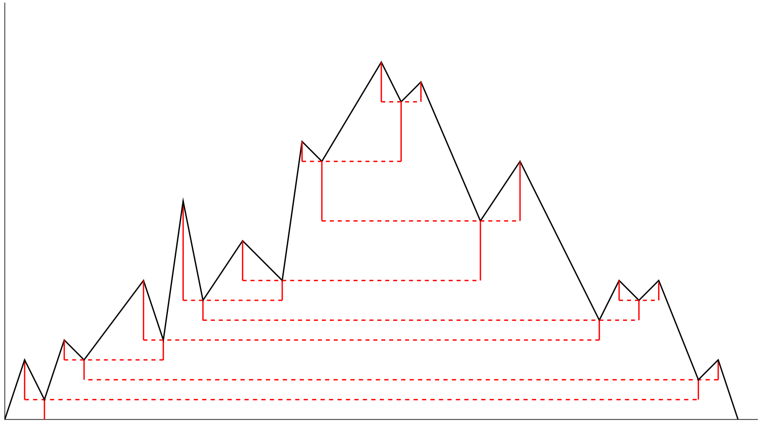
Trees from excursions

Let $h : [0, 1] \to \mathbb{R}^+$ be an excursion, that is a continuous function such that $h(0) = h(1) = 0$ and $h(x) > 0$ for $x \in (0, 1)$.

Trees from excursions

Define a distance d on $[0,1]$ via

$$d(x,y) = h(x) + h(y) - 2 \inf_{x \wedge y \le z \le x \vee y} h(z).$$



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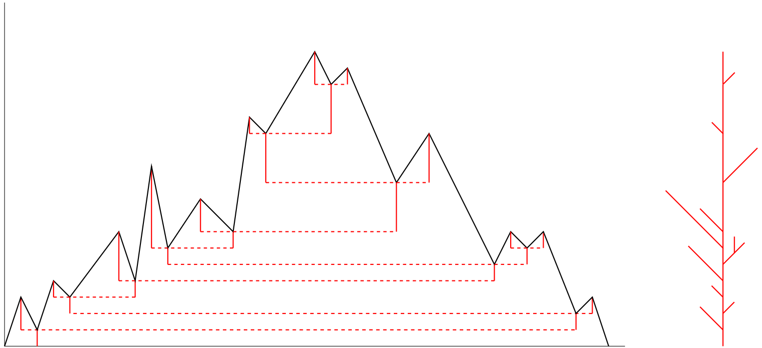

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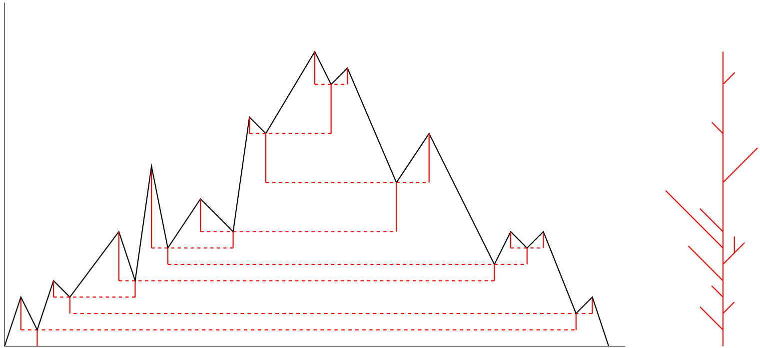

Define an equivalence relation $\sim$ by $x \sim y$ if $d(x,y) = 0$ and take the quotient $\mathcal{T}_h = [0,1]/\sim$.

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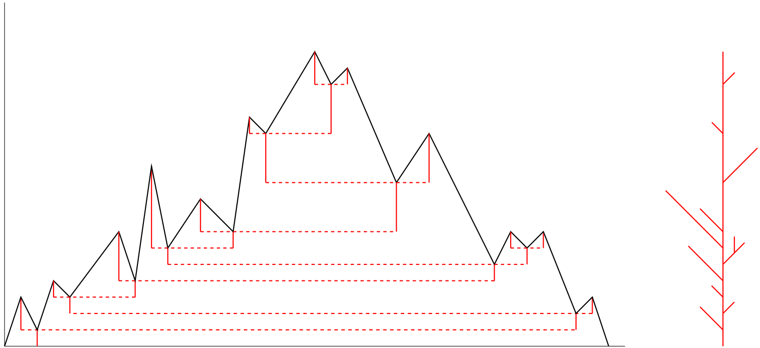

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The Brownian continuum random tree is $\mathcal{T}_h$ with $h(x) = 2e(x)$ and $(e(x), 0 \leq x \leq 1)$ a standard Brownian excursion.

The limit of the random graph

In the tree case, we rescaled distances by $1/\sqrt{m}$, where m was the number of vertices. This is the correct distance rescaling for all of the big components in the random graph.

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In the limit, surplus edges correspond to vertex-identifications (since edge-lengths have shrunk to 0). In each excursion, the points of the Poisson process tell us where these vertex-identifications should occur.

Excursions of the limit process

Consider the process $(B^{\lambda}(t), t \geq 0)$. An excursion $\tilde{e}^{(x)}$ of this process, conditioned to have length x, has a distribution specified by

$$\mathbb{E}\left[f\left(\tilde{e}^{(x)}\right)\right] = \frac{\mathbb{E}\left[f\left(e^{(x)}\right)\exp\left(\int_0^x e^{(x)}(u)du\right)\right]}{\mathbb{E}\left[\exp\left(\int_0^x e^{(x)}(u)du\right)\right]},$$

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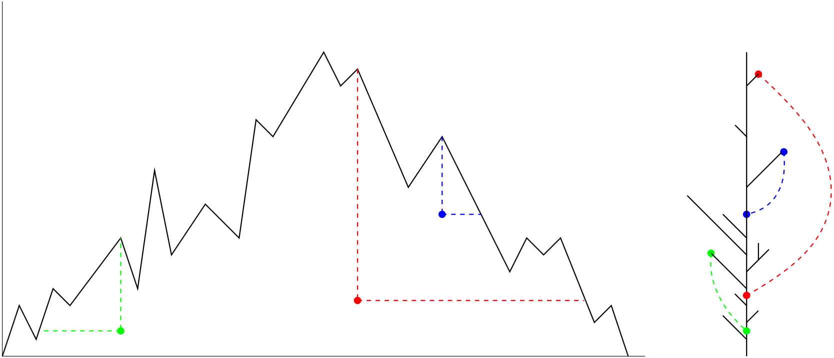
where f is any suitable test-function and $e^{(x)}$ is a Brownian excursion of length x.

We refer to $\tilde{e}^{(x)}$ as a tilted excursion and to the tree $\tilde{\mathcal{T}}$ that it encodes as a tilted tree.

Vertex identifications



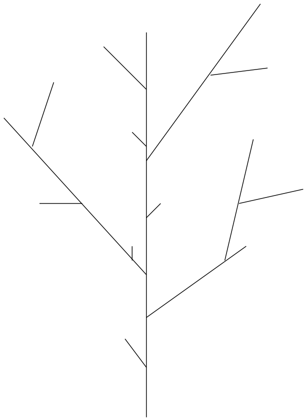
A point at (x, y) identifies the vertex v at height $h(x)$ with the vertex at distance y along the path from the root to v.

A limiting component

Note that it follows from properties of the tilted trees and of the Poisson process that we may equivalently describe the limit of a component on $\sim xn^{2/3}$ vertices as follows.

A limiting component

Sample a tilted excursion $\tilde{e}^{(x)}$ of length x and use it to create a CRT $\tilde{\mathcal{T}}$.



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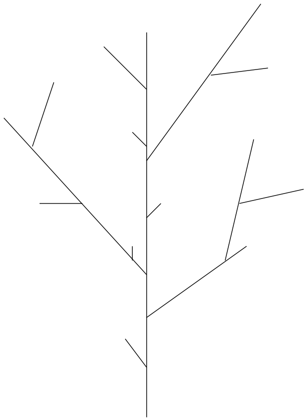
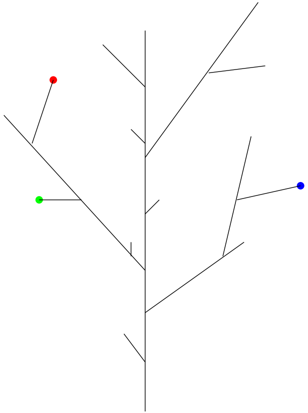


Conditional on $\tilde{e}^{(x)}$, sample a random variable P with Poisson $\left(\int_0^x \tilde{e}^{(x)}(u)du\right)$ distribution.

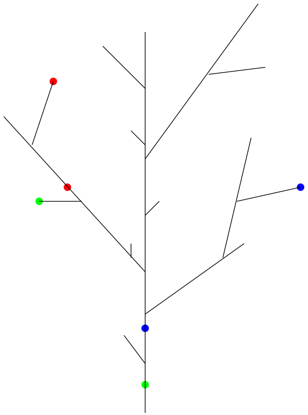
A limiting component

Conditional on $P = s$, pick s vertices of the tree $\tilde{\mathcal{T}}$ independently with density proportional to their height. (These will almost surely be leaves.)



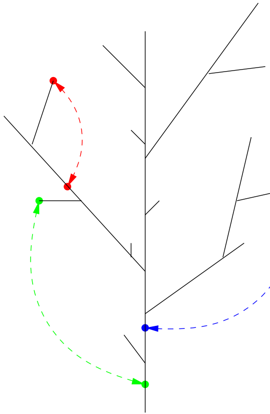
A limiting component

For each of the selected leaves, pick a uniform point on the path from the leaf to the root.



A limiting component

Identify each of the selected leaves with its chosen point.



Convergence result

Let $\mathcal{C}_1^n, \mathcal{C}_2^n, \ldots$ be the sequence of components of $G(n,p)$ in decreasing order of size, considered as metric spaces with the graph distance.

Theorem. As $n \to \infty$,

$$n^{-1/3}(\mathcal{C}_1^n, \mathcal{C}_2^n, \ldots) \xrightarrow{d} (\mathcal{C}_1, \mathcal{C}_2, \ldots),$$

where $\mathcal{C}_1, \mathcal{C}_2, \ldots$ is the sequence of metric spaces corresponding to the excursions of Aldous' marked limit process in decreasing order of length.

Here, convergence is with respect to the metric

$$d(\mathcal{A}, \mathcal{B}) := \left(\sum_{i=1}^{\infty} d_{GH}(\mathcal{A}_i, \mathcal{B}_i)^4 \right)^{1/4}.$$

Diameter

Let D_n be the diameter of $G(n, p)$ for p in the critical window, that is the largest distance between a pair of vertices lying in the same component of the graph.

Nachmias and Peres (2008) showed that $D_n = \Theta(n^{1/3})$. (Also follows from results of Addario-Berry, Broutin and Reed.)

Our convergence result allows us to prove that

$$n^{-1/3} D_n \xrightarrow{d} D$$

as $n \to \infty$, where D is an absolutely continuous random variable with finite mean.

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We explore the graph step-by-step. At each step, the vertices may be in one of four states: unexplored, current, alive or dead.

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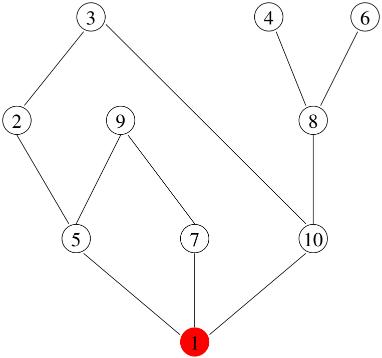
We explore the graph step-by-step. At each step, the vertices may be in one of four states: unexplored, current, alive or dead.

Let $X(k)$ be the number of vertices alive at step k.

Depth-first exploration

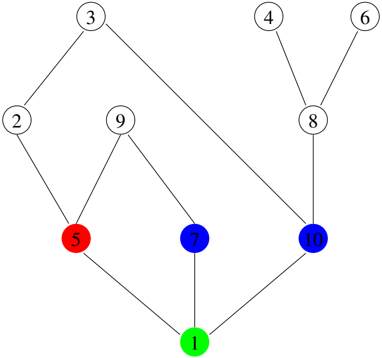
Step 0: initialization



Current: 1 Alive: none Dead: none $X(0) = 0$.

Depth-first exploration

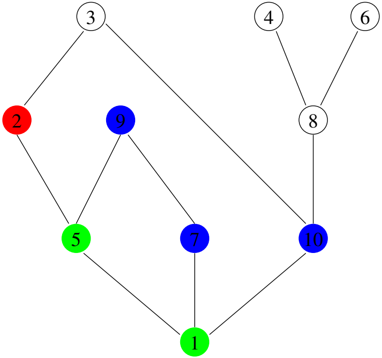
Step 1



Current: 5 Alive: 7, 10 Dead: 1 $X(1) = 2$.

Depth-first exploration

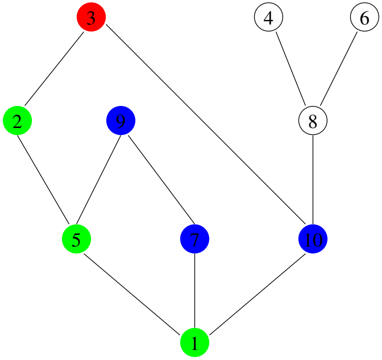
Step 2



Current: 2   Alive: 9, 7, 10   Dead: 1, 5 $X(2) = 3$.

Depth-first exploration

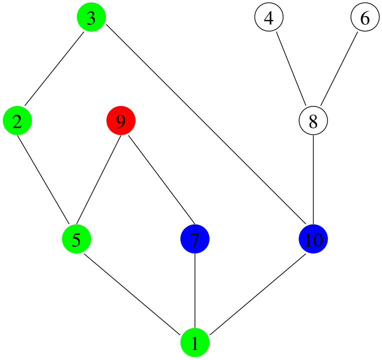
Step 3



Current: 3 Alive: 9, 7, 10 Dead: 1, 5, 2 $X(3) = 3$.

Depth-first exploration

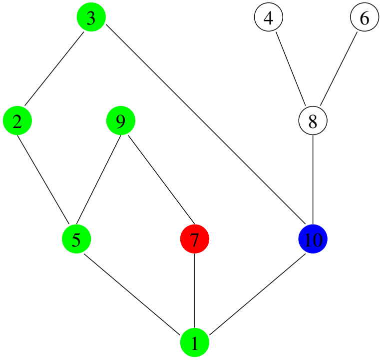
Step 4



Current: 9 Alive: 7, 10 Dead: 1, 5, 2, 3 $X(4) = 2$.

Depth-first exploration

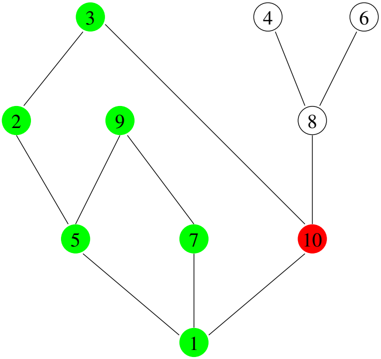
Step 5



Current: 7 Alive: 10 Dead: 1, 5, 2, 3, 9 $X(5) = 1$.

Depth-first exploration

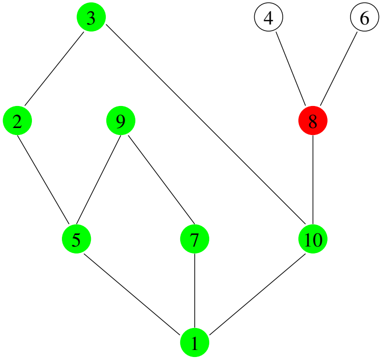
Step 6



Current: 10   Alive: none   Dead: 1, 5, 2, 3, 9, 7 $X(6) = 0$.

Depth-first exploration

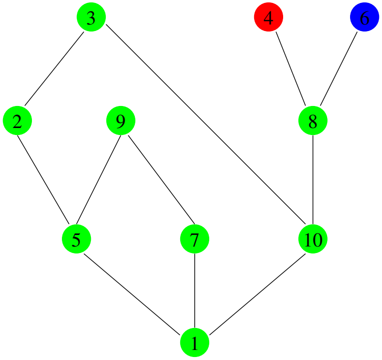
Step 7



Current: 8 Alive: none Dead: 1, 5, 2, 3, 9, 7, 10 $X(7) = 0$.

Depth-first exploration

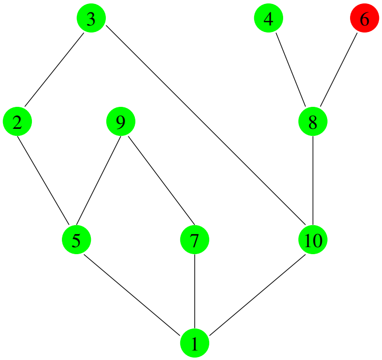
Step 8



Current: 4   Alive: 6   Dead: 1, 5, 2, 3, 9, 7, 10, 8 $X(8) = 1$.

Depth-first exploration

Step 9

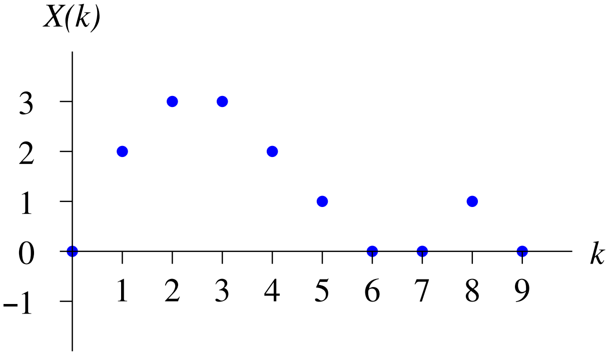


Current: 6 Alive: none Dead: 1, 5, 2, 3, 9, 7, 10, 8, 4

$X(9) = 0$.

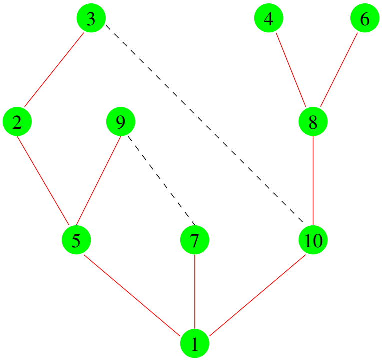
Depth-first walk

$X(k)$ = the number of vertices alive at the kth step of the depth-first exploration.



Depth-first tree

We essentially explored this tree; the dashed edges made no difference to the depth-first exploration.



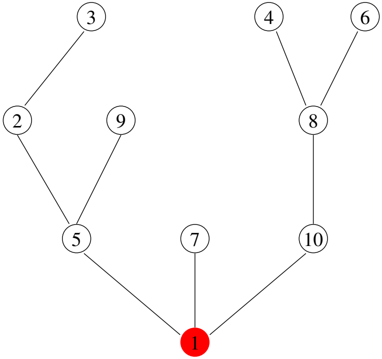
Call it the depth-first tree associated with the graph G, and write $T(G)$.

Permitted edges

For a given tree T, which connected graphs have depth-first tree T? In other words, where can we put surplus edges so that they don't change T?

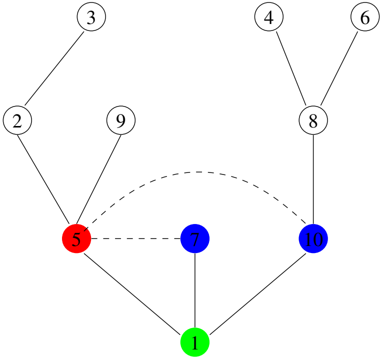
Call such edges permitted.

Depth-first walk and permitted edges



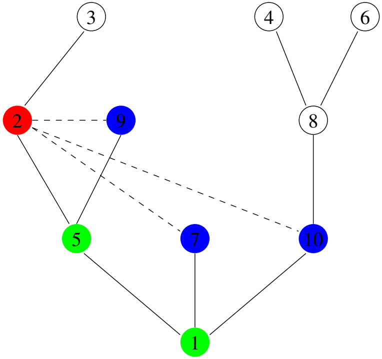
Step 0: $X(0) = 0$.

Depth-first walk and permitted edges



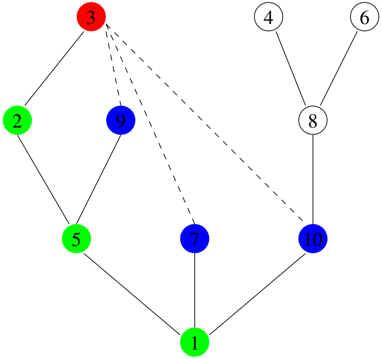
Step 1: $X(1) = 2$.

Depth-first walk and permitted edges



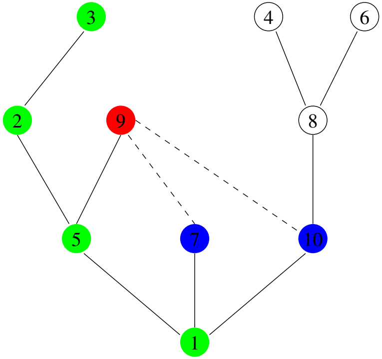
Step 2: $X(2) = 3$.

Depth-first walk and permitted edges



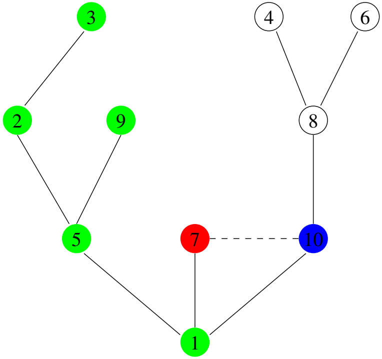
Step 3: $X(3) = 3$.

Depth-first walk and permitted edges



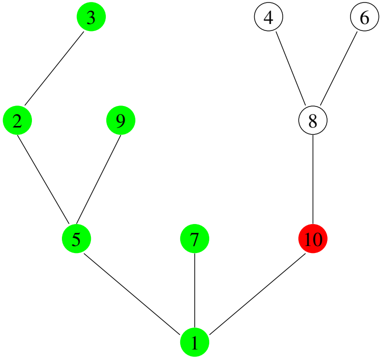
Step 4: $X(4) = 2$.

Depth-first walk and permitted edges



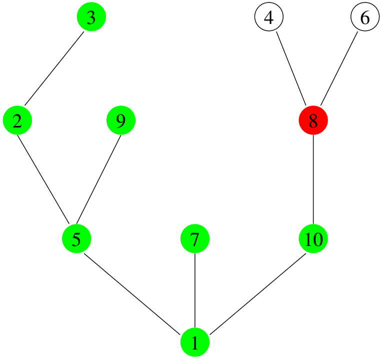
Step 5: $X(5) = 1$.

Depth-first walk and permitted edges



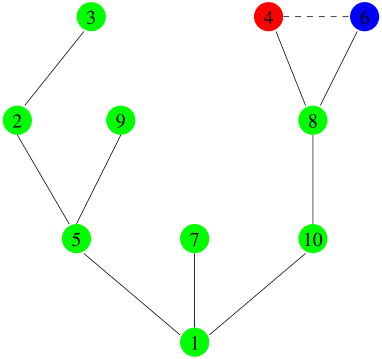
Step 6: $X(6) = 0$.

Depth-first walk and permitted edges



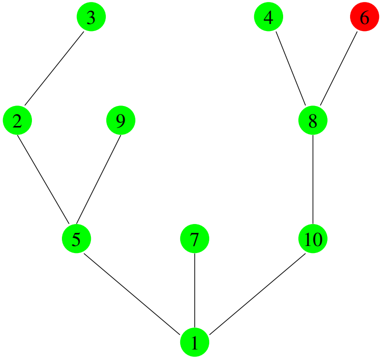
Step 7: $X(7) = 0$.

Depth-first walk and permitted edges



Step 8: $X(8) = 1$.

Depth-first walk and permitted edges



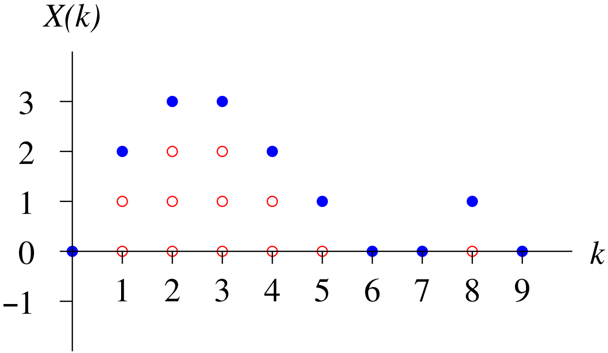
Step 10: $X(9) = 0$.

Area

At step $k \geq 0$ there are $X(k)$ permitted edges. So the total number is

$$a(T) = \sum_{k=0}^{m-1} X(k).$$

We call this the area of T.



Classifying graphs by depth-first tree

Let $\mathbb{G}_T$ be the set of graphs G such that $T(G) = T$. It follows that $|\mathbb{G}_T| = 2^{a(T)}$.

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Let $\mathbb{T}_{[m]}$ be the set of trees with label-set $[m] = \{1, 2, \ldots, m\}$. Then

$$\left\{\mathbb{G}_T : T \in \mathbb{T}_{[m]}\right\}$$

is a partition of the set of connected graphs on $[m]$.

Recipe for creating a connected graph on $[m]$

Create a connected graph $\tilde{G}_m^p$ as follows.

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$$\mathbb{P}\left(\tilde{T}_m^p = T\right) \propto (1-p)^{-a(T)}, \quad T \in \mathbb{T}_{[m]}.$$

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- Add each of the $a(\tilde{T}_m^p)$ permitted edges to $\tilde{T}_m^p$ independently with probability p.

Recipe for creating a connected graph on $[m]$

Lemma. $\tilde{G}^p_m$ has the same distribution as G^p_m, a component of $G(n, p)$ conditioned to have vertex-set $[m]$.

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Proof. For a connected graph G on $[m]$ which has $T(G) = T$ and surplus s,

$$\mathbb{P}\left(\tilde{G}_m^p = G\right) \propto (1-p)^{-a(T)} p^s (1-p)^{a(T)-s} = (p/(1-p))^s.$$

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Likewise, by the definition of $G(n,p)$,

$$\begin{aligned}\mathbb{P}\left(G_m^p = G\right) \propto \mathbb{P}\left(G(m,p) = G\right) &= p^{m+s-1}(1-p)^{\binom{m}{2}-s+1} \\ &\propto (p/(1-p))^s.\end{aligned}$$

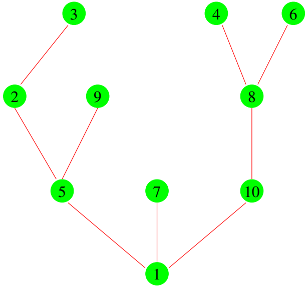
$\square$

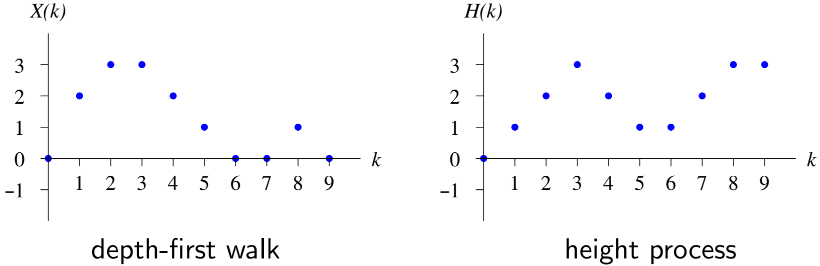
Taking limits

So we need to prove that

- the tree $\tilde{T}_m^p$ converges to a CRT coded by a tilted excursion;
- the locations of the surplus edges converge to the locations in our limiting picture.

Important aside: two different encodings of discrete trees





Important aside: two different encodings of discrete trees

Let T_m be a uniform random tree on $[m]$ and let $(X^m(k), 0 \leq k \leq m)$ and $(H^m(k), 0 \leq k \leq m)$ be its depth-first walk and height process respectively.

Theorem. (Marckert and Mokkadem (2003))
As $m \to \infty$,

$$((m^{-1/2} X^m(mt), 0 \leq t \leq 1), (m^{-1/2} H^m(mt), 0 \leq t \leq 1))$$
$$\xrightarrow{d} ((e(t), 0 \leq t \leq 1), (2e(t), 0 \leq t \leq 1)),$$

where $(e(t), 0 \leq t \leq 1)$ is a standard Brownian excursion.

Taking limits

We need to prove that

- the tree $\tilde{T}_m^p$ converges to a CRT coded by a tilted excursion;
- the locations of the surplus edges converge to the locations in our limiting picture.

We will deal with the tree first. For simplicity, we will take $p = m^{-3/2}$; the general case is similar.

Part 1: Convergence of the tree

Write $\tilde{X}^m$ for the depth-first walk associated with $\tilde{T}_m^p$, thought of as a càdlàg function $[0, m] \to \mathbb{R}^+$. Then

$$a\left(\tilde{T}_m^p\right) = \int_0^m \tilde{X}^m(s)ds.$$

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$$a\left(\tilde{T}^p_m\right) = \int_0^m \tilde{X}^m(s)ds.$$

Recall that T_m is a uniform random tree on $[m]$ and that X^m is its depth-first walk. Then

$$(m^{-1/2}X^m(mt), 0 \leq t \leq 1) \xrightarrow{d} (e(t), 0 \leq t \leq 1).$$

Part 1: Convergence of the tree

Now use the change of measure to get from $\tilde{X}^m$ to X^m: for any bounded continuous function f,

$$\mathbb{E}\left[f\left(m^{-1/2}\tilde{X}^m(mt), 0 \le t \le 1\right)\right]$$
$$= \frac{\mathbb{E}\left[f\left(m^{-1/2}X^m(mt), 0 \le t \le 1\right)(1-p)^{-\int_0^m X^m(u)du}\right]}{\mathbb{E}\left[(1-p)^{-\int_0^m X^m(u)du}\right]}$$

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&= \frac{\mathbb{E}\left[f\left(m^{-1/2}X^m(mt), 0 \le t \le 1\right)(1-p)^{-m^{3/2}\int_0^1 m^{-1/2}X^m(ms)ds}\right]}{\mathbb{E}\left[(1-p)^{-m^{3/2}\int_0^1 m^{-1/2}X^m(ms)ds}\right]}
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$$= \frac{\mathbb{E}\left[f\left(m^{-1/2}X^m(mt), 0 \le t \le 1\right)(1-p)^{-m^{3/2}\int_0^1 m^{-1/2}X^m(ms)ds}\right]}{\mathbb{E}\left[(1-p)^{-m^{3/2}\int_0^1 m^{-1/2}X^m(ms)ds}\right]}$$

Since $(m^{-1/2}X^m(mt), 0 \le t \le 1) \xrightarrow{d} (e(t), 0 \le t \le 1)$ and $p = m^{-3/2}$,

$$(1-p)^{-m^{3/2}\int_0^1 m^{-1/2}X^m(ms)ds} \xrightarrow{d} \exp\left(\int_0^1 e(u)du\right).$$

Part 1: Convergence of the tree

Taking care with the limits, we obtain

$$\mathbb{E}\left[f\left(m^{-1/2}\tilde{X}^m(mt), 0 \le t \le 1\right)\right] \to \frac{\mathbb{E}\left[f(e)\exp\left(\int_0^1 e(u)du\right)\right]}{\mathbb{E}\left[\exp\left(\int_0^1 e(u)du\right)\right]}$$

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As in the uniform case, we get jointly

$$(m^{-1/2}\tilde{H}^m(mt), 0 \leq t \leq 1) \xrightarrow{d} (2\tilde{e}(t), 0 \leq t \leq 1)$$

(where the limit is the *same* tilted excursion).

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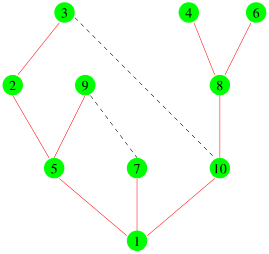
(where the limit is the *same* tilted excursion).

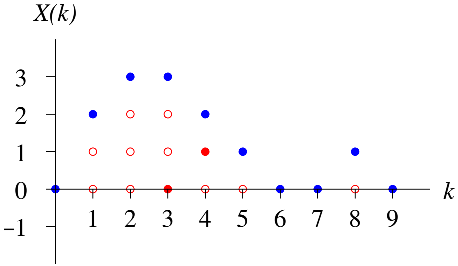
This is basically enough to prove that

$$\frac{1}{\sqrt{m}}\tilde{T}_m^p \xrightarrow{d} \tilde{\mathcal{T}}.$$

Part 2: Surplus edges

The permitted edges are in bijective correspondence with the integer points under the graph of the depth-first walk. Since each permitted edge is included independently with probability p, the surplus edges form a Binomial point process.





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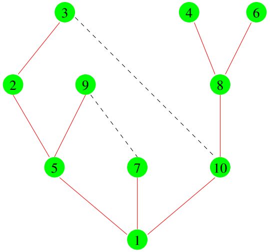
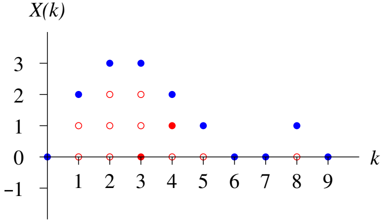
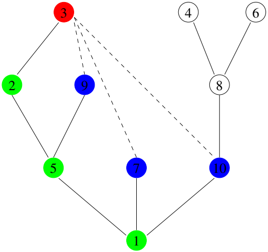
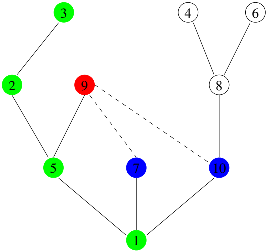




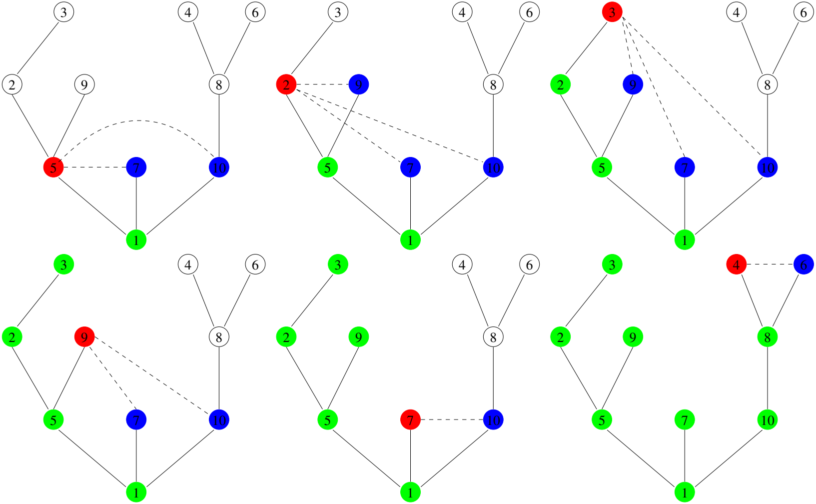




A point at (k, j) means "put an edge between the current vertex at step k and the vertex at distance j from the bottom of the list of alive vertices".

Part 2: Surplus edges

Surplus edges almost go to ancestors... In fact, they go to younger sons of ancestors of the current vertex.



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When we rescale, the distance between a vertex and one of its sons becomes small and so, in the limit, surplus "edges" do go to ancestors of the current vertex.

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The Binomial point process of surplus edges, when rescaled, straightforwardly converges to the required Poisson point process.

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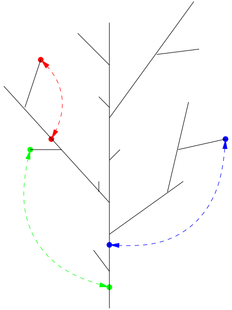
The Binomial point process of surplus edges, when rescaled, straightforwardly converges to the required Poisson point process.

The difference between the depth-first walk and the height process is also small, and so the locations of the surplus "edges" are essentially as described in our limit process.

□

Papers



The continuum limit of critical random graphs
L. Addario-Berry, N. Broutin and C. Goldschmidt
`arXiv:0903.4730 [math.PR]`.

Critical random graphs: limiting constructions and distributional properties
L. Addario-Berry, N. Broutin and C. Goldschmidt
`arXiv:0908.3629 [math.PR]`.