

The scaling limit of critical random graphs

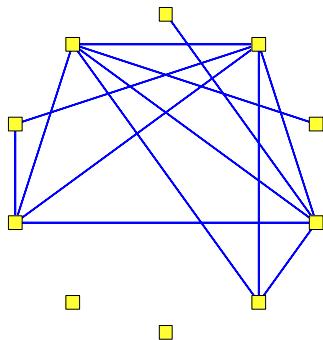
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Joint work with Louigi Addario-Berry (McGill University) and
Nicolas Broutin (INRIA Rocquencourt)

The Erdős-Rényi random graph

Take n vertices labelled by $[n] := \{1, 2, \dots, n\}$ and put an edge between any pair independently with probability p . Call the resulting model $G(n, p)$.

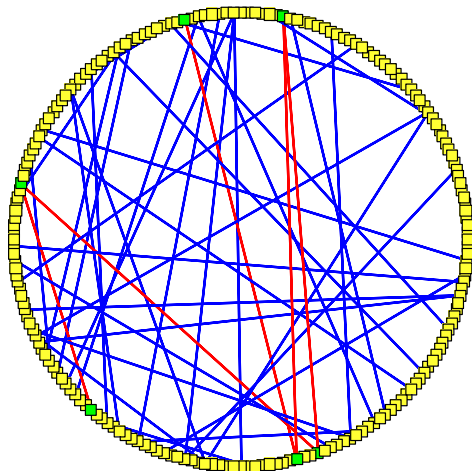
Example: $n = 10$, $p = 0.4$ (vertex labels omitted).



The phase transition

Let $p = c/n$ and consider the largest component (vertices in green, edges in red).

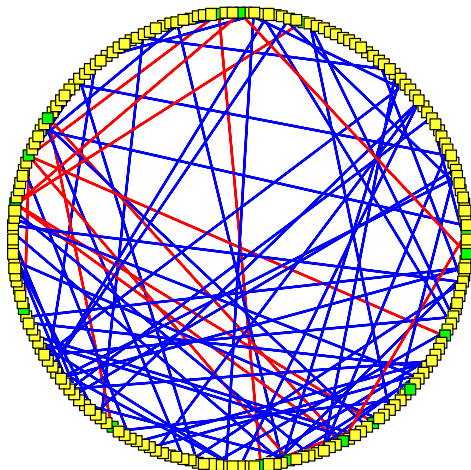
$n = 200$, $c = 0.4$



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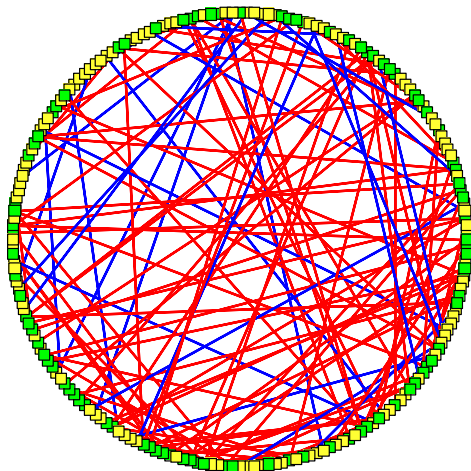
$n = 200$, $c = 0.8$



The phase transition

Let $p = c/n$ and consider the largest component (vertices in green, edges in red).

$n = 200$, $c = 1.2$



The phase transition (Erdős and Rényi (1960))

Consider $p = c/n$.

- ▶ For $c < 1$, the largest connected component has size $O(\log n)$;
- ▶ for $c > 1$, the largest connected component has size $\Theta(n)$ (and the others are all $O(\log n)$).

The critical random graph

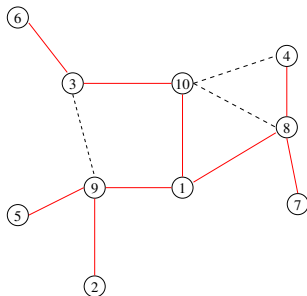
The **critical window**: $p = \frac{1}{n} + \frac{\lambda}{n^{4/3}}$, where $\lambda \in \mathbb{R}$. For such p , the largest components have size $\Theta(n^{2/3})$.

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We will also be interested in the **surplus** of a component, the number of edges more than a tree that it has.

A component with surplus 3:



Convergence of the sizes and surpluses

Fix λ and let C_1^n, C_2^n, \dots be the sequence of component sizes in decreasing order, and let S_1^n, S_2^n, \dots be their surpluses.

Write $\mathbf{C}^n = (C_1^n, C_2^n, \dots)$ and $\mathbf{S}^n = (S_1^n, S_2^n, \dots)$.

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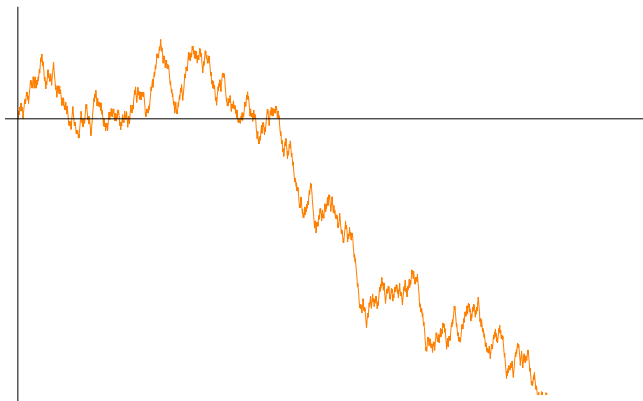
$$\ell^2 := \left\{ \mathbf{x} = (x_1, x_2, \dots) : x_1 \geq x_2 \geq \dots \geq 0, \sum_{i=1}^{\infty} x_i^2 < \infty \right\}.$$

Limiting sizes and surpluses

Let $W^\lambda(t) = W(t) + \lambda t - \frac{t^2}{2}$, $t \geq 0$, where $(W(t), t \geq 0)$ is a standard Brownian motion.

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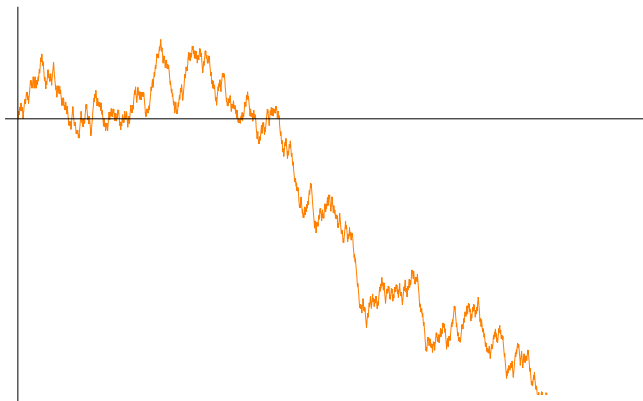
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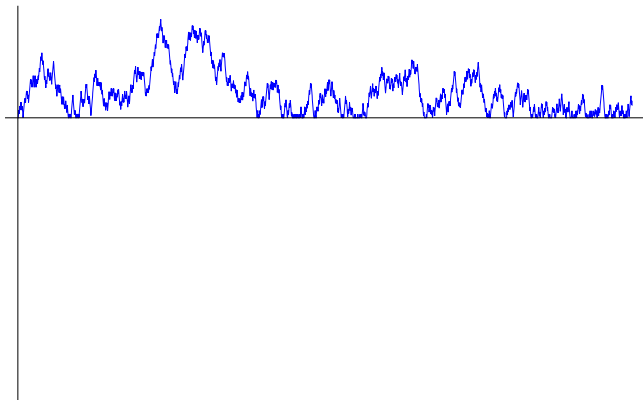
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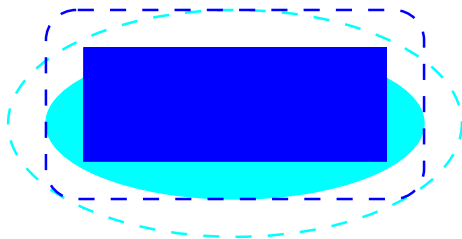
The vertex-labels are irrelevant: we are really interested in what **distances** look like in the limit. So we will give a metric space answer.

Measuring the distance between metric spaces

The **Hausdorff distance** between two compact subsets K and K' of a metric space (M, δ) is

$$d_H(K, K') = \inf\{\epsilon > 0 : K \subseteq F_\epsilon(K'), K' \subseteq F_\epsilon(K)\},$$

where $F_\epsilon(K) := \{x \in M : \delta(x, K) \leq \epsilon\}$ is the ϵ -fattening of K .



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So define the **Gromov-Hausdorff distance**

$$d_{GH}(X, X') = \inf\{d_H(\phi(X), \phi'(X'))\},$$

where the infimum is taken over all choices of metric space (M, δ) and all isometric embeddings $\phi : X \rightarrow M$, $\phi' : X' \rightarrow M$.

Our approach

Simple but important fact: a component of $G(n, p)$ conditioned to have m vertices and s surplus edges is a uniform connected graph on those m vertices with $m + s - 1$ edges.

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Our general approach is to pick out a (well-chosen) spanning tree, and then to put in the surplus edges.

There is one case which we already understand very well: when the surplus of a component is 0 and so we have a uniform random tree.

Warm-up: the tree case

Take a uniform random tree T_m on vertices labelled by $[m] = \{1, 2, \dots, m\}$ and, using the ordinary graph distance, think of it as a metric space.

Warm-up: the tree case

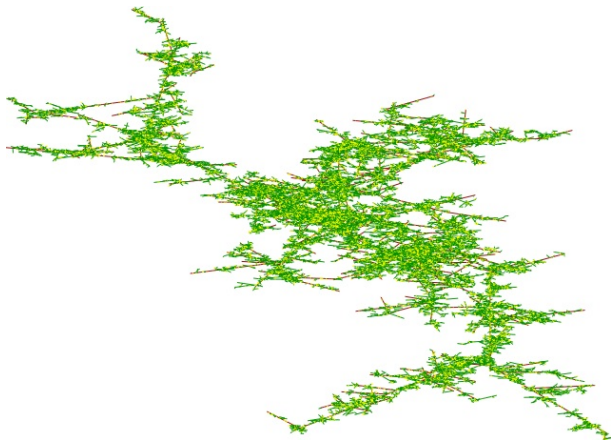
Theorem. (Aldous (1993), Le Gall (2005)) As $m \rightarrow \infty$,

$$\frac{1}{\sqrt{m}} T_m \xrightarrow{d} \mathcal{T},$$

where the convergence is in the Gromov-Hausdorff distance.

The limit \mathcal{T} is the **Brownian continuum random tree**.

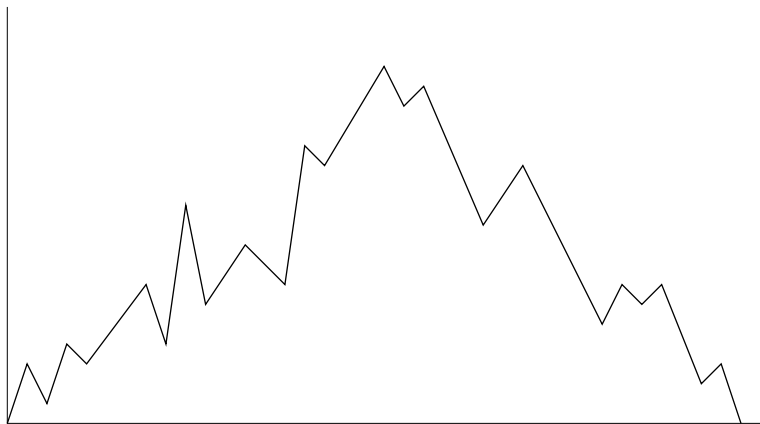
The Brownian continuum random tree



[Picture by Grégory Miermont]

Trees from excursions

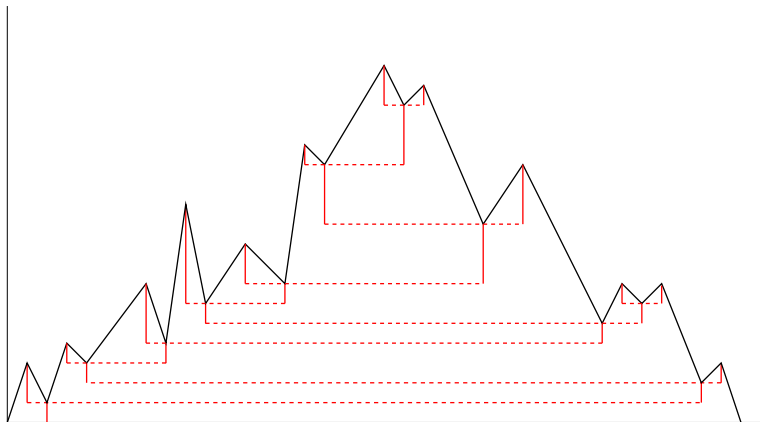
Let $h : [0, 1] \rightarrow \mathbb{R}^+$ be an **excursion**, that is a continuous function such that $h(0) = h(1) = 0$ and $h(x) > 0$ for $x \in (0, 1)$.



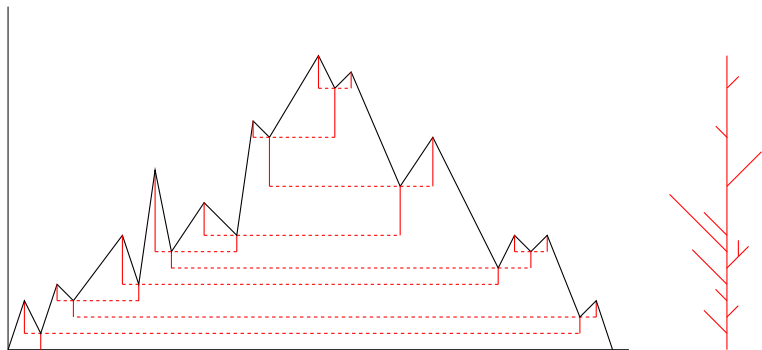
Trees from excursions

Define a distance d on $[0, 1]$ via

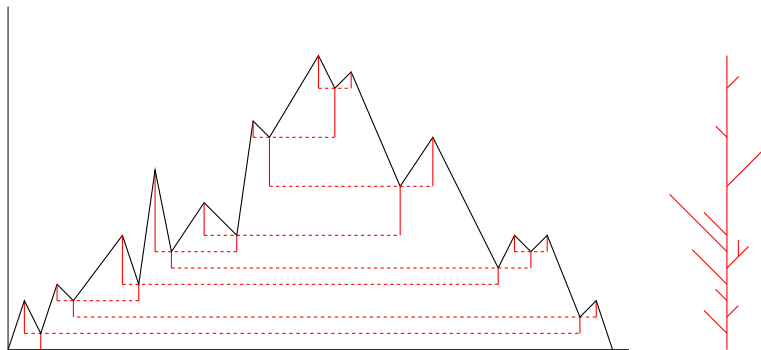
$$d(x, y) = h(x) + h(y) - 2 \inf_{x \wedge y \leq z \leq x \vee y} h(z).$$



Trees from excursions



Trees from excursions



Define an equivalence relation \sim by $x \sim y$ if $d(x, y) = 0$ and take the quotient $\mathcal{T}_h = [0, 1] / \sim$.

The Brownian continuum random tree is \mathcal{T}_h with $h(x) = 2e(x)$ and $(e(x), 0 \leq x \leq 1)$ a standard Brownian excursion.

The limit of the random graph

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In the limit, surplus edges correspond to vertex-identifications (since edge-lengths have shrunk to 0). In each excursion, the points of the Poisson process tell us where these vertex-identifications should occur.

Excursions of the limit process

Consider the process $(B^\lambda(t), t \geq 0)$. An excursion $\tilde{e}^{(x)}$ of this process, conditioned to have length x , has a distribution specified by

$$\mathbb{E} \left[f \left(\tilde{e}^{(x)} \right) \right] = \frac{\mathbb{E} \left[f \left(e^{(x)} \right) \exp \left(\int_0^x e^{(x)}(u) du \right) \right]}{\mathbb{E} \left[\exp \left(\int_0^x e^{(x)}(u) du \right) \right]},$$

where f is any suitable test-function and $e^{(x)}$ is a Brownian excursion of length x .

Excursions of the limit process

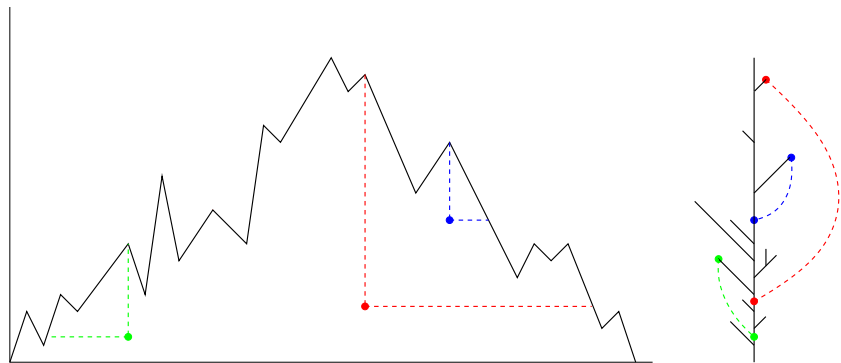
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where f is any suitable test-function and $e^{(x)}$ is a Brownian excursion of length x .

We refer to $\tilde{e}^{(x)}$ as a **tilted excursion** and to the tree $\tilde{\mathcal{T}}$ that it encodes as a **tilted tree**.

Vertex identifications



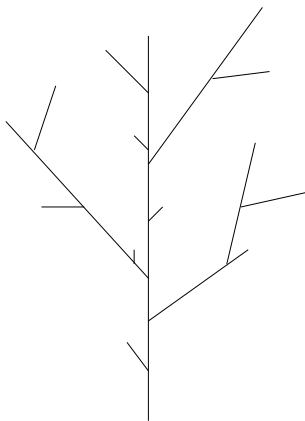
A point at (x, y) identifies the vertex v at height $h(x)$ with the vertex at distance y along the path from the root to v .

A limiting component

Note that it follows from properties of the tilted trees and of the Poisson process that we may equivalently describe the limit of a component on $\sim xn^{2/3}$ vertices as follows.

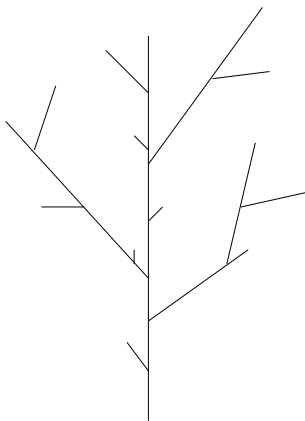
A limiting component

Sample a tilted excursion $\tilde{e}^{(x)}$ of length x and use it to create a CRT $\tilde{\mathcal{T}}$.



A limiting component

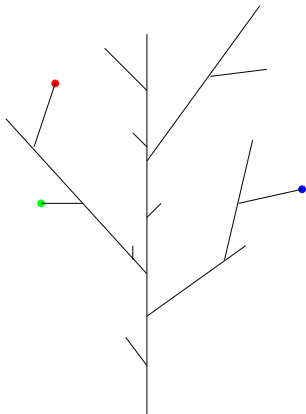
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Conditional on $\tilde{e}^{(x)}$, sample a random variable P with Poisson $(\int_0^x \tilde{e}^{(x)}(u) du)$ distribution.

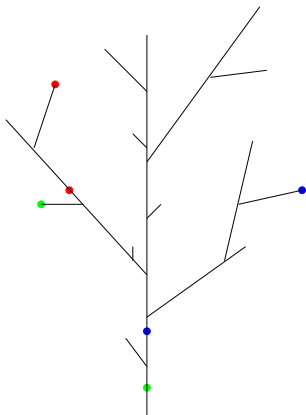
A limiting component

Conditional on $P = s$, pick s vertices of the tree \tilde{T} independently with density proportional to their height. (These will almost surely be leaves.)



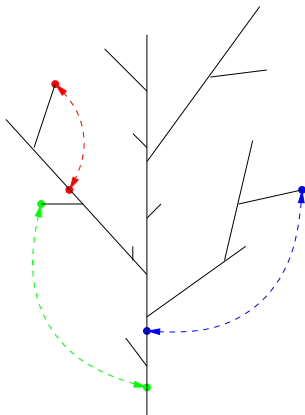
A limiting component

For each of the selected leaves, pick a uniform point on the path from the leaf to the root.



A limiting component

Identify each of the selected leaves with its chosen point.



Convergence result

Let $\mathcal{C}_1^n, \mathcal{C}_2^n, \dots$ be the sequence of components of $G(n, p)$ in decreasing order of size, considered as metric spaces with the graph distance.

Theorem. As $n \rightarrow \infty$,

$$n^{-1/3}(\mathcal{C}_1^n, \mathcal{C}_2^n, \dots) \xrightarrow{d} (\mathcal{C}_1, \mathcal{C}_2, \dots),$$

where $\mathcal{C}_1, \mathcal{C}_2, \dots$ is the sequence of metric spaces corresponding to the excursions of Aldous' marked limit process in decreasing order of length.

Here, convergence is with respect to the metric

$$d(\mathcal{A}, \mathcal{B}) := \left(\sum_{i=1}^{\infty} d_{GH}(\mathcal{A}_i, \mathcal{B}_i)^4 \right)^{1/4}.$$

Diameter

Let D_n be the **diameter** of $G(n, p)$ for p in the critical window, that is the largest distance between a pair of vertices lying in the same component of the graph.

Nachmias and Peres (2008) showed that $D_n = \Theta(n^{1/3})$. (Also follows from results of Addario-Berry, Broutin and Reed.)

Our convergence result allows us to prove that

$$n^{-1/3} D_n \xrightarrow{d} D$$

as $n \rightarrow \infty$, where D is an absolutely continuous random variable with finite mean.

Idea of proof

The key idea turns out to be study a component of $G(n, p)$ conditioned on its size but *not* on its surplus.

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We will begin by taking an arbitrary connected graph G and picking out a particular spanning tree via a **depth-first exploration**.

We explore the graph step-by-step. At each step, the vertices may be in one of four states: unexplored, **current**, **alive** or **dead**.

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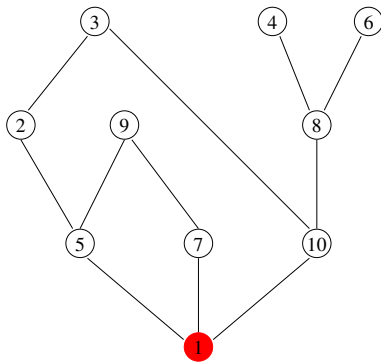
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We explore the graph step-by-step. At each step, the vertices may be in one of four states: unexplored, **current**, **alive** or **dead**.

Let $X(k)$ be the number of vertices alive at step k .

Depth-first exploration

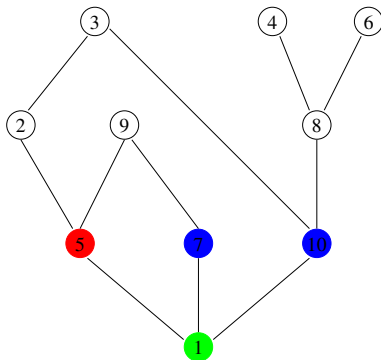
Step 0: initialization



Current: 1 Alive: none Dead: none $X(0) = 0$.

Depth-first exploration

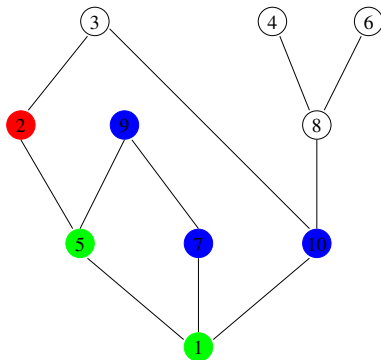
Step 1



Current: 5 Alive: 7, 10 Dead: 1 $X(1) = 2$.

Depth-first exploration

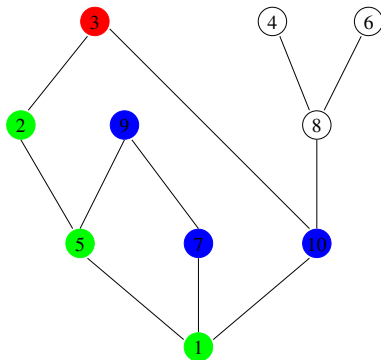
Step 2



Current: 2 Alive: 9, 7, 10 Dead: 1, 5 $X(2) = 3$.

Depth-first exploration

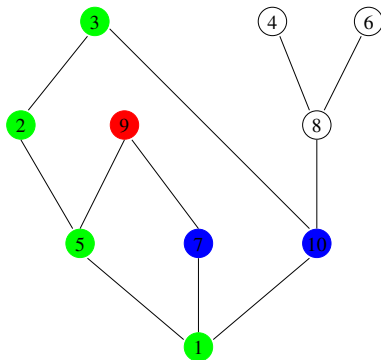
Step 3



Current: 3 Alive: 9, 7, 10 Dead: 1, 5, 2 $X(3) = 3$.

Depth-first exploration

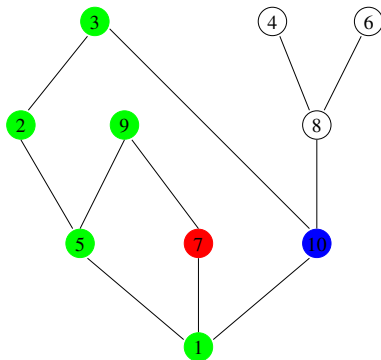
Step 4



Current: 9 Alive: 7, 10 Dead: 1, 5, 2, 3 $X(4) = 2$.

Depth-first exploration

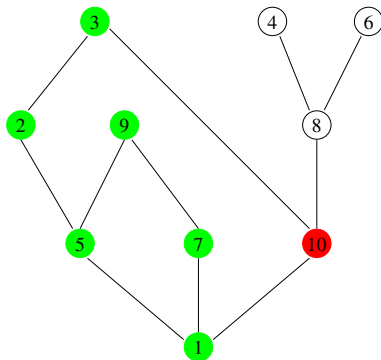
Step 5



Current: 7 Alive: 10 Dead: 1, 5, 2, 3, 9 $X(5) = 1$.

Depth-first exploration

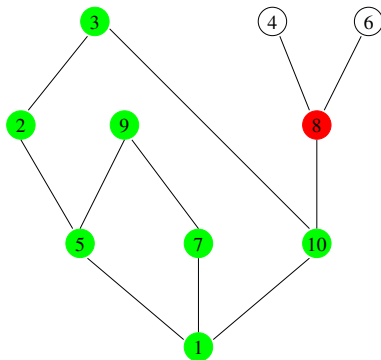
Step 6



Current: 10 Alive: none Dead: 1, 5, 2, 3, 9, 7 $X(6) = 0$.

Depth-first exploration

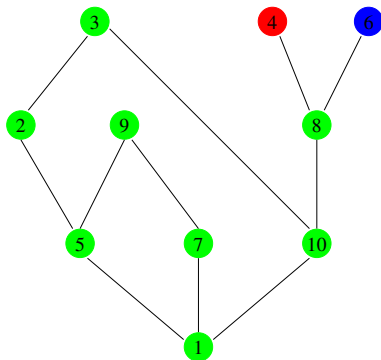
Step 7



Current: 8 Alive: none Dead: 1, 5, 2, 3, 9, 7, 10 $X(7) = 0$.

Depth-first exploration

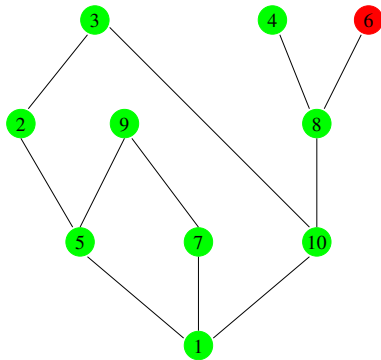
Step 8



Current: 4 Alive: 6 Dead: 1, 5, 2, 3, 9, 7, 10, 8 $X(8) = 1$.

Depth-first exploration

Step 9

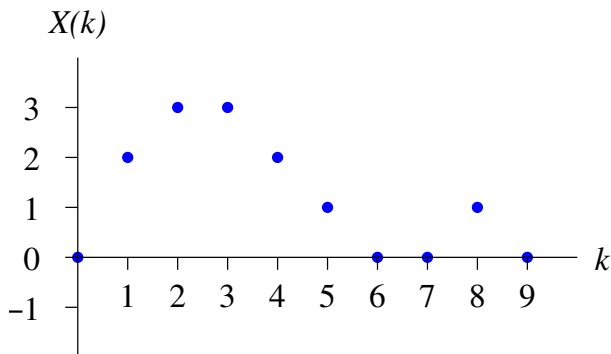


Current: 6 Alive: none Dead: 1, 5, 2, 3, 9, 7, 10, 8, 4

$X(9) = 0$.

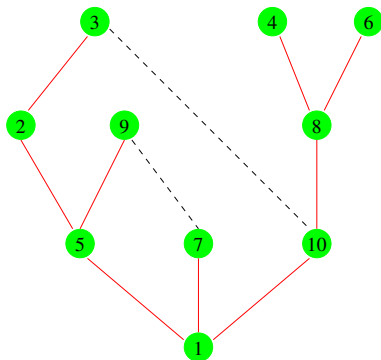
Depth-first walk

$X(k)$ = the number of vertices **alive** at the k th step of the depth-first exploration.



Depth-first tree

We essentially explored this tree; the dashed edges made no difference to the depth-first exploration.



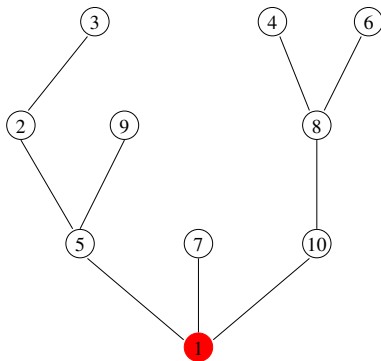
Call it the **depth-first tree** associated with the graph G , and write $T(G)$.

Permitted edges

For a given tree T , which connected graphs have depth-first tree T ? In other words, where can we put surplus edges so that they don't change T ?

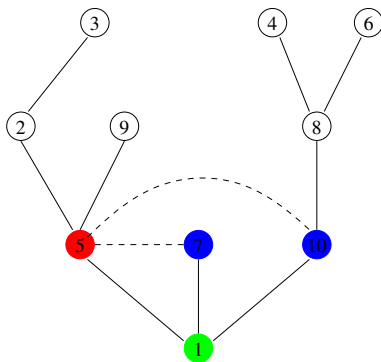
Call such edges **permitted**.

Depth-first walk and permitted edges



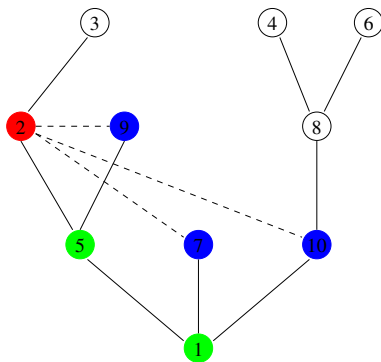
Step 0: $X(0) = 0$.

Depth-first walk and permitted edges



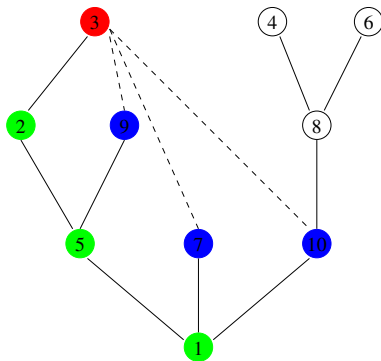
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Depth-first walk and permitted edges



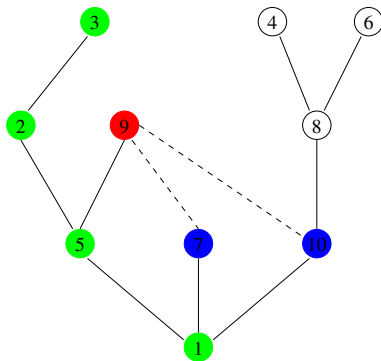
Step 2: $X(2) = 3$.

Depth-first walk and permitted edges



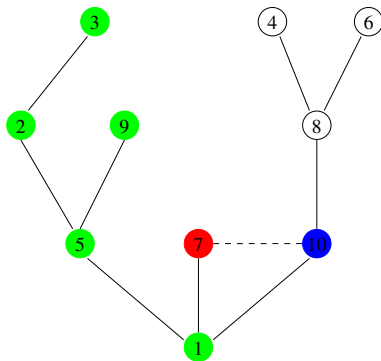
Step 3: $X(3) = 3$.

Depth-first walk and permitted edges



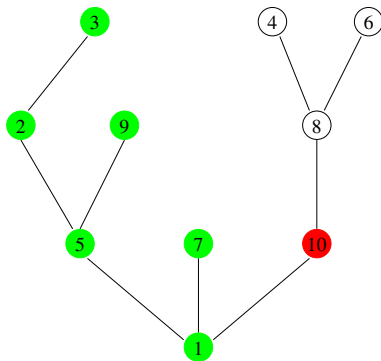
Step 4: $X(4) = 2$.

Depth-first walk and permitted edges



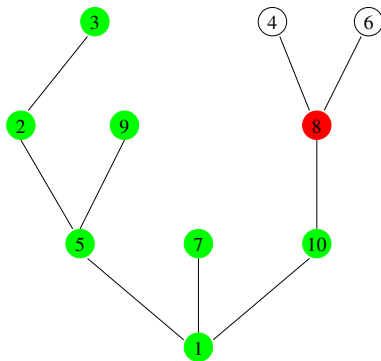
Step 5: $X(5) = 1$.

Depth-first walk and permitted edges



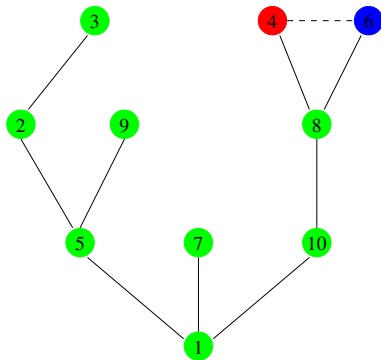
Step 6: $X(6) = 0$.

Depth-first walk and permitted edges



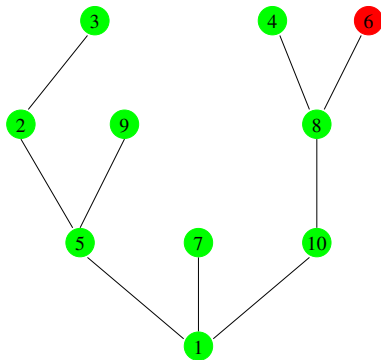
Step 7: $X(7) = 0$.

Depth-first walk and permitted edges



Step 8: $X(8) = 1$.

Depth-first walk and permitted edges



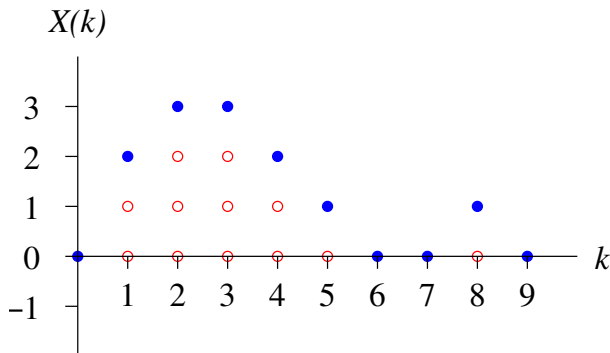
Step 10: $X(9) = 0$.

Area

At step $k \geq 0$ there are $X(k)$ permitted edges. So the total number is

$$a(T) = \sum_{k=0}^{m-1} X(k).$$

We call this the **area** of T .



Classifying graphs by depth-first tree

Let \mathbb{G}_T be the set of graphs G such that $T(G) = T$. It follows that $|\mathbb{G}_T| = 2^{a(T)}$.

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Let $\mathbb{T}_{[m]}$ be the set of trees with label-set $[m] = \{1, 2, \dots, m\}$.
Then

$$\{\mathbb{G}_T : T \in \mathbb{T}_{[m]}\}$$

is a partition of the set of connected graphs on $[m]$.

Recipe for creating a connected graph on $[m]$

Create a connected graph \tilde{G}_m^P as follows.

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$$\mathbb{P}\left(\tilde{T}_m^p = T\right) \propto (1 - p)^{-a(T)}, \quad T \in \mathbb{T}_{[m]}.$$

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- ▶ Add each of the $a(\tilde{T}_m^p)$ permitted edges to \tilde{T}_m^p independently with probability p .

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Lemma. \tilde{G}_m^p has the same distribution as G_m^p , a component of $G(n, p)$ conditioned to have vertex-set $[m]$.

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Proof. For a connected graph G on $[m]$ which has $T(G) = T$ and surplus s ,

$$\mathbb{P}\left(\tilde{G}_m^p = G\right) \propto (1-p)^{-a(T)} p^s (1-p)^{a(T)-s} = (p/(1-p))^s.$$

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Likewise, by the definition of $G(n, p)$,

$$\begin{aligned} \mathbb{P}(G_m^p = G) &\propto \mathbb{P}(G(m, p) = G) = p^{m+s-1} (1-p)^{\binom{m}{2}-s+1} \\ &\propto (p/(1-p))^s. \end{aligned}$$

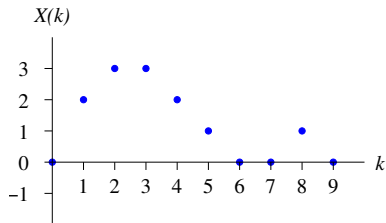
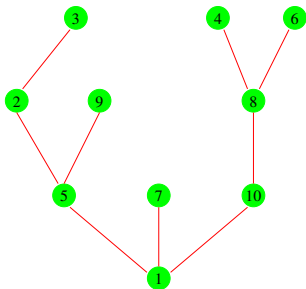
□

Taking limits

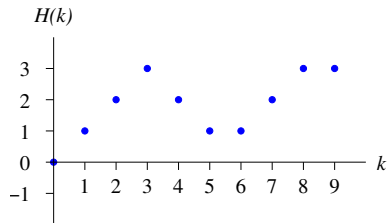
So we need to prove that

- ▶ the tree \tilde{T}_m^p converges to a CRT coded by a tilted excursion;
- ▶ the locations of the surplus edges converge to the locations in our limiting picture.

Important aside: two different encodings of discrete trees



depth-first walk



height process

Important aside: two different encodings of discrete trees

Let T_m be a uniform random tree on $[m]$ and let $(X^m(k), 0 \leq k \leq m)$ and $(H^m(k), 0 \leq k \leq m)$ be its depth-first walk and height process respectively.

Theorem. (Marckert and Mokkadem (2003))

As $m \rightarrow \infty$,

$$\begin{aligned} & ((m^{-1/2}X^m(mt), 0 \leq t \leq 1), (m^{-1/2}H^m(mt), 0 \leq t \leq 1)) \\ & \xrightarrow{d} ((e(t), 0 \leq t \leq 1), (2e(t), 0 \leq t \leq 1)), \end{aligned}$$

where $(e(t), 0 \leq t \leq 1)$ is a standard Brownian excursion.

Taking limits

We need to prove that

- ▶ the tree \tilde{T}_m^p converges to a CRT coded by a tilted excursion;
- ▶ the locations of the surplus edges converge to the locations in our limiting picture.

We will deal with the tree first. For simplicity, we will take $p = m^{-3/2}$; the general case is similar.

Part 1: Convergence of the tree

Write \tilde{X}^m for the depth-first walk associated with \tilde{T}_m^p , thought of as a càdlàg function $[0, m] \rightarrow \mathbb{R}^+$. Then

$$a\left(\tilde{T}_m^p\right) = \int_0^m \tilde{X}^m(s) ds.$$

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Recall that T_m is a uniform random tree on $[m]$ and that X^m is its depth-first walk. Then

$$(m^{-1/2}X^m(mt), 0 \leq t \leq 1) \xrightarrow{d} (e(t), 0 \leq t \leq 1).$$

Part 1: Convergence of the tree

Now use the change of measure to get from \tilde{X}^m to X^m : for any bounded continuous function f ,

$$\begin{aligned} & \mathbb{E} \left[f \left(m^{-1/2} \tilde{X}^m(mt), 0 \leq t \leq 1 \right) \right] \\ &= \frac{\mathbb{E} \left[f \left(m^{-1/2} X^m(mt), 0 \leq t \leq 1 \right) (1 - \rho)^{-\int_0^m X^m(u) du} \right]}{\mathbb{E} \left[(1 - \rho)^{-\int_0^m X^m(u) du} \right]} \end{aligned}$$

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Since $(m^{-1/2} X^m(mt), 0 \leq t \leq 1) \xrightarrow{d} (e(t), 0 \leq t \leq 1)$ and $\rho = m^{-3/2}$,

$$(1 - \rho)^{-m^{3/2} \int_0^1 m^{-1/2} X^m(ms) ds} \xrightarrow{d} \exp \left(\int_0^1 e(u) du \right).$$

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Taking care with the limits, we obtain

$$\begin{aligned}\mathbb{E} \left[f \left(m^{-1/2} \tilde{X}^m(mt), 0 \leq t \leq 1 \right) \right] &\rightarrow \frac{\mathbb{E} \left[f(e) \exp \left(\int_0^1 e(u) du \right) \right]}{\mathbb{E} \left[\exp \left(\int_0^1 e(u) du \right) \right]} \\ &= \mathbb{E} [f(\tilde{e})].\end{aligned}$$

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As in the uniform case, we get jointly

$$(m^{-1/2} \tilde{H}^m(mt), 0 \leq t \leq 1) \xrightarrow{d} (2\tilde{e}(t), 0 \leq t \leq 1)$$

(where the limit is the *same* tilted excursion).

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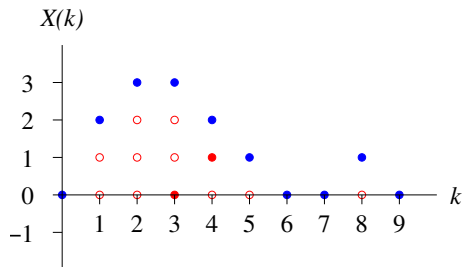
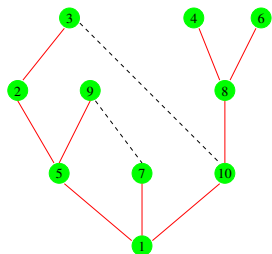
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This is basically enough to prove that

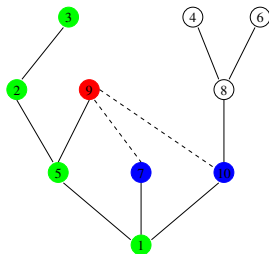
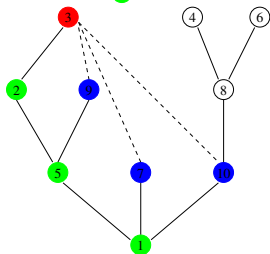
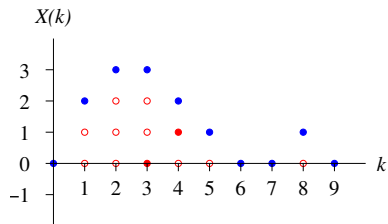
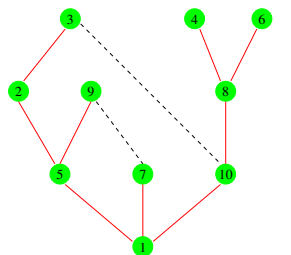
$$\frac{1}{\sqrt{m}} \tilde{T}_m^p \xrightarrow{d} \tilde{\mathcal{T}}.$$

Part 2: Surplus edges

The permitted edges are in bijective correspondence with the integer points under the graph of the depth-first walk. Since each permitted edge is included independently with probability p , the surplus edges form a Binomial point process.



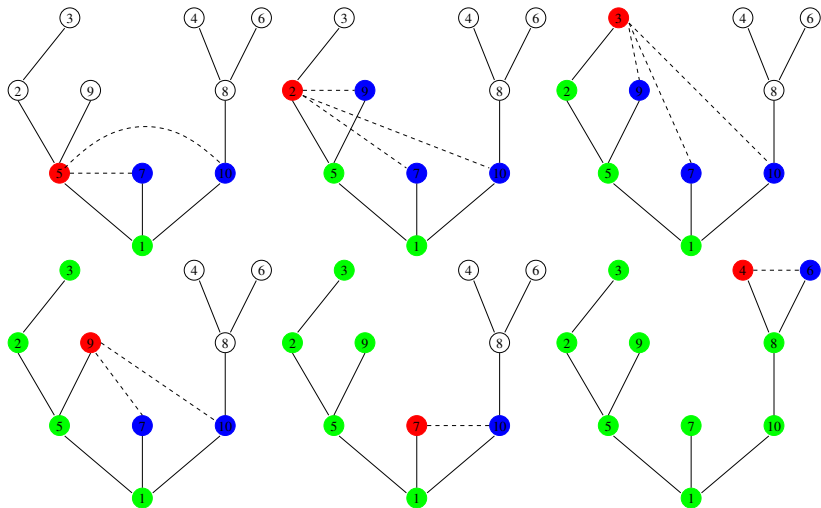
Part 2: Surplus edges



A point at (k, j) means “put an edge between the current vertex at step k and the vertex at distance j from the bottom of the list of alive vertices”.

Part 2: Surplus edges

Surplus edges almost go to ancestors... In fact, they go to younger sons of ancestors of the current vertex.



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The Binomial point process of surplus edges, when rescaled, straightforwardly converges to the required Poisson point process.

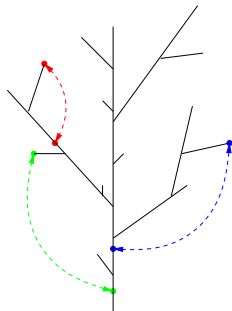
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The difference between the depth-first walk and the height process is also small, and so the locations of the surplus “edges” are essentially as described in our limit process.





The continuum limit of critical random graphs

L. Addario-Berry, N. Broutin and C. Goldschmidt
arXiv:0903.4730 [math.PR].

Critical random graphs: limiting constructions and distributional properties

L. Addario-Berry, N. Broutin and C. Goldschmidt
arXiv:0908.3629 [math.PR].