Lattice walks in a Weyl chamber of type B

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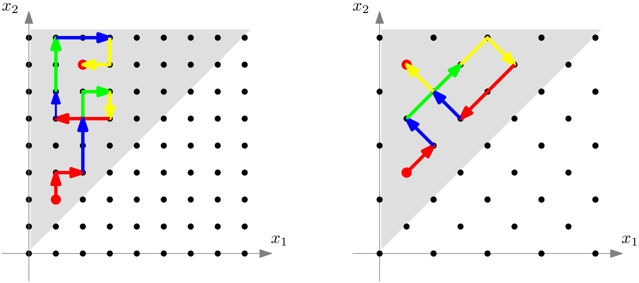
Outline

Introduction

Proof of the asymptotics for $P_n^+(\mathbf{u} \to \mathbf{v})$

Applications

The model



We consider lattice walks on a regular lattice $\mathcal{L} \subset \mathbb{R}^k$ that are confined to the region

$$\mathcal{W}^0 = \{(x_1, \ldots, x_k) \in \mathcal{L} \;:\; 0 < x_1 < \cdots < x_k\}\,.$$

The walks are required to be reflectable. (This restricts $\mathcal{L}$ as well as the steps the walks may consist of.)

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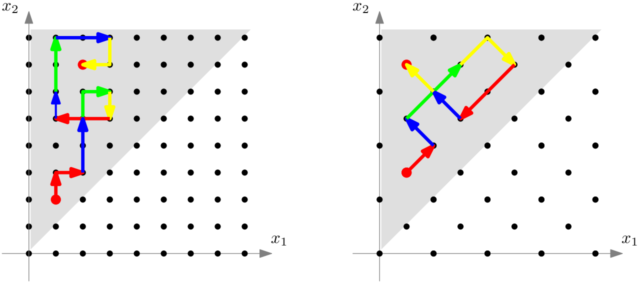

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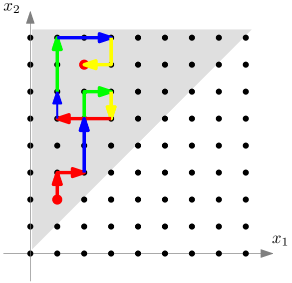

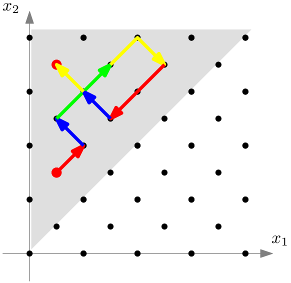

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Motivation

Motivation for studying these objects:

- *k*-noncrossing tangled diagrams
- Vicious walkers models:
  - lock step model
  - random turns model
- Young tableaux, rhombus tilings, plane partitions, random matrix theory,. . .

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Some notation

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$$\mathcal{W} = \left\{(x_1, \ldots, x_k) \in \mathbb{R}^k \ : \ 0 \leq x_1 \leq \cdots \leq x_k\right\}$$

Let $\left\{\mathbf{b}^{(1)}, \ldots, \mathbf{b}^{(k)}\right\}$ denote the canonical basis in $\mathbb{R}^k$, and set

$$\Delta = \left\{\mathbf{b}^{(j+1)} - \mathbf{b}^{(j)} \ : \ 1 \leq j < k\right\} \cup \left\{\mathbf{b}^{(1)}\right\}.$$

The set Δ is a *root system* of the reflection group of type B_k generated by the reflections in the hyperplanes

$$x_{j+1} - x_j = 0 \quad \text{for } 1 \leq j < k \qquad \text{and} \qquad x_1 = 0.$$

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Atomic step sets and composite step sets

Definition

Let $\mathcal{A} \subseteq \mathbb{R}^k$ be a finite set and denote by $\mathcal{L}$ the $\mathbb{Z}$-lattice spanned by $\mathcal{A}$. Then the set $\mathcal{A}$ is said to be an atomic step set if and only if

- If $\mathbf{a} \in \mathcal{A}$ then $r_\alpha(\mathbf{a}) \in \mathcal{A}$ for all $\alpha \in \Delta$.
- If $\mathbf{u} \in \mathcal{W}^0 \cap \mathcal{L}$ and $\mathbf{a} \in \mathcal{A}$ then $\mathbf{u} + \mathbf{a} \in \mathcal{W}$.

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A finite set $\mathcal{S}$ consisting of finite sequences of elements of an atomic step set is said to be an composite step set if and only if

$$(\mathbf{a}^{(1)}, \ldots, \mathbf{a}^{(m)}) \in \mathcal{S} \implies (r_\alpha(\mathbf{a}^{(1)}), \ldots, r_\alpha(\mathbf{a}^{(j)}), \mathbf{a}^{(j+1)}, \ldots, \mathbf{a}^{(m)}) \in \mathcal{S}$$

for all $\alpha \in \Delta$ and $j = 1, \ldots, m$.

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Some more notation

We will always use the following notation:

- The set $\mathcal{A}$ denotes an atomic step set.
- The $\mathbb{Z}$-lattice spanned by $\mathcal{A}$ is denoted by $\mathcal{L}$.
- By $\mathcal{S}$ we denote a composite step set over $\mathcal{A}$.
- $w : \mathcal{S} \to \mathbb{R}_+$ denotes a weight function such that $w(\mathbf{s}) = w(\hat{\mathbf{s}})$ whenever $\mathbf{s}$ and $\hat{\mathbf{s}}$ have the same length (as sequences over $\mathcal{A}$).

The step generating function

We associate

$$\mathbf{a} = (a_1, \ldots, a_k) \in \mathcal{A} \qquad \longleftrightarrow \qquad \mathbf{z}^{\mathbf{a}} := \prod_{j=1}^{k} z_j^{a_j}$$

and

$$\mathbf{s} = (\mathbf{a}^{(1)}, \ldots, \mathbf{a}^{(m)}) \in \mathcal{S} \qquad \longleftrightarrow \qquad w(\mathbf{s}) \prod_{j=1}^{k} \mathbf{z}^{\mathbf{a}^{(j)}}.$$

The step generating function $S(z_1, \ldots, z_k)$ is defined by

$$S(z_1, \ldots, z_k) = \sum_{\mathbf{s} \in \mathcal{S}} w(\mathbf{s}) \prod_{\mathbf{g} \in \mathbf{s}} \mathbf{z}^{\mathbf{g}}.$$

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If $S(z_1, \ldots, z_k)$ is a step generating function, then

- The generating function for all paths of length n starting in $\mathbf{0} \in \mathcal{L}$ (and arbitrary end point) is given by $S(z_1, \ldots, z_k)^n$.
- For $\mathbf{c} = (c_1, \ldots, c_k) \in \mathcal{L}$, the coefficient $[z_1^{c_1} \ldots z_k^{c_k}] S(z_1, \ldots, z_k)^n$ equals the weighted sum of all paths of length n from $\mathbf{0} \in \mathcal{L}$ to $\mathbf{c} \in \mathcal{L}$.
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Statement of the problem

Let $\mathbf{u}, \mathbf{v} \in \mathcal{W}^0 \cap \mathcal{L}$. We are interested in

- $P_n^+(\mathbf{u} \to \mathbf{v})$, the generating function of n-step walks from $\mathbf{u}$ to $\mathbf{v}$ confined to $\mathcal{W}^0$.
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Known results

Answers to special instances of these questions can be found in the literature:

- Asymptotic growth order of
  - k-noncrossing tangled diagrams (Zeilberger et al. [1])
  - Vicious walkers with free end point (Grabiner [2])
- Precise asymptotics for special configurations of the lock step model of vicious walkers (Krattenthaler [4] et al., Rubey [5])

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Asymptotics for $P_n^+(\mathbf{u} \to \mathbf{v})$

Theorem

Let $\mathcal{M}$ we denote the set of maximal points of $(\varphi_1, \ldots, \varphi_k) \mapsto |S(e^{i\varphi_1}, \ldots, e^{i\varphi_k})|$. We have the asymptotics

$$P_n^+(\mathbf{u}\to\mathbf{v}) = |\mathcal{M}| S(1,\ldots,1)^n \left(\frac{2}{\pi}\right)^{k/2} \left(\frac{S(1,\ldots,1)}{nS''(1,\ldots,1)}\right)^{k^2+k/2} \times \frac{\left(\prod_{1\le j<m\le k}(v_m^2 - v_j^2)(u_m^2 - u_j^2)\right)\left(\prod_{j=1}^{k} v_j u_j\right)}{\left(\prod_{j=1}^{k}(2j-1)!\right)} \left(1 + O(n^{-1/4})\right)$$

as $n \to \infty$ in the set $\{n \,:\, P_n^+(\mathbf{u}\to\mathbf{v}) > 0\}$.

Asymptotics for $P_n^+(\mathbf{u})$

Theorem

We have the asymptotics

$$P_n^+(\mathbf{u}) = S(1,\ldots,1)^n \left(\frac{2}{\pi}\right)^{k/2} \left(\frac{S(1,\ldots,1)}{nS''(1,\ldots,1)}\right)^{k^2/2} \times \left(\prod_{j=1}^{k} \frac{u_j (j-1)!}{(2j-1)!}\right) \left(\prod_{1\le j<m\le k} (u_m^2 - u_j^2)\right) \left(1 + O\left(n^{-1/4}\right)\right)$$

as $n \to \infty$. Here, $S''(1,\ldots,1)$ denotes the second derivative of $S(z_1,\ldots,z_k)$ with respect to any of the z_j.

Outline of the proof

Techniques: reflection principle, generating functions, saddlepoint method

- Integral formula for the number of "good walks" $(P_n^+(\mathbf{u} \rightarrow \mathbf{v}))$.
- What do composite step sets look like? What properties do the associated step generating functions have?
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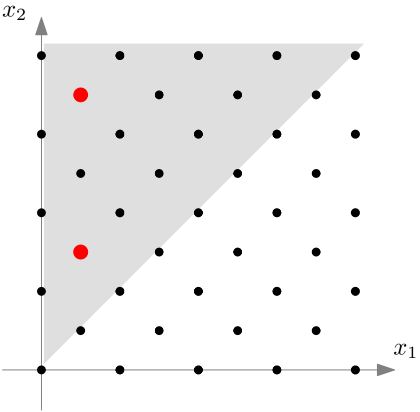
Exact enumeration: The reflection principle

Theorem (Gessel, Zeilberger)

We have

$$P_n^+(\mathbf{u} \to \mathbf{v}) = \sum_{r \in B_k} (-1)^{l(r)} P_n(r(\mathbf{u}) \to \mathbf{v}).$$



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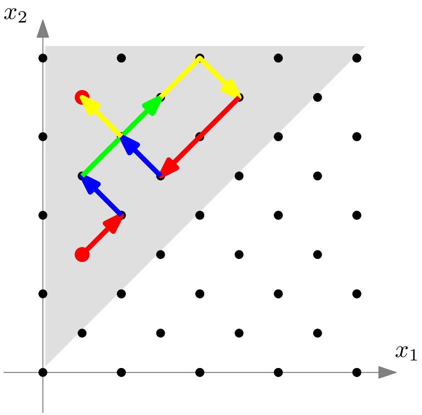

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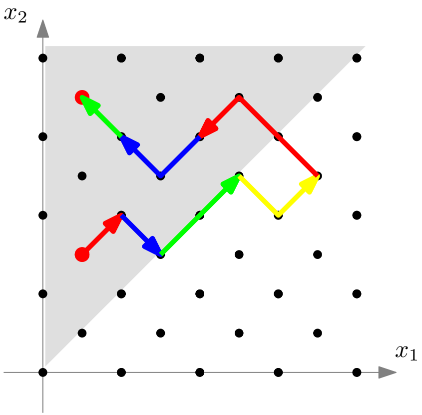

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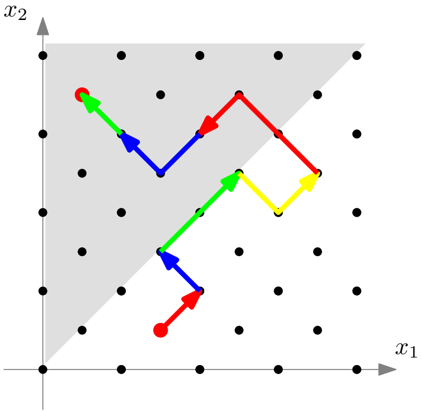


An integral representation

Lemma

For any two lattice points $\mathbf{u}, \mathbf{v} \in \mathcal{W}^0 \cap \mathcal{L}$ *we have*

$$P_n^+(\mathbf{u} \to \mathbf{v}) =$$

$$\frac{1}{(2\pi i)^k} \int \cdots \int_{|z_1|=\cdots=|z_k|=1} \det_{1 \le j,m \le k} \left(z_j^{u_m} - z_j^{-u_m} \right) S(z_1, \dots, z_k)^n \left(\prod_{j=1}^{k} \frac{dz_j}{z_j^{v_j+1}} \right).$$

Proof of the integral representation

The reflection principle gives us

$$P_n^+(\mathbf{u}\to\mathbf{v}) =$$

$$\sum_{\substack{\sigma\in\mathfrak{S}_k \\ \varepsilon_1,\dots,\varepsilon_k\in\{-1,+1\}}} \left(\prod_{j=1}^{k}\varepsilon_j\right) \operatorname{sgn}(\sigma) \left[z_1^{v_1-\varepsilon_1 u_{\sigma(1)}} \dots z_k^{v_k-\varepsilon_k u_{\sigma(k)}}\right] S(z_1,\dots,z_k)^n.$$

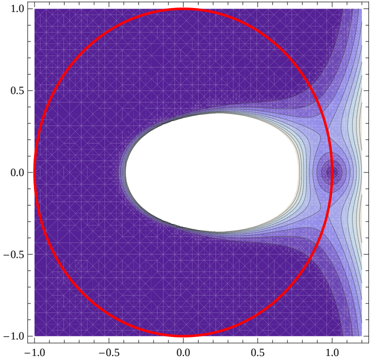
Now, the result follows from Cauchy's integral formula followed by an interchange of summation and integration and the identity

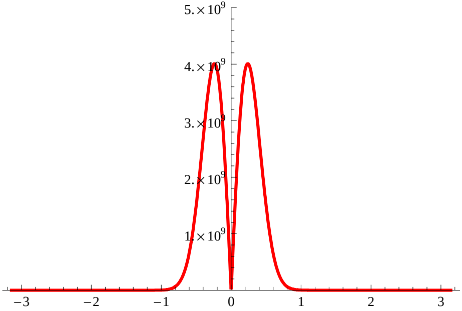
$$\sum_{\substack{\sigma\in\mathfrak{S}_k \\ (\varepsilon_1,\dots,\varepsilon_k)\in\{-1,+1\}^k}} \left(\prod_{j=1}^{k}\varepsilon_j\right) \operatorname{sgn}(\sigma) \left(\prod_{j=1}^{k} z_j^{\varepsilon_j u_{\sigma(j)}}\right) = \det_{1\le j,m\le k}\left(z_j^{u_m} - z_j^{-u_m}\right).$$

Example: 2-noncrossing tangled diagrams

For 2-noncrossing tangled diagrams, the integral derived on the previous two pages is given by ($\mathbf{a} = (1, 2)$)

$$P_n^+(\mathbf{a} \to \mathbf{a}) = \frac{1}{2\pi i} \int_{|z|=1} \left(z - \frac{1}{z}\right) \left(1 + z + \frac{1}{z} + \left(z + \frac{1}{z}\right)^2\right)^n \frac{dz}{z}.$$





Step generating functions

Lemma (Grabiner and Magyar)

If the finite set $\mathcal{G} \subset \mathbb{R}^k$ is reflectable, then $\mathcal{G}$ is isomorphic either to

$$\left\{\pm\mathbf{b}^{(1)}, \pm\mathbf{b}^{(2)}, \ldots, \pm\mathbf{b}^{(k)}\right\} \qquad or \qquad \left\{\sum_{j=1}^{k} \varepsilon_j \mathbf{b}^{(j)} \ : \ \varepsilon_1, \ldots, \varepsilon_k \in \{-1, +1\}\right\}.$$

Corollary

The generating function for an atomic step set is either equal to

$$\sum_{j=1}^{k}\left(z_j + \frac{1}{z_j}\right) \qquad or \qquad \prod_{j=1}^{k}\left(z_j + \frac{1}{z_j}\right).$$

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Corollary

The generating function for an atomic step set is either equal to

$$\sum_{j=1}^{k}\left(z_j + \frac{1}{z_j}\right) \qquad or \qquad \prod_{j=1}^{k}\left(z_j + \frac{1}{z_j}\right).$$

Step generating functions

Lemma

Let $\mathcal{S}$ be an admissible set of steps. Then there exists a polynomial $P(x)$ with non-negative coefficients such that the associated step generating function $S(z_1, \ldots, z_k)$ is equal to either

$$P\left(\sum_{j=1}^{k}\left(z_j + \frac{1}{z_j}\right)\right) \quad or \quad P\left(\prod_{j=1}^{k}\left(z_j + \frac{1}{z_j}\right)\right).$$

Proof (Sketch).

- If $\mathcal{S}$ contains one step of length ℓ, then $\mathcal{S}$ contains all steps of length ℓ.
- By definition, all these steps have the same weight.

□

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Step generating functions

Lemma

Let $\mathcal{S}$ be an admissible set of steps, and let $S(z_1, \dots, z_k)$ denote the corresponding weighted step generating function. Then, all maxima of the function $(\varphi_1, \dots, \varphi_k) \mapsto |S(e^{i\varphi_1}, \dots, e^{i\varphi_k})|$ lie within the set $\{0, \pi\}^k$. The point $(\varphi_1, \dots, \varphi_k) = (0, \dots, 0)$ is always a maximum.

Proof (Sketch).

From the last lemma we deduce that $S(e^{i\varphi_1}, \dots, e^{i\varphi_k})$ is either of the form $P\left(2\sum_{j=1}^k \cos(\varphi_j)\right)$ or $P\left(2^k \prod_{j=1}^k \cos(\varphi_j)\right)$ for some non-negative polynomial $P(x)$. Consequently, we have $|S(e^{i\varphi_1}, \dots, e^{i\varphi_k})| \le S(1, \dots, 1)$. □

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Proof of the main theorem

$$P_n^+(\mathbf{u}\to\mathbf{v}) = \frac{1}{(2\pi i)^k} \int\limits_{|z_1|=\cdots=|z_k|=1} \det_{1\le j,m\le k}\left(z_j^{u_m} - z_j^{-u_m}\right) S(z_1,\ldots,z_k)^n \prod_{j=1}^{k} \frac{dz_j}{z_j^{v_j+1}}$$

$$= \int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi} \det_{1\le j,m\le k}\Big(\sin(u_m\varphi_j)\Big) S\left(e^{i\varphi_1},\ldots,e^{i\varphi_k}\right)^n \left(\prod_{j=1}^{k} e^{-iv_j\varphi_j}\frac{d\varphi_j}{\pi}\right)$$

The dominant contribution is captured by

$$\sum_{\hat\varphi\in\mathcal{M}} \int\cdots\int_{U_n(\hat\varphi)} \det_{1\le j,m\le k}\Big(\sin(u_m\varphi_j)\Big) S\left(e^{i\varphi_1},\ldots,e^{i\varphi_k}\right)^n \left(\prod_{j=1}^{k} e^{-iv_j\varphi_j}\frac{id\varphi_j}{\pi}\right),$$

where $\mathcal{M} \subseteq \{0,\pi\}^k$ is the set of maxima of $|S(e^{i\varphi_1},\ldots,e^{i\varphi_k})|$, and $U_n(\hat\varphi) = \left\{\varphi \in [0,2p)^k \ : \ |\hat\varphi - \varphi|_\infty < n^{-5/12}\right\}$.

Proof of the main theorem

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$$= \int\limits_{-\pi}^{\pi}\ldots\int\limits_{-\pi}^{\pi} \det_{1\le j,m\le k}\Big(\sin(u_m\varphi_j)\Big) S\left(e^{i\varphi_1},\ldots,e^{i\varphi_k}\right)^n \left(\prod_{j=1}^{k} e^{-iv_j\varphi_j}\frac{d\varphi_j}{\pi}\right)$$

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$U_n(\hat\varphi) = \left\{\varphi \in [0,2p)^k \;:\; |\hat\varphi - \varphi|_\infty < n^{-5/12}\right\}$.

Proof of the main theorem

Fix a point $\hat{\varphi} \in \mathcal{M}$ and consider the integral

$$\int_{U_n(\hat{\varphi})} \det_{1\le j,m\le k}\Big(\sin(u_m\varphi_j)\Big) S\left(e^{i\varphi_1},\dots,e^{i\varphi_k}\right)^n \left(\prod_{j=1}^{k} e^{-iv_j\varphi_j}\frac{id\varphi_j}{\pi}\right)$$

$$= \left(\frac{2}{\pi}\right)^k \int_0^{n^{-\frac{5}{12}}} \cdots \int_0^{n^{-\frac{5}{12}}} \det_{1\le j,m\le k}\left((-1)^{(v_m+u_j)\frac{\hat{\varphi}_j}{\pi}} \sin(v_m\varphi_j)\sin(u_j\varphi_j)\right)$$

$$\times\, S\left(e^{i(\varphi_1+\hat{\varphi}_1)},\dots,e^{i(\varphi_k+\hat{\varphi}_k)}\right)^n \left(\prod_{j=1}^{k} d\varphi_j\right).$$

Proof of the main theorem

For this range of integration, the Taylor series expansion of $\log\left|S\left(e^{i(\varphi_1+\hat{\varphi}_1)}, \ldots, e^{i(\varphi_k+\hat{\varphi}_k)}\right)\right|$ centered at $\hat{\varphi}$ satisfies

$$\log\left|S\left(e^{i(\varphi_1+\hat{\varphi}_1)}, \ldots, e^{i(\varphi_k+\hat{\varphi}_k)}\right)\right| = \log S(1, \ldots, 1) - \Lambda\left(\sum_{j=1}^{k} \frac{\varphi_j^2}{2}\right) + O\left(n^{-5/4}\right)$$

as $n \to \infty$, where $\Lambda = \frac{S''(1,\ldots,1)}{S(1,\ldots,1)} > 0$ and $S''(z_1, \ldots, z_k) = \frac{\partial^2}{\partial z_1^2} S(z_1, \ldots, z_k)$.

Proof of the main theorem

Conseqently, the integral can be rewritten as

$$\left(\frac{2}{\pi}\right)^k S(e^{i\hat{\varphi}_1}, \ldots, e^{i\hat{\varphi}_k})^n$$
$$\times \det_{1 \le j,m \le k} \left((-1)^{(v_m+u_j)\frac{\hat{\varphi}_j}{\pi}} \int_0^{n^{-5/12}} \sin(v_m \vartheta) \sin(u_j \vartheta) e^{-n\Lambda \frac{\vartheta^2}{2}} d\vartheta \right) \left(1 + O(n^{-\frac{1}{4}})\right)$$

as $n \to \infty$.

It can be checked that, whenever we have $P_n^+(\mathbf{u} \to \mathbf{v}) > 0$, then the factors $(-1)^{(v_m+u_j)\frac{\varphi_j}{\pi}}$ can be factorised out of the determinant, and cancel the sign of $S(e^{i\hat{\varphi}_1}, \ldots, e^{i\hat{\varphi}_k})^n$.

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Proof of the main theorem

As a consequence of the considerations above we know that, if $\mathbf{u}, \mathbf{v} \in \mathcal{L}$ and $n \in \mathbb{N}$ are chosen such that $P_n^+(\mathbf{u} \to \mathbf{v}) > 0$, then we have

$$P_n^+(\mathbf{u} \to \mathbf{v}) = |\mathcal{M}| S(1, \ldots, 1)^n \times \det_{1 \le j,m \le k} \left(\frac{2}{\pi} \int_0^{n^{-\frac{5}{12}}} \sin(v_m \vartheta) \sin(u_j \vartheta) e^{-n\Lambda \frac{\vartheta^2}{2}} \, d\vartheta \right) \left(1 + O(n^{-\frac{1}{4}})\right),$$

where $|\mathcal{M}|$ denotes the number of points of maximum modulus of $S(e^{i\varphi_1}, \ldots, e^{i\varphi_k})$.

Proof of the main theorem

Now, the change of variables $u = n\Lambda\vartheta^2/2$ gives

$$\frac{2}{\pi}\int_0^{n^{-5/12}} \vartheta^{2\ell} e^{-n\Lambda\vartheta^2/2} d\vartheta = \frac{2}{n\Lambda\pi}\int_0^{\Lambda n^{1/6}/2} \left(\frac{2u}{n\Lambda}\right)^{\ell-1/2} e^{-u} du$$

$$\sim \frac{1}{\pi}\left(\frac{2}{n\Lambda}\right)^{\ell+1/2} \Gamma\left(\ell + \frac{1}{2}\right)$$

as $n \to \infty$, which shows that

$$\frac{2}{\pi}\int_0^{n^{-\frac{5}{12}}} \sin(v_m\vartheta)\sin(u_j\vartheta) e^{-n\Lambda\frac{\vartheta^2}{2}} d\vartheta$$

can be expanded into an asymptotic series in decreasing powers of n.

Proof of the main theorem

$$\det_{1\le j,m\le k}\left(\frac{2}{\pi}\int_0^{n^{-\frac{5}{12}}}\sin(v_m\vartheta)\sin(u_j\vartheta)e^{-n\Lambda\frac{\vartheta^2}{2}}\,d\vartheta\right)$$

Interpreting this determinant as a polynomial (a truncated asymptotic series) with respect to the variables $v_1,\ldots,v_k,u_1,\ldots,u_k$, we see that it is divisible by

$$\left(\prod_{1\le j<m\le k}(v_m^2-v_j^2)(u_m^2-u_j^2)\right)\left(\prod_{j=1}^{k}v_ju_j\right).$$

The coefficient is easily seen to be

$$\left(\prod_{j=1}^{k}(2j-1)!\right)^{-2}\left(\frac{2}{n\Lambda}\right)^{k^2+k/2}\det_{1\le j,m\le k}\left(\frac{1}{\pi}\Gamma\left(m+j-\frac{1}{2}\right)\right),$$

where

$$\det_{1\le j,m\le k}\left(\frac{1}{\pi}\Gamma\left((i+j-\frac{1}{2}\right)\right)=\pi^{-k/2}2^{-k^2}\prod_{j=1}^{k}(2j-1)!.$$

Proof of the main theorem

$$\det_{1\le j,m\le k}\left(\frac{2}{\pi}\int_0^{n^{-\frac{5}{12}}}\sin(v_m\vartheta)\sin(u_j\vartheta)e^{-n\Lambda\frac{\vartheta^2}{2}}\,d\vartheta\right)$$

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Asymptotics for $P_n^+(\mathbf{u} \to \mathbf{v})$

Theorem

Let $\mathcal{M}$ we denote the set of maximal points of $(\varphi_1, \ldots, \varphi_k) \mapsto |S(e^{i\varphi_1}, \ldots, e^{i\varphi_k})|$. We have the asymptotics

$$P_n^+(\mathbf{u}\to\mathbf{v}) = |\mathcal{M}| S(1,\ldots,1)^n \left(\frac{2}{\pi}\right)^{k/2} \left(\frac{S(1,\ldots,1)}{nS''(1,\ldots,1)}\right)^{k^2+k/2} \times \frac{\left(\prod_{1\le j<m\le k}(v_m^2 - v_j^2)(u_m^2 - u_j^2)\right)\left(\prod_{j=1}^{k} v_j u_j\right)}{\left(\prod_{j=1}^{k}(2j-1)!\right)} \left(1 + O(n^{-1/4})\right)$$

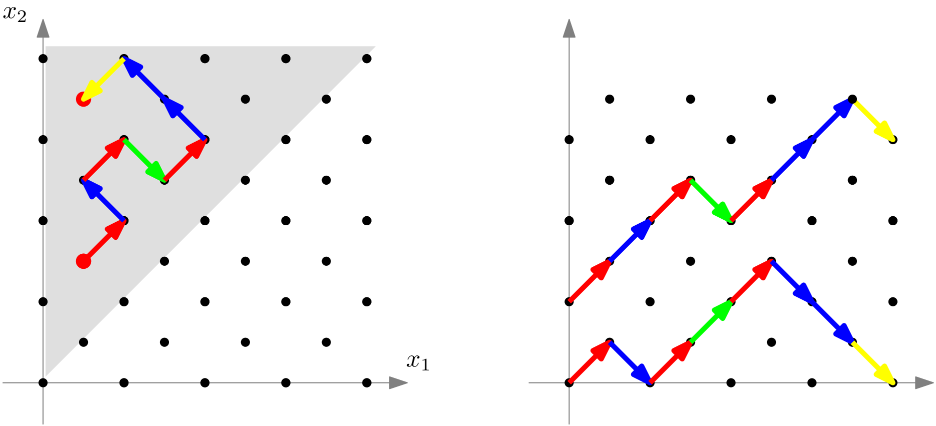
as $n \to \infty$ in the set $\{n \ : \ P_n^+(\mathbf{u}\to\mathbf{v}) > 0\}$.

Introduction

Proof of the asymptotics for $P_n^+(\mathbf{u} \to \mathbf{v})$

Applications

Lock step model of vicious walkers: Illustration



Lock step model of vicious walkers: Asymptotics

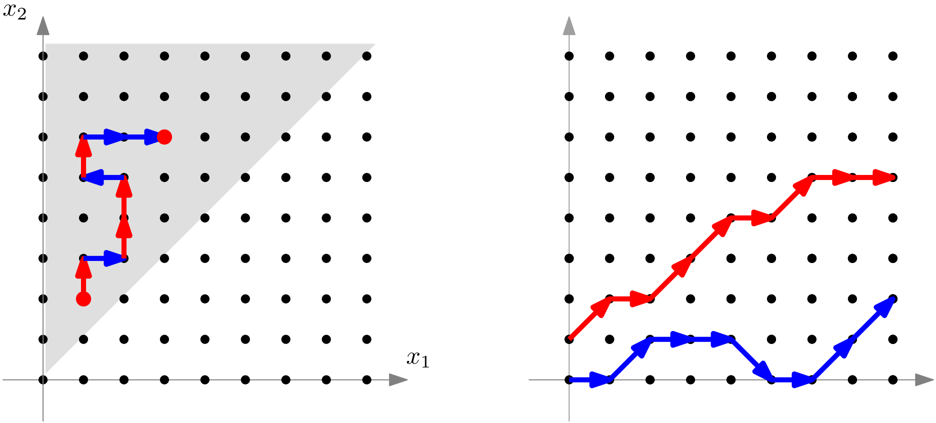
$$S(z_1, \ldots, z_k) = \prod_{j=1}^{k} \left(z_j + \frac{1}{z_j} \right)$$

The number of vicious walkers of length n with k walkers that start at $(0, u_1 - 1), \ldots, (0, u_k - 1)$ and end at $(n, v_1 - 1), \ldots, (n, v_k - 1)$ (we assume that $u_1 + v_1 \equiv n \mod 2$) is asymptotically equal to

$$2^{nk+3k/2} \pi^{-k/2} n^{-k^2-k/2} \frac{\left(\prod_{1 \le j < m \le k} (v_m^2 - v_j^2)(u_m^2 - u_j^2) \right) \left(\prod_{j=1}^{k} v_j u_j \right)}{\left(\prod_{j=1}^{k} (2j-1)! \right)}$$

as $n \to \infty$.

Random turns model of vicious walkers: Illustration



Random turns model of vicious walkers: Asymptotics

$$S(z_1, \ldots, z_k) = \sum_{j=1}^{k} \left(z_j + \frac{1}{z_j} \right)$$

The number of k vicious walkers in the random turns model, where the j-th walker starts at $(0, u_j - 1)$ and, after n steps ends at $(n, v_j - 1)$, is asymptotically equal to

$$2(2k)^n \left(\frac{2}{\pi}\right)^{k/2} \left(\frac{k}{n}\right)^{k^2+k/2} \frac{\left(\prod_{1 \le j < m \le k} (v_m^2 - v_j^2)(u_m^2 - u_j^2) \right) \left(\prod_{j=1}^{k} v_j u_j \right)}{\left(\prod_{j=1}^{k} (2j-1)! \right)}$$

as $n \to \infty$.

$(k+1)$-noncrossing tangled diagrams with isolated points

The number of $(k+1)$ tangled diagrams with isolated points equals the number of simple lattice walks of length n in $0 < x_1 < \cdots < x_k$ starting and ending in $\mathbf{a} = (1, 2, \ldots, k)$ with step generating function

$$S(z_1, \ldots, z_k) = 1 + \left(\sum_{j=1}^{k} z_j + \frac{1}{z_j}\right) + \left(\sum_{j=1}^{k} z_j + \frac{1}{z_j}\right)^2.$$

Conseqently, we have as $n \to \infty$

$$P_n^+(\mathbf{a} \to \mathbf{a}) \sim (1 + 2k + 4k^2)^n \left(\frac{2}{\pi}\right)^{k/2} \left(\frac{1 + 2k + 4k^2}{n(2 + 8k)}\right)^{k^2 + k/2} \left(\prod_{j=1}^{k} (2j-1)!\right).$$

$(k+1)$-noncrossing tangled diagrams without isolated points

The number of $(k+1)$ tangled diagrams without isolated points equals the number of simple lattice walks of length n in $0 < x_1 < \cdots < x_k$ starting and ending in $\mathbf{a} = (1, 2, \ldots, k)$ with step generating function

$$S(z_1, \ldots, z_k) = \left(\sum_{j=1}^{k} z_j + \frac{1}{z_j} \right) + \left(\sum_{j=1}^{k} z_j + \frac{1}{z_j} \right)^2 .$$

Conseqently, we have as $n \to \infty$

$$P_n^+(\mathbf{a} \to \mathbf{a}) \sim (2k + 4k^2)^n \left(\frac{2}{\pi} \right)^{k/2} \left(\frac{2k + 4k^2}{n(2 + 8k)} \right)^{k^2 + k/2} \left(\prod_{j=1}^{k} (2j-1)! \right)$$

as $n \to \infty$.

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