Fraction-free Computation of Simultaneous Padé Approximants

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Outline

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Simultaneous Padé Approximants

Given vector of power series $(f_1(z), f_2(z), ..., f_m(z)) \in \mathbb{K}[[z]]^{1\times m}$ and a multi-index of integers $\vec{n} = (n_1, ..., n_m)$,

Find polynomials $\mathfrak{P}_1(z), ..., \mathfrak{P}_m(z)$ with :

- for $k = 2, 3, ..., m$: approximation $\frac{\mathfrak{P}_k(z)}{\mathfrak{P}_1(z)} \approx \frac{f_k(z)}{f_1(z)}$,
- for $k = 1, ..., m$: degree bounds for $\mathfrak{P}_k(z)$,

Simultaneous Padé Approximants (more precise)

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$$f_k(z)\mathfrak{P}_1(z) - f_1(z)\mathfrak{P}_k(z) = \mathcal{O}(z^{|\vec{n}|+1}), \quad |\vec{n}| = n_1 + n_2 + ... + n_m,$$

- for $k = 1, ..., m$: degree bounds for $\mathfrak{P}_k(z)$, namely

$$\deg \mathfrak{P}_k(z) \leq |\vec{n}| - n_k.$$

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• homogeneous system with $(m-1)(|\vec{n}|+1)$ equations and $(m-1)(|\vec{n}|+1)+1$ unknowns $\Longrightarrow \exists$ solution.

• $\mathfrak{P}(\vec{n}, z) = \mathfrak{P}(z) = (\mathfrak{P}_1(z), ..., \mathfrak{P}_m(z))^t$ is also called type 2 Hermite-Padé approximant or "german polynomials".

History and Applications

- Introduced by Hermite (transcendence of e), used by Lindemann (transcendence of π)
- formalized by Padé (see book [Baker & Graves-Morris]), especially the case $m = 2$ of Padé approximants.
- landmark paper of Mahler "Perfect systems" (1968)

- Applications for inversion formulas of striped Toeplitz matrices [Labahn 92], linear system solving by vector rational reconstruction problems [Olesh & Storjohann 07], etc.

Vector Hermite-Padé approximants

A related problem is to find for $F(z) \in \mathbb{K}[[z]]^{m \times m}$ a vector $P(z) \in \mathbb{K}[z]^{m \times 1}$ with certain degree constraints such that $F(z)P(z) = \mathcal{O}(z^{\vec{\sigma}})$, i.e.,

$$j\text{th row of } F(z)P(z) \text{ is } \mathcal{O}(z^{\sigma_j}) \text{ for } j = 1, ..., m.$$

Example 1:

$$F(z) = \begin{bmatrix} f_1(z) & f_2(z) & \cdots & \cdots & f_m(z) \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \end{bmatrix}, \quad \deg P_k \leq n_k - 1, \quad \vec{\sigma} = (|\vec{n}| - 1, 0, \ldots., 0)$$

Hermite-Padé approximants (of type 1) or "latin polynomials".

Example 2:

$$\mathfrak{F}(z) = \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ -f_2(z) & f_1(z) & 0 & \cdots & 0 \\ -f_3(z) & 0 & f_1(z) & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ -f_m(z) & 0 & \cdots & 0 & f_1(z) \end{bmatrix}, \quad \vec{\sigma} = (0, |\vec{n}| + 1, \ldots, |\vec{n}| + 1)$$

simultaneous Padé approximants as before.

The work of Mahler

Under very strong regularity assumptions ("perfect" systems):
arrange m neighboring HP-approximants in $m \times m$ matrix

$$M(\vec{n}, z) = \left[P(\vec{n} + \vec{e}_1, z), \ldots, P(\vec{n} + \vec{e}_m, z)\right], \text{ degrees } \begin{bmatrix} = n_1 & < n_1 & \cdots & \cdots & < n_1 \\ < n_2 & = n_2 & < n_2 & \cdots & < n_2 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ < n_m & \cdots & \cdots & < n_m & = n_m \end{bmatrix}$$

normalized s.t. entries on diagonal monic.

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normalized s.t. entries on diagonal monic.
Arrange m neighboring SP-approximants in $m \times m$ matrix

$$\mathfrak{M}(\vec{n}, z) = \left[\mathfrak{P}(\vec{n} - \vec{e}_1, z), \ldots, \mathfrak{P}(\vec{n} - \vec{e}_m, z)\right],$$

$$\text{degrees } \begin{bmatrix} = |\vec{n}| - n_1 & < |\vec{n}| - n_1 & \cdots & \cdots & < |\vec{n}| - n_1 \\ < |\vec{n}| - n_2 & = |\vec{n}| - n_2 & < |\vec{n}| - n_2 & \cdots & < |\vec{n}| - n_2 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ < |\vec{n}| - n_m & \cdots & \cdots & < |\vec{n}| - n_m & = |\vec{n}| - n_m \end{bmatrix}$$

normalized s.t. entries on diagonal monic.

The work of Mahler

Under very strong regularity assumptions ("perfect" systems):
arrange m neighboring HP-approximants in $m \times m$ matrix

$$M(\vec{n}, z) = \left[P(\vec{n}+\vec{e}_1, z), \ldots, P(\vec{n}+\vec{e}_m, z)\right], \text{ degrees} \begin{bmatrix} = n_1 & < n_1 & \cdots & \cdots & < n_1 \\ < n_2 & = n_2 & < n_2 & \cdots & < n_2 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ < n_m & \cdots & \cdots & < n_m & = n_m \end{bmatrix}$$

normalized s.t. entries on diagonal monic.
Arrange m neighboring SP-approximants in $m \times m$ matrix

$$\mathfrak{M}(\vec{n}, z) = \left[\mathfrak{P}(\vec{n}-\vec{e}_1, z), \ldots, \mathfrak{P}(\vec{n}-\vec{e}_m, z)\right],$$

$$\text{degrees} \begin{bmatrix} = |\vec{n}| - n_1 & < |\vec{n}| - n_1 & \cdots & \cdots & < |\vec{n}| - n_1 \\ < |\vec{n}| - n_2 & = |\vec{n}| - n_2 & < |\vec{n}| - n_2 & \cdots & < |\vec{n}| - n_2 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ < |\vec{n}| - n_m & \cdots & \cdots & < |\vec{n}| - n_m & = |\vec{n}| - n_m \end{bmatrix}$$

normalized s.t. entries on diagonal monic.

Duality $M(\vec{n}, z)\mathfrak{M}(\vec{n}, z)^t = z^{|\vec{n}|} I_m$.
Ingredient $F(z)\mathfrak{F}(z)^t = f_1(z) I_m$ [B-L, JCAM 97]

The work of Mahler (continued)

Mahler also shows how to determine $U_j(\vec{n}, z) \in \mathbb{K}[z]^{m \times m}$ from $M(\vec{n}, z)$ such that

$$M(\vec{n} + \vec{e}_j, z) = M(\vec{n}, z)\, U_j(\vec{n}, z).$$

Since $M(\vec{0}, z) = I_m$, this gives a recursive way to compute $M(\vec{n}, z)$ for any multi-index $\vec{n}$.

By duality, one may deduce $\mathfrak{U}_j(\vec{n}, z) \in \mathbb{K}[z]^{m \times m}$ such that

$$\mathfrak{M}(\vec{n} + \vec{e}_j, z) = \mathfrak{M}(\vec{n}, z)\, \mathfrak{U}_j(\vec{n}, z).$$

Dependency on $\mathfrak{M}(\vec{n}, z)$ only implicit in paper of Mahler...

The σ–basis algorithm

Idea: Generalize work of Mahler to vector HP, but drop all regularity assumptions [B-L, SIMAX 94].

- For fixed $F(z)$ and fixed order vector $\vec{\sigma}$, consider the $\mathbb{K}[z]$ module of all $P(z) \in \mathbb{K}[z]^m$ having order $\mathcal{O}(z^{\vec{\sigma}})$.
- determine basis $\in \mathbb{K}[z]^{m \times m}$ of this module with particular degree constraints allowing to parametrize all vectors $P(z) \in \mathbb{K}[z]^m$ in module with degree $\leq \vec{n} + (d, ..., d)$
- compute such $\vec{\sigma}$-bases recursively by increasing in each step one component of the order by 1.

$M(\vec{n} + (d, ..., d), z)$ for $d \in \mathbb{Z}$ does the job, but does not always exist! Also, it is cheaper to weaken a bit the degree constraints for our bases.

Summary of algorithms

- (Vector) Hermite-Padé :
  - Beckermann-Labahn (SIMAX - 1994)
  - P. Giorgi, C-P. Jeannerod, G. Villard (ISSAC 2003)
    - fast polynomial matrix arithmetic
  - Beckermann-Labahn (SIMAX 2000)
    - fraction-free arithmetic
  - Zhou-Labahn (ISSAC 2009)
    - fast arithmetic
- Simultaneous Padé
  - Olesky and Storjohann (2007)

Cost for SP fraction-free algorithms

If input has size $O(\kappa)$ and $N = |\vec{n}|$ then :

- Fraction-Free Gaussian Elimination (FFGE) of Bareiss :
  - Bit complexity of operations: $O(\kappa^2 m^6 N^5)$
- B-L [SIMAX 2000] :
  - Bit complexity of operations: $O(\kappa^2 m^5 N^4)$
  - Size of objects : $O(\kappa m N)$
- B-L [ISAAC 2009] : Today
  - Bit complexity of operations: $O(\kappa^2 m^2 N^4)$
  - Size of objects : $O(\kappa N)$

Outline

Why fraction-free ?

Imagine that $\mathbb{K} = \mathbb{Q}$ (or $\mathbb{K} = \mathbb{C}(a_1, ..., a_n)$).

- Computation is easier in integral domain $\mathbb{Z}$ (or $\mathbb{C}[a_1, ..., a_n]$), no expensive GCD computations for simplifying fractions.
- We want to divide only through predictable factors in order to stay in our integral domain, and control coefficient growth.

In what follows: all data in integral domain, of size $\mathcal{O}(\kappa)$.

How to solve homog. systems without fractions?

For the system

$$\begin{bmatrix} c_{2,1} & c_{2,2} & \cdots & c_{2,m} \\ c_{3,1} & c_{3,2} & \cdots & c_{3,m} \\ \vdots & \vdots & & \vdots \\ c_{m,1} & c_{m,2} & \cdots & c_{m,m} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

we compute the Cramer solution via FFGE

$$c_{j,\ell}^{k+1} = \frac{1}{c_{k,k-1}^{k-1}} \left(c_{j,\ell}^{k} c_{k+1,k}^{k} - c_{j,k}^{k} c_{k+1,\ell}^{k} \right),$$

$$c_{j,\ell}^{1} = c_{j,\ell}, \quad c_{1,0}^{0} = 1$$

(Sylvester identity), gives

Cramer solution for homogeneous systems

Cramer solution

$$y_1 = \det \begin{bmatrix} c_{2,1} & c_{2,2} & c_{2,3} & \cdots & c_{2,m} \\ c_{3,1} & c_{3,2} & c_{3,3} & \cdots & c_{3,m} \\ \vdots & \vdots & \vdots & & \vdots \\ c_{m,1} & c_{m,2} & c_{m,3} & \cdots & c_{m,m} \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

Cramer solution for homogeneous systems

Cramer solution

$$y_{2} = \det \begin{bmatrix} c_{2,1} & c_{2,2} & c_{2,3} & \cdots & c_{2,m} \\ c_{3,1} & c_{3,2} & c_{3,3} & \cdots & c_{3,m} \\ \vdots & \vdots & \vdots & & \vdots \\ c_{m,1} & c_{m,2} & c_{m,3} & \cdots & c_{m,m} \\ 0 & 1 & 0 & \cdots & 0 \end{bmatrix}$$

Cramer solution for homogeneous systems

Cramer solution

$$y_m = \det \begin{bmatrix} c_{2,1} & c_{2,2} & c_{2,3} & \cdots & c_{2,m} \\ c_{3,1} & c_{3,2} & c_{3,3} & \cdots & c_{3,m} \\ \vdots & \vdots & \vdots & & \vdots \\ c_{m,1} & c_{m,2} & c_{m,3} & \cdots & c_{m,m} \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

Each component of Cramer solution is of size $\mathcal{O}(m\kappa)$.
Also, unique solution up to normalization iff Cramer solution is non-trivial.

Cramer solution for Hermite-Padé

$$P_{1}(\vec{n}, z) = \det$$

$$\left[\begin{array}{ccc|ccc|ccc}
f_{1,0} & & & f_{2,0} & & & f_{3,0} & & \\
\vdots & \ddots & & \vdots & \ddots & & \vdots & \ddots & \\
\vdots & & f_{1,0} & \vdots & & f_{2,0} & \vdots & & f_{3,0} \\
\vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
f_{1,|\vec{n}|-2} & \cdots & f_{1,|\vec{n}|-n_1-1} & f_{2,|\vec{n}|-2} & \cdots & f_{2,|\vec{n}|-n_2-1} & f_{3,|\vec{n}|-2} & \cdots & f_{3,|\vec{n}|-n_3-1} \\
\hline
z^0 & \cdots & z^{n_1-1} & 0 & \cdots & 0 & 0 & \cdots & 0
\end{array}\right]$$

$$= \pm K(\vec{n} - \vec{e}_1) z^{n_1 - 1} + \text{lower degrees}$$

Cramer solution for Hermite-Padé

$$P_2(\vec{n}, z) = \det$$

$$\left[\begin{array}{ccc|ccc|ccc}
f_{1,0} & & & f_{2,0} & & & f_{3,0} & & \\
\vdots & \ddots & & \vdots & \ddots & & \vdots & \ddots & \\
\vdots & & f_{1,0} & \vdots & & f_{2,0} & \vdots & & f_{3,0} \\
\vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
f_{1,|\vec{n}|-2} & \cdots & f_{1,|\vec{n}|-n_1-1} & f_{2,|\vec{n}|-2} & \cdots & f_{2,|\vec{n}|-n_2-1} & f_{3,|\vec{n}|-2} & \cdots & f_{3,|\vec{n}|-n_3-1} \\
\hline
0 & \cdots & 0 & z^0 & \cdots & z^{n_2-1} & 0 & \cdots & 0
\end{array}\right]$$

$$= \pm K(\vec{n} - \vec{e}_2) z^{n_2-1} + \text{lower degrees}$$

Residual for Cramer solution for Hermite-Padé

$$f_1(z)P_1(\vec{n},z) + f_2(z)P_2(\vec{n},z) + f_3(z)P_3(\vec{n},z) = \det$$

$$\left[\begin{array}{ccc|ccc|ccc}
f_{1,0} & & & f_{2,0} & & & f_{3,0} & & \\
\vdots & \ddots & & \vdots & \ddots & & \vdots & \ddots & \\
\vdots & & f_{1,0} & \vdots & & f_{2,0} & \vdots & & f_{3,0} \\
\vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
f_{1,|\vec{n}|-2} & \cdots & f_{1,|\vec{n}|-n_1-1} & f_{2,|\vec{n}|-2} & \cdots & f_{2,|\vec{n}|-n_2-1} & f_{3,|\vec{n}|-2} & \cdots & f_{3,|\vec{n}|-n_3-1} \\
\hline
f_1(z)z^0 & \cdots & f_1(z)z^{n_1-1} & f_2(z)z^0 & \cdots & f_2(z)z^{n_2-1} & f_3(z)z^0 & \cdots & f_3(z)z^{n_2-1}
\end{array}\right]$$

$$= K(\vec{n})z^{|\vec{n}|-1} + \text{higher order}$$

Cramer solution for SP seen as vector HP

$$\mathfrak{P}_{1}(\vec{n}, z) = \det \left[\begin{array}{ccc|ccc|ccc}
-f_{2,0} & & & f_{1,0} & & & & & \\
\vdots & \ddots & & \vdots & \ddots & & & & \\
\vdots & & -f_{2,0} & \vdots & & f_{1,0} & & & \\
\vdots & & \vdots & \vdots & & \vdots & & & \\
-f_{2,|\vec{n}|} & \cdots & -f_{2,n_1} & f_{1,|\vec{n}|} & \cdots & f_{1,n_2} & & & \\
\hline
-f_{3,0} & & & & & & f_{1,0} & & \\
\vdots & \ddots & & & & & \vdots & \ddots & \\
\vdots & & -f_{3,0} & & & & \vdots & & f_{1,0} \\
\vdots & & \vdots & & & & \vdots & & \vdots \\
-f_{3,|\vec{n}|} & \cdots & -f_{3,n_1} & & & & f_{1,|\vec{n}|} & \cdots & f_{1,n_3} \\
\hline
z^0 & \cdots & z^{|\vec{n}|-n_1} & 0 & \cdots & 0 & 0 & \cdots & 0
\end{array}\right]$$

Simplifies (only?) for $f_1(z) = 1$...

Cramer solution for SP seen as vector HP, $f_1(z) = 1$

$\mathfrak{P}_1(\vec{n}, z) =$

$$\det \left[\begin{array}{cccc}
-f_{2,|\vec{n}|-n_2+1} & -f_{2,|\vec{n}|-n_2+0} & \cdots & -f_{2,n_1-n_2+1} \\
-f_{2,|\vec{n}|-n_2+2} & -f_{2,|\vec{n}|-n_2+1} & \cdots & -f_{2,n_1-n_2+2} \\
\vdots & \vdots & & \vdots \\
-f_{2,|\vec{n}|} & -f_{2,|\vec{n}|-1} & \cdots & -f_{2,n_1} \\
\hline
-f_{3,|\vec{n}|-n_3+1} & -f_{3,|\vec{n}|-n_3+0} & \cdots & -f_{3,n_1-n_3+1} \\
-f_{3,|\vec{n}|-n_3+2} & -f_{3,|\vec{n}|-n_3+1} & \cdots & -f_{3,n_0-n_3+2} \\
\vdots & \vdots & & \vdots \\
-f_{3,|\vec{n}|} & -f_{3,|\vec{n}|-1} & \cdots & -f_{3,n_1} \\
\hline
z^0 & \cdots & \cdots & z^{|\vec{n}|-n_1}
\end{array} \right]$$

Coefficients of size $\mathcal{O}(|\vec{n}|\kappa)$ for $f_1(z) = 1$ versus size $\mathcal{O}(m|\vec{n}|\kappa)$ for general $f_1(z)$.

New formula always of size $\mathcal{O}(|\vec{n}|\kappa)$...

New Cramer solution for SP

$$\mathfrak{P}_{\color{red}1}(\vec{n}, z) = \det$$

$$\left[\begin{array}{ccc|ccc|ccc|c}
f_{1,0} & & & f_{2,0} & & & f_{3,0} & & & \\
\vdots & \ddots & & \vdots & \ddots & & \vdots & \ddots & & \\
\vdots & & f_{1,0} & \vdots & & f_{2,0} & \vdots & & f_{3,0} & \\
\vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \\
f_{1,|\vec{n}|} & \cdots & f_{1,|\vec{n}|-n_1} & f_{2,|\vec{n}|} & \cdots & f_{2,|\vec{n}|-n_2} & f_{3,|\vec{n}|} & \cdots & f_{3,|\vec{n}|-n_3} & \\
\hline
z^0 & \cdots & z^{n_1} & & & & & & & {\color{red}1} \\
& & & z^0 & \cdots & z^{n_2} & & & & {\color{red}0} \\
& & & & & & z^0 & \cdots & z^{n_3} & {\color{red}0}
\end{array}\right]$$

$$= \pm K(\vec{n} + \vec{e}_1) z^{|\vec{n}| - n_1} + \text{lower degrees}$$

New Cramer solution for SP

$$\mathfrak{P}_2(\vec{n}, z) = \det$$

$$\left[\begin{array}{ccc|ccc|ccc|c}
f_{1,0} & & & f_{2,0} & & & f_{3,0} & & & \\
\vdots & \ddots & & \vdots & \ddots & & \vdots & \ddots & & \\
\vdots & & f_{1,0} & \vdots & & f_{2,0} & \vdots & & f_{3,0} & \\
\vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \\
f_{1,|\vec{n}|} & \cdots & f_{1,|\vec{n}|-n_1} & f_{2,|\vec{n}|} & \cdots & f_{2,|\vec{n}|-n_2} & f_{3,|\vec{n}|} & \cdots & f_{3,|\vec{n}|-n_3} & \\
\hline
z^0 & \cdots & z^{n_1} & & & & & & & 0 \\
 & & & z^0 & \cdots & z^{n_2} & & & & 1 \\
 & & & & & & z^0 & \cdots & z^{n_3} & 0
\end{array}\right]$$

$$= \pm K(\vec{n} + \vec{e}_2) z^{|\vec{n}| - n_2} + \text{lower degrees}$$

ℓth residual for new Cramer solution for SP

$$\ell = 2:\ -f_{2}(z)\mathfrak{P}_1(\vec{n},z) + f_1(z)\mathfrak{P}_{2}(\vec{n},z) = \det$$

$$\left[\begin{array}{ccc|ccc|ccc|c}
f_{1,0} & & & f_{2,0} & & & f_{3,0} & & & \\
\vdots & \ddots & & \vdots & \ddots & & \vdots & \ddots & & \\
\vdots & & f_{1,0} & \vdots & & f_{2,0} & \vdots & & f_{3,0} & \\
\vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \\
f_{1,|\vec{n}|} & \cdots & f_{1,|\vec{n}|-n_1} & f_{2,|\vec{n}|} & \cdots & f_{2,|\vec{n}|-n_2} & f_{3,|\vec{n}|} & \cdots & f_{3,|\vec{n}|-n_3} & \\
\hline
z^0 f_1(z) & \cdots & z^{n_1} f_1(z) & & & & & & & -1 \\
 & & & z^0 f_2(z) & \cdots & z^{n_2} f_2(z) & & & & 1 \\
 & & & & & & z^0 & \cdots & z^{n_3} & 0
\end{array}\right]$$

ℓth residual for new Cramer solution for SP

$\ell = 2$: $-f_2(z)\mathfrak{P}_1(\vec{n},z) + f_1(z)\mathfrak{P}_2(\vec{n},z) = \det$

$$\left[\begin{array}{ccc|ccc|ccc|c}
f_{1,0} & & & f_{2,0} & & & f_{3,0} & & & \\
\vdots & \ddots & & \vdots & \ddots & & \vdots & \ddots & & \\
\vdots & & f_{1,0} & \vdots & & f_{2,0} & \vdots & & f_{3,0} & \\
\vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \\
f_{1,|\vec{n}|} & \cdots & f_{1,|\vec{n}|-n_1} & f_{2,|\vec{n}|} & \cdots & f_{2,|\vec{n}|-n_2} & f_{3,|\vec{n}|} & \cdots & f_{3,|\vec{n}|-n_3} & \\
\hline
z^0 f_1(z) & \cdots & z^{n_1} f_1(z) & z^0 f_2(z) & \cdots & z^{n_2} f_2(z) & z^0 f_3(z) & \cdots & z^{n_3} f_3(z) & 0 \\
 & & & z^0 f_2(z) & \cdots & z^{n_2} f_2(z) & & & & 1 \\
 & & & & & & z^0 & \cdots & z^{n_3} & 0
\end{array}\right]$$

$= O(z^{|\vec{n}|+1})$, more precisely

ℓth residual for new Cramer solution for SP

$$\ell = 2\text{:}\ -f_2(z)\mathfrak{P}_{\color{red}1}(\vec{n}, z) + f_1(z)\mathfrak{P}_{\color{red}2}(\vec{n}, z) = \pm z^{|\vec{n}|+1} \det$$

$$\left|\begin{array}{c|ccc|c|ccc|ccc}
f_{1,0} & & & & \color{red}{f_{2,0}} & & & & & & \\
\hline
f_{1,1} & f_{1,0} & & & \color{red}{f_{2,1}} & f_{2,0} & & & f_{3,0} & & \\
\vdots & \vdots & \ddots & & \vdots & \vdots & \ddots & & \vdots & \ddots & \\
\vdots & \vdots & & f_{1,0} & \vdots & \vdots & & f_{2,0} & \vdots & & f_{3,0} \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
f_{1,|\vec{n}|+1} & f_{1,|\vec{n}|} & \cdots & f_{1,|\vec{n}|-n_1+1} & \color{red}{f_{2,|\vec{n}|+1}} & f_{2,|\vec{n}|} & \cdots & f_{2,|\vec{n}|-n_2+1} & f_{3,|\vec{n}|} & \cdots & f_{3,|\vec{n}|-n_3+1}
\end{array}\right|$$

$$+O(z^{|\vec{n}|+2}).$$

HP Mahler-Cramer systems (provided that $K(\vec{n}) \neq 0$)

Up to normalization there exist unique m neighboring HP-approximants in $m \times m$ matrix

$$M(\vec{n}, z) = \left[P(\vec{n} + \vec{e}_1, z), ..., P(\vec{n} + \vec{e}_m, z)\right],$$

with degrees

$$\begin{bmatrix} = n_1 & < n_1 & \cdots & \cdots & < n_1 \\ < n_2 & = n_2 & < n_2 & \cdots & < n_2 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ < n_m & \cdots & \cdots & < n_m & = n_m \end{bmatrix},$$

with coefficients of entries in integral domain if leading coefficient of diagonal entries $= d(\vec{n}) = \pm K(\vec{n})$.

All coefficients are of size $\mathcal{O}(|\vec{n}|\kappa)$.

[B-L, 2000]

SP Mahler-Cramer systems (provided that $K(\vec{n}) \neq 0$)

Up to normalization there exist unique m neighboring SP-approximants in $m \times m$ matrix

$$\mathfrak{M}(\vec{n}, z) = \left[\mathfrak{P}(\vec{n} - \vec{e}_1, z), ..., \mathfrak{P}(\vec{n} - \vec{e}_m, z)\right],$$

with degrees

$$\begin{bmatrix} = |\vec{n}| - n_1 & < |\vec{n}| - n_1 & \cdots & \cdots & < |\vec{n}| - n_1 \\ < |\vec{n}| - n_2 & = |\vec{n}| - n_2 & < |\vec{n}| - n_2 & \cdots & < |\vec{n}| - n_2 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ < |\vec{n}| - n_m & \cdots & \cdots & < |\vec{n}| - n_m & = |\vec{n}| - n_m \end{bmatrix},$$

with coefficients of entries in integral domain if leading coefficient of diagonal entries $d(\vec{n}) = \pm K(\vec{n})$.

All coefficients are of size $\mathcal{O}(|\vec{n}|\kappa)$.

[B-L, 2009]

Outline

Computing SP Mahler-Cramer systems recursively

One has $\mathfrak{M}(\vec{0}, z) = I_m$. Suppose that $d(\vec{n}) = \pm K(\vec{n}) \neq 0$.

- How to check that $K(\vec{n} + \vec{e}_\pi) \neq 0$?
  This is true for at least one $\pi \in \{1, 2, ..., m\}$ since $f_1(0) \neq 0$.
- How to compute $\mathfrak{M}(\vec{n} + \vec{e}_\pi, z)$ from $\mathfrak{M}(\vec{n}, z)$? Column-wise
  - $d(\vec{n})\mathfrak{P}(\vec{n}, z) = \sum_{j=1}^{m} y_j \mathfrak{P}(\vec{n} - \vec{e}_j, z)$
    and $d(\vec{n} + \vec{e}_\pi) = y_\pi$.
  - for $j \neq \pi$:
    $d(\vec{n})\mathfrak{P}(\vec{n} + \vec{e}_\pi - \vec{e}_j, z) = y_\pi z \mathfrak{P}(\vec{n} - \vec{e}_j, z) - b_j \mathfrak{P}(\vec{n}, z)$.

How to find the y_j? Step 1

In order to obtain the correct degrees/order conditions for SP approximants of type $\vec{n}$, we require that

$$\begin{bmatrix} c_{2,1} & c_{2,2} & \cdots & c_{2,m} \\ c_{3,1} & c_{3,2} & \cdots & c_{3,m} \\ \vdots & \vdots & & \vdots \\ c_{m,1} & c_{m,2} & \cdots & c_{m,m} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

where

$$-f_\ell(z)\mathfrak{P}_1(\vec{n}-\vec{e}_j,z) + f_1(z)\mathfrak{P}_\ell(\vec{n}-\vec{e}_j,z) = z^{|\vec{n}|}c_{\ell,j} + \mathcal{O}(z^{|\vec{n}|+1})$$

How to find the y_j? Step 2

We have seen that there exist $\delta_\ell, \epsilon_j \in \{\pm 1\}$ such that $\delta_\ell\, \epsilon_j\, c_{\ell,j} = \det$

$$\begin{array}{c|ccc|c|ccc|cccc}
f_{1,0} & & & & f_{2,0} & & & & & & & \\ \hline
f_{1,1} & f_{1,0} & & & f_{2,1} & f_{2,0} & & & f_{3,0} & & & \\
\vdots & \vdots & \ddots & & \vdots & \vdots & \ddots & & \vdots & \ddots & & \\
\vdots & \vdots & & f_{1,0} & \vdots & \vdots & & f_{2,0} & \vdots & & f_{3,0} & \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots & \\
f_{1,|\vec{n}|+1} & f_{1,|\vec{n}|} & \cdots & f_{1,|\vec{n}|-n_1+1} & f_{2,|\vec{n}|+1} & f_{2,|\vec{n}|} & \cdots & f_{2,|\vec{n}|-n_2+1} & f_{3,|\vec{n}|} & \cdots & f_{3,|\vec{n}|-n_3+1} & \\ \hline
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1
\end{array}$$

(here $m = 3$, $\ell = 2$, $j = 3$) and thus Cramer solution

$$y_j = \det \begin{bmatrix} c_{2,1} & c_{2,2} & c_{2,3} & \cdots & c_{2,m} \\ c_{3,1} & c_{3,2} & c_{3,3} & \cdots & c_{3,m} \\ \vdots & \vdots & \vdots & & \vdots \\ c_{m,1} & c_{m,2} & c_{m,3} & \cdots & c_{m,m} \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix} = \pm [f_1(0) K(\vec{n})]^{m-2} K(\vec{n} + \vec{e}_j).$$

How to find the y_j? Step 3

Thus solution of full rank system

$$\begin{bmatrix} c_{2,1} & c_{2,2} & \cdots & c_{2,m} \\ c_{3,1} & c_{3,2} & \cdots & c_{3,m} \\ \vdots & \vdots & & \vdots \\ c_{m,1} & c_{m,2} & \cdots & c_{m,m} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

with FFGE and first pivot $c_{1,0}^0 = f_1(0)d(\vec{n})$ gives components $y_j = \pm K(\vec{n} + \vec{e}_j)$.

At least one $y_\pi \neq 0$.

What to do with the y_j?

Let $y_\pi = \pm K(\vec{n} + \vec{e}_\pi) \neq 0$.

- Cramer approximant $\mathfrak{P}(\vec{n}, z)$ is unique SP approximant of type $\vec{n}$ up to scaling.
  It's πth component is of degree $|\vec{n}| - n_\pi$, with leading coefficient $\pm y_\pi$.
- $$\mathfrak{P}(z) = \sum_{j=1}^{m} y_j \mathfrak{P}(\vec{n} - \vec{e}_j, z)$$
  satisfies the degree/order conditions for an SP approximant of type $\vec{n}$.
  It's πth component is of degree $|\vec{n}| - n_\pi$, with leading coefficient $y_\pi d(\vec{n})$.

Conclusion: $\mathfrak{P}(z) = d(\vec{n})\, \mathfrak{P}(\vec{n}, z)$.

How to get the other columns of the new Mahler-Cramer system?

Let $y_\pi = \pm K(\vec{n} + \vec{e}_\pi) \neq 0$, and $j \neq \pi$.

- Cramer approximant $\mathfrak{P}(\vec{n} + \vec{e}_\pi - \vec{e}_j, z)$ is unique SP approximant of type $\vec{n} + \vec{e}_\pi - \vec{e}_j$ up to scaling.
  It's jth component is of degree $|\vec{n}| - n_j$, with leading coefficient $\pm y_\pi$.
- $\mathfrak{P}(z) = z\, y_\pi\, \mathfrak{P}(\vec{n} - \vec{e}_j, z) - b_j\, \mathfrak{P}(\vec{n}, z)$ with

$$b_j = \text{coeff}(\mathfrak{P}_\pi(\vec{n} - \vec{e}_j, z), z^{|\vec{n} - \vec{e}_j| - n_\pi})$$

  satisfies the degree/order conditions for an SP approximant of type $\vec{n} + \vec{e}_\pi - \vec{e}_j$.
  It's jth component is of degree $|\vec{n} + \vec{e}_\pi - \vec{e}_j| - (n_j - 1)$, with leading coefficient $y_\pi\, d(\vec{n})$.

Conclusion: $\mathfrak{P}(z) = d(\vec{n})\, \mathfrak{P}(\vec{n} + \vec{e}_\pi - \vec{e}_j, z)$.

Computation Process

Input : $\vec{n}$, $f_1(z), f_2(z), ..., f_m(z)$, $f_1(0) \neq 0$.
Output : closest normal multi-indices $\vec{v}^{(0)}, \vec{v}^{(1)}, ..., |\vec{v}^{(k)}| = k$,

$$\mathfrak{M}(\vec{v}^{(0)}, z) = I_m \to \mathfrak{M}(\vec{v}^{(1)}, z) \to \mathfrak{M}(\vec{v}^{(2)}, z) \to ...$$

and $\vec{v}^{(k+1)} = \vec{v}^{(k)} + \vec{e}_\pi$ s.t. $K(\vec{v}^{(k+1)}) \neq 0$, and

$$\vec{n}_\pi - \vec{v}_\pi^{(k)} = \max\left\{\vec{n}_\pi - \vec{v}_j^{(k)} : K(\vec{v}^{(k)} + \vec{e}_j) \neq 0\right\}.$$

Order bases in shifted Popov form allow to parametrize all $\vec{n} + (d, d, ..., d)$ SP approximants for $d \in \mathbb{Z}$.

A small example...

Consider $\vec{n} = (3, 4, 3)$ and

$$\begin{aligned} f_1(z) &= 3 + 3\,z + 6\,z^2 + 18\,z^3 + 72\,z^4 + 360\,z^5 + O(z^{11}) \\ f_2(z) &= 1 + 8\,z^3 + 64\,z^6 + 512\,z^9 + O(z^{11}) \\ f_3(z) &= 1 - z + z^2 - z^3 + z^4 - z^5 + O(z^{11}) \end{aligned}$$

Closest normal indices

$$(0,0,0), (0,1,0), (1,1,0), (1,2,0), (1,2,1), (2,2,1), (2,3,1), \cdots$$

After step 4 the closest normal index is $\vec{v}^{(4)} = [1, 2, 1]$ with the SP Mahler-Cramer system given by

$$\mathfrak{M}(\vec{v}^{(4)}, z) = \begin{bmatrix} 48\,z^3 + 33\,z^2 + 30\,z + 3 & -9\,z^2 - 126\,z - 27 & -54\,z^2 - 36\,z - 18 \\ 9\,z + 1 & 48\,z^2 - 33\,z - 9 & -6\,z - 6 \\ -8\,z^2 + 8\,z + 1 & 72\,z^2 - 24\,z - 9 & 48\,z^3 - 6 \end{bmatrix}.$$

Columns SP approximants of index $(0, 2, 1)$, $(1, 1, 1)$ and $(1, 2, 0)$.

To construct $\mathfrak{M}(\vec{v}^{(5)}, z)$ the C matrix for the next step is given by

$$\begin{bmatrix} -6 & 54 & -252 \\ 210 & -738 & -252 \end{bmatrix}, \quad \text{with kernel} \quad \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1386 \\ 378 \\ 48 \end{bmatrix}.$$

New pivot index is $\pi = 1$. After replacing column 1 of $\mathfrak{M}(\vec{v}^{(4)}, z)$ by the linear combination of all three columns, we get a new column 1 which has order 5. Multiplying columns 2 and 3 by z then implies that all the columns now have order 5. In this case the degrees of the resulting matrix polynomial are

$$\begin{bmatrix} 3 & 3 & 3 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}$$

and the leading coefficient of the 1, 1 term is 48×1386. Eliminating the highest terms in row 1 of columns 2 and 3 using cross multiplication with the new column 1 then gives degrees of the form

$$\begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}.$$

Correct degree bounds for the SP Mahler-Cramer system for the closest normal point $\vec{v}^{(5)} = (2, 2, 1)$, it remains to divide by 48.

[B-L 2009] versus [B-L 2000]

At each iteration for Hermite-Padé [B-L 2000]

- Increase order of other columns using pivot column π
- Increase order of pivot column π
- Normalize order basis to get special shifted *Popov* form

At each iteration for Simultaneous Padé [B-L 2009]

- Increase order of pivot column using fraction-free Gaussian elimination on first term of residual
- Increase order of the other columns using the new pivot column
- Normalize order basis to get special shifted *Popov* form

Future Research

- Fraction-Free $\rightarrow$ modular methods
- Use alternative order basis algorithm for noncommutative case of Ore matrix polynomials

Code in Maple available at
www.cs.uwaterloo.ca/$\sim$glabahn/pade-code .